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Robust optimization of dynamic consumption streams and applications

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Prototypical choice problem throughout the talk

- given **time interval** for consumption
- **exogeneous uncertain endowments** and investment in risky assets traded in a **financial market**
- consumption set consists of **consumption streams** (synonymously **consumption rate processes**) over the time interval
- consumer has **variational preferences** (according to Maccheroni et al. (2006))
  - consumer uncertain about the model driving the endowments and the financial market
  - consumer is risk- and ambiguity averse
  - consumer makes an assessment of the plausibility of possible market models
  - **numerical representation:**

    $U(c) = \inf_{\mathrm{Q}}(E_{\mathrm{Q}}\,[u(c)] + \gamma(\mathrm{Q}))$ for any consumption rate process c
    * u concave **utility index**
    * γ **penalty function** assessing the plausibilities of priors with $\gamma(\mathrm{Q}) = \infty$ worst
  - special cases:
    * $\gamma(\bar{\mathrm{Q}}) = 0$ for a unique $\bar{\mathrm{Q}}$ and $\gamma(\mathrm{Q}) = \infty$ for $\mathrm{Q} \neq \bar{\mathrm{Q}}$

      $\rightarrow$ expected utility representation
    * $\gamma(\mathrm{Q}) < \infty \Rightarrow \gamma(\mathrm{Q}) = 0$

      $\rightarrow$ robust expected utility representation (cf. Gilboa/Schmeidler (1989))

$\rightarrow$ **robust optimization**

Issues of the talk

- **microeconomic aspects:**
  - optimality conditions
    (review of results from project + some recent completions)
  - new phenomena of individual behaviour in comparison with consumers whoses preferences have expected utility representation w.r.t. objective probability?

- **macroeconomic aspects:**
  - variational preferences responsible for market failure?
  - representative consumers in financial markets?

- **prospect:**

  consumption choice in presence of exogeneous shocks

Contents

1 The single-agent consumption model

- **The financial market:**
  - filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathrm{P})$ („usual conditions“)
  - d assets with joint discounted price process $(S(t))_{0 \leq t \leq T}$ being a semimartingale
  - market is arbitrage free in the sense that equivalent local martingale measures exist

  Remark: time-discrete financial markets subsumable under present framework

- **consumers' choice conditions:**
  - exogeneous endowment:
    - initial capital $w_0 > 0$
    - state-dependent endowment rate process $(e_t)_{0 \le t \le T}$ (nonnegative, progressively measurable)
  - consumption (partially) financiable by state-dependent investment process $(\vartheta_t)_{0 \le t \le T}$ (predictable) with a credit limit (admissible):

    net gains $\int_0^t \vartheta_s \, dS_s$ $(0 \le t \le T)$ have a state-independent simultaneous lower bound
  - consumer chooses an aggregation function over the time:

    μ distribution on $[0, T]$ with
    - restriction to $[0, T[$ has a density w.r.t. the uniform distribution to $[0, T[$
    - state-dependent cumulated endowment $\int_0^T e_t \, \mu(dt)$ is bounded P $-$a.s.

- **consumption set:**
  - consumption-investment plan $\big(c := (c_t)_{0\le t\le T}, \vartheta := (\vartheta_t)_{0\le t\le T}\big)$:
    - c consumption rate process (nonnegative, progressively measurable)

      ϑ admissible investment process

$$\downarrow$$

    - value process $(V_t^{c,\vartheta})_{0\le t\le T}$ w.r.t. (c,ϑ)

$$V_t^{c,\vartheta} := w_0 + \int_0^t \vartheta_s \, dS_s + \int_0^t (e_s - c_s)\mu(ds)$$

  - admissible consumption rate process $c := (c_t)_{0\le t\le T}$:

    ex. consumption-investment plan (c,ϑ) with $V_T^{c,\vartheta} \ge 0$ P −a.s.

$$\Updownarrow$$ (Karatzas/Zitkovic (2003))

