Charity Auctions for the Happy Few

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What is a charity auction?

Contrary to non-charity auctions, here the amount paid is “never lost”. An investor, who bought a Dior perfume for 60 000 yuans said in the *Beijing Review*:

*I would never buy perfume for this amount normally, but this time it is for charity. I feel very happy.*

Motivations Model First-Price APA Second-Price APA Revenue Comparisons Min Bids Imposed

Literature


Experimental papers: Orzen (2005), Carpenter, Homes and Matthews (EJ, 2008), Onderstal and Schram (IER, 2008)
Motivations

The purpose of this paper is to determine whether or not all-pay auctions can raise higher revenue for charity when the asymmetry between bidders is strong.

A lot of charity auctions are conducted among rich people during charity dinners. Potential bidders are acquaintances or know one another well. A complete information environment is well suited.
Allocation Rule

An object is sold and allocated to one of the potential bidders \( N = \{1, \ldots, n\} \) contingent upon their bids \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \).

An all-pay auction is a pair \((a, t)\)

\[ a = (a_1, \ldots, a_n) : \mathbb{R}_+^n \rightarrow [0, 1]^n \] is such that the winner \( i \) gets the object iff \( a_i(x) = 1 \) given the bids and \( \sum_{i=1}^n a_i(x) = 1 \) for all \( x \).

\[
\begin{cases}
  a_i(x) = \frac{1}{\# Q(x)} & \text{if } i \in Q(x) \\
  a_i(x) = 0 & \text{otherwise}
\end{cases}
\]

where \( Q(x) := \{j|j = \arg \max\{x_k, k \in N\}\} \).
Payment Rule

\[ t = (t_1, \ldots, t_n) : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+ \]

In the first-price all-pay auction, each bidder pays his own bid

\[ t_i(x) = x_i \quad \forall i \in N \]

In the second-price all-pay auction the winner pays the second highest bid and the losers their own bid

\[ t_i(x) = x^{(2)} \text{ if } i \in Q(x) \]
\[ t_i(x) = x_i \text{ otherwise} \]

with \( x^{(2)} \) the second order statistic of sample \( (x_1, \ldots, x_n) \).
Denote $h_i(t(x))$ the externality that the bidder $i$ takes advantage of such that

$$h_i(t(x)) = h_i \left( \sum_{j=1}^{n} t_j(x) \right) = \alpha_i \sum_{j=1}^{n} t_j(x)$$

where $\alpha_i \geq 0$.

The bidder $i$’s utility is given by

$$U_i(x) = \tilde{U}_i(a_i, t) = v_i a_i(x) - t_i(x) + \alpha_i \sum_{j=1}^{n} t_j(x)$$
Assumptions

Assumption 1 (A1)

\( \tilde{U}_i(a_i, t) \) is a continuous and differentiable function in the transfer functions \( t_j \) for all \( j \).

Assumption 2 (A2)

\( \forall x_i \geq 0 \quad \frac{\partial \tilde{U}_i}{\partial t_i(x)}(a_i, t) < 0 \) equivalent to \( \alpha_i \sum_{j=1}^{n} \frac{dt_j(x)}{dt_i(x)} < 1 \).
**Expected Utility**

\[
EU_i(x_i, X_{-i}) = v_i \prod_{j \neq i} F_j(x_j) - \int_{\mathbb{R}_+^{n-1}} t_i(x) \prod_{j \neq i} dF_j(x_j)
\]

\[
+ \alpha_i \int_{\mathbb{R}_+^{n-1}} \sum_{j=1}^{n} t_j(x) \prod_{j \neq i} dF_j(x_j)
\]

with \( X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \)

\[
\frac{v_1}{1 - \alpha_1} > \frac{v_2}{1 - \alpha_2} > \ldots > \frac{v_n}{1 - \alpha_n}
\]

is common knowledge.
Equilibrium in the First-Price All-Pay Auction

Assumption A2 implies that $\alpha_i < 1$
A bidder takes part in the auction if $\mathbb{E}U_i(x, X_{-i}) \geq \alpha_i \sum_{j \neq i} \mathbb{E}X_j$

Bidder $i$’s expected utility with $n$ potential competitors is given by

$$\mathbb{E}U_i(x_i, X_{-i}) = \prod_{j \neq i} F_j(x_j)v_i - (1 - \alpha_i)x_i + \alpha_i \sum_{j \neq i} \mathbb{E}X_j$$
Proposition 1

