Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Non-asymptotic confidence bounds via multiplier bootstrap

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Outline

Generalized regression

- A random sample of a fixed size n from unknown distribution:

$$\boldsymbol{Y} = (Y_1, \ldots, Y_n)^\top \in \mathbb{R}^n,$$

$$Y_i \sim P_i\big(f(X_i)\big), \text{ independent.}$$

- Deterministic design $X_1, \ldots, X_n \in \mathbb{R}^d$.

- Unknown function $f(\cdot) : \mathbb{R}^d \to \mathbb{R}$.

- Mean regression

$$Y_i \sim f(X_i) + \varepsilon_i, \quad \varepsilon_i \text{ are i.i.d.}$$

- Logistic regression

$$Y_i \sim Bernoulli(f(X_i)).$$

- Quantile regression

$$P\left(Y_i - f(X_i) < 0\right) = \tau$$

for $\tau \in (0, 1)$.

- Let $\boldsymbol{Y} \sim I\!P$ (unknown).
- Fix some known parametric family:

$$\{I\!P(\boldsymbol{\theta}) \ll \mu_0, \boldsymbol{\theta} \in \Theta \subset I\!R^p\}\,.$$

- The target parameter $\boldsymbol{\theta}^* \in \Theta$ is defined by the projection:

$$\begin{aligned} \boldsymbol{\theta}^* &\stackrel{\text{def}}{=} \operatorname*{argmin}_{\boldsymbol{\theta}} KL(I\!P, I\!P(\boldsymbol{\theta})) \\ &= \operatorname*{argmin}_{\boldsymbol{\theta}} \int \log\left\{\frac{dI\!P}{dI\!P(\boldsymbol{\theta})}\right\} dI\!P. \end{aligned}$$

To find a sharp confidence set $\mathcal{E}$ s.t. for a given coverage probability $1-\alpha \in (0,1)$ and for a fixed sample size n :

$$I\!P\{\boldsymbol{\theta}^* \in \mathcal{E}\} \geq 1-\alpha.$$

Outline

Maximum likelihood method:

- Log-likelihood function

$$L(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \log \frac{d\mathbb{P}(\boldsymbol{\theta})}{d\mu_0}(\boldsymbol{Y})$$

$$= \sum_{i=1}^{n} \log \frac{dP_i(\boldsymbol{\theta})}{d\mu_0}(Y_i).$$

- The target parameter:

$$\boldsymbol{\theta}^* \stackrel{\text{def}}{=} \operatorname*{argmax}_{\boldsymbol{\theta}} \mathbb{E} L(\boldsymbol{\theta}) = \operatorname*{argmin}_{\boldsymbol{\theta}} KL(\mathbb{P}, \mathbb{P}(\boldsymbol{\theta})).$$

- (Quasi) Maximum likelihood estimate (qMLE):

$$\widetilde{\boldsymbol{\theta}} \stackrel{\text{def}}{=} \operatorname*{argmax}_{\boldsymbol{\theta}} L(\boldsymbol{\theta}).$$

Wilks phenomenon for the likelihood ratio:

For $I\!P \in \{P(\boldsymbol{\theta})\}$

$$L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*) \approx \|\boldsymbol{\xi}\|^2/2, \qquad \boldsymbol{\xi} \stackrel{w}{\to} \mathcal{N}(0, \boldsymbol{I}_p)$$

with $n \to \infty$.

p is the dimension of parametric set Θ,

$\boldsymbol{\xi} \stackrel{\text{def}}{=} D_0^{-1} \nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta}^*)$ normalized score,

$D_0^2 \stackrel{\text{def}}{=} -\nabla^2_{\boldsymbol{\theta}} I\!E L(\boldsymbol{\theta}^*)$ full Fisher Information matrix.

Likelihood-based confidence set:

$$\mathcal{E}(\mathfrak{z}) \stackrel{\text{def}}{=} \{\boldsymbol{\theta} : L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}) \leq \mathfrak{z}\},$$

the level $\mathfrak{z} > 0$ can be found by the Wilks theorem:

$$I\!P\left\{\boldsymbol{\theta}^* \in \mathcal{E}(\mathfrak{z})\right\} \to P\left\{\chi^2_p \leq 2\mathfrak{z}\right\}.$$

- The likelihood ratio is asymptotically pivotal,
- $\mathcal{E}(\mathfrak{z})$ does not require to estimate the variance,
- pivotality fails under model misspecification $I\!P \notin \{I\!P(\boldsymbol{\theta})\}$,
- the convergence speed is slow.

Example: Mean Gaussian regression

$$Y_i = \theta^* + \varepsilon_i, \quad Y_i \sim \mathcal{N}(\theta^*, 1), \text{ i.i.d.}$$

- The log-likelihood function is quadratic:

$$L(\theta) = -\frac{1}{2}\sum_{i=1}^{n}(Y_i - \theta)^2 + C$$

- The MLE:

$$\widetilde{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim \mathcal{N}(\theta^*, \frac{1}{\sqrt{n}})$$

- Wilks Theorem:

$$2\left\{L(\widetilde{\theta}) - L(\theta^*)\right\} = n(\widetilde{\theta} - \theta^*)^2 \sim \chi^2_1.$$

If $Y_i \nsim \mathcal{N}(\theta^*, 1)$, then the distribution of $L(\widetilde{\theta}) - L(\theta^*)$ differs from χ^2_1.