  - **budget constraint:**

    consumption rate process $(c_t)_{0\le t\le T}$ satisfies

    - state-dependent cumulated consumption $\int_0^T c_t\mu(dt)$ is finite P −a.s.
    - E $[\int_0^T f \; c_t \; \mu(dt)] \le w_0 + E \; [\int_0^T f \; e_t \; \mu(dt)]$ for every pricing density f

- **consumers' preferences:**
  - intertemporal strictly concave and strictly increasing utility indices $u_t : [0, \infty[\to \mathbb{R}$ $(t \in [0, T])$
  - penalty function γ for priors:
    - associated with a convex risk measure ρ
      - $\rho(X) = \sup\limits_{\mathrm{Q}}(-E_{\mathrm{Q}}[X] - \gamma(\mathrm{Q}))$
      - $\gamma(\mathrm{Q}) = \sup\limits_{X}(-E_{\mathrm{Q}}[X] - \rho(X))$

      (for criteria see V.K. SFB 2007-010)
    - $\rho(X) > 0$ for $X \leq 0, \; X \neq 0$
    - $\rho(X) = -E_{\mathrm{Q}}[X] - \gamma(\mathrm{Q})$ for some prior Q (X arbitrary)

      $\Updownarrow$ (V.K. SFB 2007-010)

      $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$

- numerical representation U of the preferences on consumption rate processes:

$$U((c_t)_{0\le t\le T}) = \inf_{Q}\big(E_Q\ [\int_0^T u_t(c_t)\ \mu(dt)] + \gamma(Q)\big)$$

optimization problem

maximize $U((c_t)_{0\le t\le T})$

s.t. $(c_t)_{0\le t\le T}$ is admissible

- some special cases: μ_1 uniform distribution on $[0, T]$, $\mu_2(\{T\}) = 1$

  * $\mu = \mu_2$:

    $U((c_t)_{0\le t\le T}) = \inf_{Q}\big(E_Q[u_T(c_T)] + \gamma(Q)\big)$

  * $\mu = \frac{1}{T+1}(\mu_1 + \mu_2)$:

    · $U((c_t)_{0\le t\le T}) = \frac{1}{T+1}\inf_{Q}\big(E_Q\ [\int_0^T u_t(c_t)]\ dt + E_Q\ [u_T(c_T)] + \gamma(Q)\big)$

    · $U((c_t)_{0\le t\le T}) = \frac{1}{T+1}\inf_{Q}\big(\sum_{t=0}^{T} E_Q\ [u_t(c_t)\ dt] +\ \gamma(Q)\big)$

    for discrete time set $\{0, 1, ..., T\}$ whenever $u_t = u_k$ if $k \le t < k+1$

2 The reduction to expected utility maximization and fuzzy perception of the financial market

- optimization problem calls for application of minimax results
  (e.g. Kindler (1979), König (1980))
  important observation: level sets $\{\gamma \leq \delta\}$ $(\delta \in \mathbb{R})$ compact w.r.t. „suitable" topology
  (cf. V.K. SFB 2006-081, 2007-010)

$$\downarrow$$

- **Theorem 2.1:**
  Let $\sup\{U(c) \mid c$ admissible$\} < \infty$. Then
  - $\sup\{U(c) \mid c$ admissible$\} =$
    $\sup\{E_{\bar{\mathrm{Q}}}\,[\int_0^T u_t(c_t)\ \mu(dt)] + \gamma(\bar{\mathrm{Q}}) \mid (c_t)_{0\leq t\leq T}$ admissible$\}$ for some $\bar{\mathrm{Q}}$
  - $\sup\{E_{\bar{\mathrm{Q}}}\,[\int_0^T u_t(c_t)\ \mu(dt)] + \gamma(\bar{\mathrm{Q}}) \mid (c_t)_{0\leq t\leq T}$ admissible$\} =$
    $E_{\bar{\mathrm{Q}}}\,[\int_0^T u_t(\bar{c}_t)\ \mu(dt)] + \gamma(\bar{\mathrm{Q}})$ for any solution $(\bar{c}_t)_{0\leq t\leq T}$ of the optimization problem