There is a unique Nash equilibrium and the mixed strategies are

\[ F_1(x) = \frac{1 - \alpha_2}{v_2} x \quad \forall x \in \left[ 0, \frac{v_2}{1 - \alpha_2} \right] \]

\[ F_2(x) = 1 - \frac{1 - \alpha_1}{1 - \alpha_2} \frac{v_2}{v_1} + \frac{1 - \alpha_1}{v_1} x \quad \forall x \in \left( 0, \frac{v_2}{1 - \alpha_2} \right) \]

All others bidders use the pure strategy of zero and do not take part in the auction: \( F_j(0) = 1 \) for \( j \in \{3, \ldots, n\} \). The expected revenue is given by \( ER = \frac{1}{2} \frac{v_2}{1 - \alpha_2} \left( \frac{1 - \alpha_1}{v_1} \frac{v_2}{1 - \alpha_2} + 1 \right) \).

Corollary 1

All bidders obtain a positive payoff. Indeed, the bidder with the two highest adjusted-value obtains a positive payoff

\[ U_1^* = v_1 - \frac{1 - \alpha_1}{1 - \alpha_2} v_2 + \frac{\alpha_1}{2} \frac{1 - \alpha_1}{v_1} \left( \frac{v_2}{1 - \alpha_2} \right)^2 \]

and \( U_2^* = \frac{v_2}{2} \frac{\alpha_2}{1 - \alpha_2} \) and their competitors get \( U_i^* = \alpha_i \frac{v_2}{1 - \alpha_2} \left( \frac{1 - \alpha_1}{v_1} \frac{v_2}{1 - \alpha_1} + 1 \right) \) for \( i \in \{3, \ldots, n\} \).
Non-linear Externalities

\[ EU_1(x_1, X_2) = F_2(x_1)v_1 - x_1 + \mathbb{E}_{X_2} h_1(x_1, X_2) \]
\[ EU_2(x_2, X_1) = F_1(x_2)v_2 - x_2 + \mathbb{E}_{X_1} h_2(X_1, x_2) \]

Bidder \( i \) takes part to the auction if \( EU_i(x_i, X_j) \geq \mathbb{E}_{X_j} h_i(0, X_j) \)

**Proposition 2**

*Given A1 – A2 and given that the two bidders have a common support \([0, b]\), the mixed strategy equilibrium exists.*
Equilibrium in the Second-Price All-Pay Auction

\[ X_i \subseteq [0, +\infty), \text{ assumption } A2 \text{ allows us to write that } \alpha_i < 1/2 \]

Proposition 3

*Only two bidders among } n \text{ participate actively to the auction. Their mixed strategies are given by an exponential distribution defined as follows,*

\[ F_i \sim \mathcal{E}\left(\frac{1 - 2\alpha_j}{v_j}\right) \text{ and } \text{ER} = \frac{2v_i v_j}{(1 - 2\alpha_i)v_j + (1 - 2\alpha_j)v_i} \]
For $n = 2$

$$EU_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1 - 2\alpha_i)x) dF_j(x) - (1 - 2\alpha_i)x_i(1 - F_j(x_i))$$

For $n > 2$

Note $G_i(x) = \prod_{j \neq i} F_j(x)$.

$$EU_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1 - \alpha_i)x_i) dG_i(x_i) - (1 - \alpha_i)x_i(1 - G_i(x_i))$$

$$+ \alpha_i \sum_{l \neq i} \int_{\mathbb{R}^+} x_l \left(1 - 1_{x_i \leq x_l} \prod_{k \neq l, i} F_k(x_l)\right) dF_l(x_l)$$

$$+ \alpha_i \sum_{l \neq i} \left(\int_{\mathbb{R}^+} \int_{x_i}^{x_l} \sum_{k \neq l, i} x_k \prod_{m \neq i, k, l} F_m(x_k) dF_k(x_k) dF_l(x_l)\right)$$

$$+ x_i \prod_{m \neq i, l} F_m(x_i)(1 - F_l(x_i))$$
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\[ EU_i(x_i, X_{-i}) = v_i \prod_{j \neq i} F_j(x_j) - (1 - \alpha_i) \int_{R_+^{n-1}} t_i(x) \prod_{j \neq i} dF_j(x_j) \]

\[ + \alpha_i \int_{R_+^{n-1}} \sum_{l=1}^{n} t_l(x) \prod_{j \neq i} dF_j(x_j) \]

\[ \sum_{l \neq i} \int_{R_+^{n-1}} t_l(x) \prod_{j \neq i} dF_j(x_j) = \sum_{l \neq i} \left\{ \int_{R_+^{n-1}} x_l \mathbb{1}_{\exists k/x_l < x_k} \prod_{j \neq i} dF_j(x_j) \right\} \]

\[ + \int_{R_+^{n-1}} x^{(2)} l \mathbb{1}_{\forall k \neq l} \prod_{j \neq i} dF_j(x_j) \]
Summary