- analogous result if **no credit** is allowed for consumer

- **fuzzy perception of the financial market:**

  Theorem 2.1 $\rightarrow$ reduction to expected utility maximization w.r.t. some subjective $\bar{\mathrm{Q}}$

  $$\boxed{\text{relationship between } \bar{\mathrm{Q}} \text{ and } \mathrm{P}?}$$

  for illustration: Black Scholes market

  $$dS(t) = (\alpha_t - r_t)S(t)\ dt + \sigma_t\ dW(t)$$

  with one asset and **deterministic** drift, interest rate and volatility processes $(\alpha_t)_t, (r_t)_t, (\sigma_t)_t$ resp.

  - best to expect $\bar{\mathrm{Q}} \approx \mathrm{P}$, i.e. $\bar{\mathrm{Q}}, \mathrm{P}$ have the same unlikely events:

    consumer perceives a Black Scholes market but with possibly **uncertain** drift process

  - second best to expect $\bar{\mathrm{Q}} \ll \mathrm{P}$, i.e. unlikely event w.r.t. P is unlikely w.r.t. $\bar{\mathrm{Q}}$:

    possibly the consumer does not recognize a Black Scholes market

  $$\boxed{\text{does fuzzy perception causes harmful economic effects?}}$$

3 Structure of optimal consumption rate processes

reduction to expected utility maximization w.r.t. some prior $\bar{\mathrm{Q}}$

assumption (3.1):

all plausible priors (finite penalty) $\ll \mathrm{P}$

$\rightarrow$ $\bar{\mathrm{Q}} \ll \mathrm{P}$, but $\bar{\mathrm{Q}} \approx \mathrm{P}$ is **not** guaranteed (for counterexample: Schied (2007a) (also SFB 2005-051))

$\rightarrow$ known result of expected utility maximization in Karatzas/Zitkovic (2003) are not applicable

But line of reasoning may be adapted with auxiliary techniques from Schied (2007a). For this purpose

assumption (3.2):

$\mathrm{Q} \approx \mathrm{P}$ with $\sup\{E_{\mathrm{Q}}\,[\int_0^T u_t(c_t)\ \mu(dt)] \mid (c_t)_{0\leq t\leq T}$ admissible$\} < \infty$

for some plausible prior Q

additional assumptions on utility indices:

assumption (3.3):

each u_t $(t \in \mathbb{R})$ continuously differentiable fulfilling some boundness requirements and Inada conditions (behaviour of marginal utilities near 0 and ∞)

assumption (3.4):

the utility indices fulfill a simultaneous asymptotic elasticity

- **Theorem 3.1 (Wittmüß SFB 2006-063)**

  Let $f_{\bar{\mathrm{Q}}}$ be density of $\bar{\mathrm{Q}}$ w.r.t. P . Under assumptions (3.1)-(3.5)

  - there exists a unique solution $\bar{c} = (\bar{c})_{0 \leq t \leq T}$, and there are a real number y and some $\mathrm{Q} \ll \mathrm{P}$ with density f_{Q} such that for all t

  $$u_t'(\bar{c}_t) = y \, \frac{f_{\mathrm{Q}}^t}{f_{\bar{\mathrm{Q}}}^t} \; \bar{\mathrm{Q}} - \text{a.s.}$$

  $f_{\mathrm{Q}}^t, f_{\bar{\mathrm{Q}}}^t$ conditional densities of $f_{\mathrm{Q}}, f_{\bar{\mathrm{Q}}}$ (w.r.t. $\mathcal{F}_t$) at time t

  - in case of complete market with pricing density f_* this reads

  $$u_t'(\bar{c}_t) = y \, \frac{f_*^t}{f_{\bar{\mathrm{Q}}}^t} \; \bar{\mathrm{Q}} - \text{a.s.}$$

- **Remarks:**

  - f_{Q} from Theorem 3.1 only close to the set of pricing densities (closure w.r.t. some suitable topology)
  - in the case of complete market and $\bar{\mathrm{Q}} \approx \mathrm{P}$ the optimality condition retains relationship between marginal utilities of optimal consumption streams and so called pricing kernel in the classical setting of Black Scholes market with expected utility maximizing agents (cf. e.g. Giacomini/Handel/Härdle SFB 2006-020)
  - in Hernandez-Hernandez/Schied (2007) and Schied (2007b) optimal consumption rate processes as solutions of partial differential equations

    $\rightarrow$ calculation by numerical algorithms

4 Equilibrium and representative agent

Do variational preferences cause market failures?

- **caveat by Rigotti/Shannon (Riggotti/Shannon (2006)):**
  - pure exange economy (without financial market)
  - consumption without budget constraints
  - all agents have variational preferences on the consumption set

$\downarrow$

if there is a „representative“ prior with simultaneous lowest penalty over all agent
– existence of Pareto allocation
– risk sharing rules for agents

$\downarrow$

representative agent

crucial idea: find Pareto allocation via „representative“prior

problem in presence of financial markets:

for optimization of consumption every agent implicitly chooses a „relevant“ prior, which might vary over agents

$\rightarrow$ does the existence of a Pareto allocation imply a competetive equilibrium?

Equilibrium market

- **assumptions:**
  - financial market is complete with pricing density f_* and conditional densities $f_*^t \; 0 \leq t \leq T$
  - I consumers
  - consumer i has
    * initial capital w_0^i, random endowment rate process $e^i := (e_t^i)_{0 \leq t \leq T}$ and aggregation function μ^i
    * variational preferences on consumption set in terms of utility indices u_t^i $(t \in [0, T])$ and penalty function γ_i for priors

  all conditions and optimization problem as in single agent model!

- optimal consumption rate processes $c^i := (c_t^i)_{0 \leq t \leq T}$

- equilibrium market:
  - clearing of commodity market:

$$\sum_{i=1}^{I} c_t^i = \sum_{i=1}^{I} (e_t^i + w_0^i) \ \mathrm{P}-\text{a.s. } (t \in [0,T])$$

  - clearing of stock market:

    for any investment process $\vartheta^i := (\vartheta_t^i)_{0 \le t \le T}$ with $V_T^{c^i,\vartheta^i} \ge 0 \ \mathrm{P}-\text{a.s. } (i = 1, \dots, I)$

$$\sum_{i=1}^{I} \vartheta_t^i = 0 \ \mathrm{P}-\text{a.s. } (t \in [0,T])$$

    holds

- $\bar{\mathrm{Q}}^1, \dots, \bar{\mathrm{Q}}^I$ subjective probability corresponding to the individual optimization problems

  $\downarrow$ Theorem 3.1

  there are positive numbers $y_1, \dots, y_I$ with

$$c_t^i = I_t^i\left(y_i \frac{f_*^t}{f_{\bar{\mathrm{Q}}_i}^t}\right) \ \bar{\mathrm{Q}}_i - \text{a.s. } (t \in [0,T]; i = 1, \dots, I),$$

  where I_t^i inverse of $u_t^{i\prime}$, $f_{\bar{\mathrm{Q}}_i}$ density of $\bar{\mathrm{Q}}_i$ with respective conditional densities $f_{\bar{\mathrm{Q}}_i}^t$ $(t \in [0,T])$

- **Risk Sharing Theorem:**

  Let $\bar{\mathrm{Q}}_i \approx \mathrm{P}$ for $i = 1, \ldots, I$.