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<thead>
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Table: Revenues and expected revenues for every design
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Table: Revenues and expected revenues for every design

Definition

The level of asymmetry between bidders’ valuations will be considered very high if $v_1 - v_2 > 2\alpha(v_1 + v_2)$, high if $v_1 - v_2 > 2\alpha v_1$, low if $v_1 - v_2 < 2\alpha v_1 - v_1 + v_2 \frac{v_2}{v_1}$ and medium if $2\alpha v_1 > v_1 - v_2 > 2\alpha v_1 - v_1 + v_2 \frac{v_2}{v_1}$.
Revenue Comparisons

Proposition 4

We assume that $\alpha_i = \alpha \forall i$ and that the bidder with the highest adjusted-value takes part in the second-price all-pay auction. Then, this design gives the highest revenues:

$$ER^{AP2} > RW^{P2} \text{ iff the level of asymmetry is not very high,}$$

$$ER^{AP2} > RW^{P1} \text{ and } ER^{AP2} > ER^{AP1}$$

All other things being equal, $ER^{AP1} > RW^{P2} \text{ iff the level of asymmetry between valuations is low, } RW^{P2} > ER^{AP1} > RW^{P1} \text{ iff this level is medium, and } RW^{P1} > ER^{AP1} \text{ iff it is high.}$

Given $v_1$ and $v_2$, first-price all-pay auction is dominated by first and second-price winner-pay auctions when the bidders’ altruism level is less than $\frac{1}{2} (1 - \frac{v_2}{v_1})$ and only by second-price winner-pay auction when it is less $1 - \frac{1}{2} \frac{v_2}{v_1} \left(\frac{v_2}{v_1} + 1\right)$ and superior to $\frac{1}{2} (1 - \frac{v_2}{v_1})$. 
Illustration

From left to right, $\frac{v_2}{v_1}$ varies from 0.9 to its limit in zero with a 0.1 step.

Figure: $E R^{AP1} > R^{WP2}$

Figure: $E R^{AP1} > R^{WP1}$
Equilibrium
Expected Revenue

Proposition 5

Imposing a minimal bid to every bidder permits to improve the first-price all-pay auction’s efficiency compared to the first-price winner-pay auction. There is a threshold $t$ above which the all-pay auction dominates the winner-pay auction when the values’ asymmetry is considered high.
Expected Revenue: Illustration

Figure: Expected revenue with a high asymmetry
Figure 1: $E_R$ for $\alpha > \max\left\{\frac{v_1}{v_1+v_2}, \frac{v_1-v_2}{2r_1}\right\}$

Figure 2: $E_R$ for $\frac{v_1}{v_1+v_2} > \alpha > \frac{v_1-v_2}{2r_1}$
Thank you for your attention!
\[ \begin{align*}
\mathbb{E}U_1(x_1, X_2) &= F_2(x_1) (v_1 + \mathbb{E}_{X_2}(h_1(x_1, X_2) | X_2 \leq x_1) - x_1) \\
&\quad + (1 - F_2(x_1)) (\mathbb{E}_{X_2}(h_1(x_1, X_2) | X_2 \geq x_1) - x_1)
\end{align*} \]

\[ \begin{align*}
\mathbb{E}U_2(x_2, X_1) &= F_1(x_2) (v_2 + \mathbb{E}_{X_1}(h_2(X_1, x_2) | X_1 \leq x_2) - x_2) \\
&\quad + (1 - F_1(x_2)) (\mathbb{E}_{X_1}(h_2(X_1, x_2) | X_1 \geq x_2) - x_2)
\end{align*} \]

with \( \mathbb{E}_{X_2}(h_1(x_1, X_2) | X_2 \leq x_1) = \frac{1}{F_2(x_1)} \int_0^{x_1} h_1(x_1, x_2) dF_2(x_2) \)