  If we have a equilibrium market, then there exist positive numbers $y_1, \ldots, y_I$ such that

$$E\,[\int_0^T f_* I_t^i(y_i \frac{f_*^t}{f_{\bar{\mathrm{Q}}_i}^t})\ \mu(dt)] = w_0^i + E\,[\int_0^T f_* e_t^i\ \mu(dt)] \text{ for } i = 1, \ldots, I$$

$$\sum_{i=1}^{I} e_t^i = \sum_{i=1}^{I} I_t^i(y_i \frac{f_*^t}{f_{\bar{\mathrm{Q}}_i}^t})\ \ \mathrm{P}\text{-a.s. for } t \in [0,1]$$

  The converse is true if the covariation matrix of the assets is positive definite P -a.s. at every time t. In either case the numbers $y_1, \ldots, y_I$ are just the same as that from the respective optimality conditions.

- **Remarks:**
  - risk sharing rule $y_1, \ldots, y_k \rightarrow$ aggregation of the individual preferences with weights $y_1, \ldots, y_k \rightarrow$ „preference of the representative agent"
  - questionable whether risk sharing rules exist when $\bar{\mathrm{Q}}_i \not\approx \mathrm{P}$ for at least one i
  - interesting open problem: does an equilibrium exist for different $\bar{\mathrm{Q}}_1, \ldots, \bar{\mathrm{Q}}_I$?

5 Robust approaches to the optimization of consumption rate processes in presence of possible exogeneous shocks

- **shock variable:**
  - positive continuously distributed random variable τ with distribution function F_τ and survivor Funktion (tail function) $\bar{F}_\tau := 1 - F_\tau$

    (e.g. lifetime of consumer, lifetime of market up to collapse)
  - shock is supposed to be **exogeneous** $\rightarrow$ τ is assumed to be **stochastically independent** of the admissible consumption rate processes

- $\gamma(\mathrm{P}) := 0$, and $\gamma(\mathrm{Q}) := \infty$ for $\mathrm{Q} \neq \mathrm{P}$

- F_τ known $\rightarrow$ utility of a consumption rate process $(c_t)_{0 \leq t \leq T}$ under shock τ

$$E_{\mathrm{P}}\left[\int_0^{\min\{t,\tau\}} u_t(c_t)\ \mu(dt)\right] = E_{\mathrm{P}}\left[\int_0^T \bar{F}_\tau(t) u_t(c_t)\ \mu(dt)\right]$$

$$\downarrow$$

Theorems 2.1, 3.1 may be applied to utility indices $\tilde{u}_t := \bar{F}_\tau(t) u_t$ $(t \in [0, T])$

- distribution F_τ unknown $\rightarrow$ uncertainty about time horizon
  - plausibility of a distribution function F with $F(0) = 0$ expressed by penalty $\gamma(F)$
    $\rightarrow$ utility of a consumption rate process $(c_t)_{0 \leq t \leq T}$ under shock τ

$$\inf_F \big(E_{\mathrm{P},F} \, [\int_0^{\min\{t,\tau\}} u_t(c_t) \, \mu(dt)] + \gamma(F) \big) = \inf_F \big(E_{\mathrm{P}} \, [\int_0^T \bar{F}(t) u_t(c_t) \, \mu(dt)] + \gamma(F) \big)$$

  - in case of time-independent utility index u alternatively:

$$\inf_F \left(\frac{E_{\mathrm{P},F} \, [\int\limits_0^{\min\{t,\tau\}} u(c_t) \, \mu(dt)]}{\int\limits_0^T \bar{F}(t) \mu(dt)} + \gamma(F) \right) = \inf_F \big(E_{\nu_F \otimes \mathrm{P}} \, [u(c)] + \gamma(F) \big),$$

    where ν_F denotes the distribution on $[0, T]$ defined by $\nu_F(]-\infty, x]) = \dfrac{\int\limits_0^x \bar{F}(t)\mu(dt)}{\int\limits_0^T \bar{F}(t)\mu(dt)}$

    for $x \in [0, 1] \rightarrow$ variational preferences