It can also be written as

\[ \begin{align*}
\mathbb{E}U_1(x_1, X_2) &= F_2(x_1)v_1 - x_1 + \mathbb{E}_{X_2} h_1(x_1, X_2) \\
\mathbb{E}U_2(x_2, X_1) &= F_1(x_2)v_2 - x_2 + \mathbb{E}_{X_1} h_2(X_1, x_2)
\end{align*} \]
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\[ T : F(x) \mapsto TF(x) \text{ and} \]
\[ TF(x) \equiv \lambda x - \lambda \int_0^b h(x, y)f(y)dy + \text{ constant} \quad (1) \]

The set \( D = \{ F \in C[0, b] \mid \|F\| \leq 1 \} \), which includes all the continuous distribution functions, is closed and convex but not compact. Thus, to prove that (1) has a solution, we apply the following Schauder’s second theorem:

**Theorem (Schauder, 1930)**

*If \( D \) is a closed convex subset of a normed space and \( E \) a relatively compact subset of \( D \), then every continuous mapping of \( D \) to \( E \) has a fixed-point.*

To apply this theorem, we need to prove two parts.

- \( T(D) \equiv E = \{ TF \mid F \in D \} \) is relatively compact which is equivalent to prove \( E \) is uniformly bounded and equicontinuous;
- Second, \( T \) is a continuous mapping from \( D \) to \( E \).
\[ \mathbb{E}U_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1 - 2\alpha_i)x) dF_j(x) - (1 - 2\alpha_i)x_i(1 - F_j(x_i)) \]

Note \( G_i(x) = \prod_{j \neq i} F_j(x) \).

\[ \mathbb{E}U_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1 - \alpha_i)x) dG_i(x) - (1 - \alpha_i)x_i(1 - G_i(x_i)) \]

\[ + \alpha_i \sum_{l \neq i} \int_{\mathbb{R}^+} x_l \left( 1 - \mathbb{1}_{x_i \leq x_l} \prod_{k \neq l, i} F_k(x_l) \right) dF_l(x_l) \]

\[ + \alpha_i \sum_{l \neq i} \left( \int_{\mathbb{R}^+} \int_{x_i}^{x_l} \sum_{k \neq l, i} x_k \prod_{m \neq i, k, l} F_m(x_k) dF_k(x_k) dF_l(x_l) \right) \]

\[ + x_i \prod_{m \neq i, l} F_m(x_i)(1 - F_l(x_i)) \]
Equilibrium

Proposition 6

Given the bidders’ adjusted-values, \((1 - \alpha_1 t)\tilde{x}_1\) and \((1 - \alpha_2 t)\tilde{x}_2\), there is a unique Nash equilibrium. The bidders’ strategies for all \(x \in (tv_1; \bar{x}]\) are

\[
F_1(x) = \alpha_2 t + \frac{x}{\tilde{x}_2} \quad \text{and} \quad F_2(x) = 1 + \frac{x - \bar{x}}{\tilde{x}_1}.
\]

Every bidder has one mass point: it is \(tv_1\) for bidder 1 and \(tv_2\) for bidder 2.

A bidder’s decision is given by her probability to participate,

\[
1 - F_1(tv_1) = 1 - \alpha_2 t - \frac{tv_1}{\tilde{x}_2} \quad \text{and} \quad 1 - F_2(tv_2) = \frac{\bar{x} - tv_1}{\tilde{x}_1}
\]
Expected Revenue

Additionally, if the maximum bid $\tilde{x}$ is inferior to bidder 1’s minimum bid $\bar{x} \leq tv_1$, offering a higher bid than their minimum bid is dominated for all bidders. Hence, $\mathbb{E}R = t(v_1 + v_2)$ for all $t \geq \bar{t}$ where

$$\bar{t} \equiv \frac{\tilde{x}_2}{v_1 + \alpha_2 \tilde{x}_2}.$$ 

Proposition 7

*Given the distribution functions $F_1(.)$, $F_2(.)$ at equilibrium, the expected revenue raised for charity is*

$$
\begin{aligned}
\begin{cases}
\bar{x}^2 \tilde{x}_1 + \tilde{x}_2 + (tv_1)^2 \frac{\tilde{x}_1 - \tilde{x}_2}{2\tilde{x}_1 \tilde{x}_2} + t^2 v_1 \alpha_2 + tv_2 \left(1 + \frac{tv_1 - \tilde{x}}{\tilde{x}_1}\right) & \text{if } t < \bar{t} \\
t(v_1 + v_2) & \text{otherwise}
\end{cases}
\end{aligned}
$$

with $\bar{t} \equiv \frac{\tilde{x}_2}{v_1 + \alpha_2 \tilde{x}_2}$.