- It holds with probability $\geq 1 - 2\mathrm{e}^{-\mathrm{y}}$ (conditions come later)

$$\left| 2\left\{ L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*) \right\} - \|\boldsymbol{\xi}\|^2 \right| \leq \Delta_{\mathtt{sq}}(\mathrm{y}) \asymp \sqrt{(p+\mathrm{y})^3/n},$$

$$\left| \sqrt{2\left\{ L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*) \right\}} - \|\boldsymbol{\xi}\| \right| \leq \Delta(\mathrm{y}) \asymp (p+\mathrm{y})/\sqrt{n}.$$

- Deviation bound for the approximating term:

$$I\!P\left(\|\boldsymbol{\xi}\|^2 \geq p + 6\mathrm{y} \right) \leq 2\mathrm{e}^{-\mathrm{y}}.$$

- Allows the model to be misspecified,
- finite sample result,
- distribution of $\|\boldsymbol{\xi}\|$ in general depends on $I\!P$ and $\boldsymbol{\theta}^*$,
- the deviation bound is not sharp enough for a confidence set.

- Estimation of the distribution of the likelihood ratio statistic with a multiplier bootstrap procedure.

- Theoretical justification of the multiplier bootstrap using:

  - Uniform non-asymptotic local quadratic approximation of the likelihood ratio statistic (Wilks-type bound) for the Y and bootstrap worlds.

  - Non-asymptotic Gaussian approximation to connect the Y and the bootstrap worlds.

Outline

The idea:

To mimic the distribution of $L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*)$ using a multiplier bootstrap:

- Take an i.i.d. sample $u_1, \ldots, u_n$ independent of the data $\boldsymbol{Y}$:
  $I\!\!E(u_i) = \mathrm{Var}(u_i) = 1\,,\ I\!\!E \exp(u_i) < \infty$ (e.g. $\sim exp(1),\ \mathcal{N}(1,1)$).

- Bootstrap the likelihood function:

$$L^{\circledcirc}(\boldsymbol{\theta}) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} \log \frac{dP_i^{\circledcirc}(\boldsymbol{\theta})}{d\mu_0}(Y_i) u_i$$

  $^{\circledcirc}$ denotes the conditional probability with the fixed sample $\boldsymbol{Y}$.

“$\boldsymbol{Y}$ world”	conditional “bootstrap world“
quasi MLE	
$\widetilde{\boldsymbol{\theta}} \stackrel{\mathrm{def}}{=} \operatorname{argmax}_{\boldsymbol{\theta}} L(\boldsymbol{\theta})$	$\widetilde{\boldsymbol{\theta}}^{\circ} \stackrel{\mathrm{def}}{=} \operatorname{argmax}_{\boldsymbol{\theta}} L^{\circ}(\boldsymbol{\theta})$
target	
$\boldsymbol{\theta}^* \stackrel{\mathrm{def}}{=} \operatorname{argmax}_{\boldsymbol{\theta}} I\!\!E L(\boldsymbol{\theta})$	$\widetilde{\boldsymbol{\theta}} \stackrel{\mathrm{def}}{=} \operatorname{argmax}_{\boldsymbol{\theta}} I\!\!E^{\circ} L^{\circ}(\boldsymbol{\theta})$
likelihood ratio	
$L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*)$	$L^{\circ}(\widetilde{\boldsymbol{\theta}}^{\circ}) - L^{\circ}(\widetilde{\boldsymbol{\theta}})$

- The true point in bootstrap world is exactly qMLE $\widetilde{\boldsymbol{\theta}}$.
- The bootstrap side is computable!
- The “bootstrap world” is inside of the parametric model, which, however, may be wrong.

- Wild bootstrap:
  [Wu, 1986], [Beran, 1986], [Hardle and Mammen, 1993], [Mammen, 1993];
  covers our approach for the case of a linear model and quadratic log-likelihood.
- Multiplier bootstrap for resampling of an objective function:
  - [Jin, Ying and Wei, 2001] use the multiplier bootstrap procedure for perturbing U-process of the k-th degree in order to estimate the covariance matrix of the U-process' minimiser.
  - [Lavergne and Patilea, 2013] use multiplier bootstrap for resampling of the smooth minimum distance statistic.
- Non-asymptotic results (very few):
  - [Arlot, Blanchard and Roquian, 2010] consider a sample of high dimensional i.i.d. Gaussian vectors, a non-asymptotical confidence set is constructed using exchangeable weighted bootstrap scheme.
  - [Chernozhukov, Chetverikov and Kato, 2013] approximate non-asymptotically the distribution of the maximum of a sum of independent high-dimensional vectors using Gaussian multiplier bootstrap.

We use the following approximating diagram:

$$\sqrt{2\Big\{L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*)\Big\}} \overset{(p+\mathrm{y})/\sqrt{n}}{\approx} \|\boldsymbol{\xi}\|$$

$$\wr\!\wr$$

$$\sqrt{2\Big\{L^\circledcirc(\widetilde{\boldsymbol{\theta}}^\circledcirc) - L^\circledcirc(\widetilde{\boldsymbol{\theta}})\Big\}} \overset{(p+\mathrm{y})/\sqrt{n}}{\approx} \|\boldsymbol{\xi}^\circledcirc\|$$

The approximation $\approx$ holds between the distributions on two different probability spaces:

$$\mathcal{L}\left(\|\boldsymbol{\xi}\|\right) \approx \mathcal{L}\left(\|\boldsymbol{\xi}^\circledcirc\| \,\middle|\, \boldsymbol{Y}\right).$$

Validity of the bootstrap [Spokoiny and Zhilova, 2014a]

It holds for all $\alpha \in (0,1)$ and the computable bootstrap $(1-\alpha)$-quantile

$$\mathfrak{z}_\alpha^\circ \stackrel{\mathrm{def}}{=} \min\left\{\mathfrak{z} \geq 0 : I\!P^\circ\left(\sqrt{2\left\{\sup_{\boldsymbol{\theta}} L^\circ(\boldsymbol{\theta}) - L^\circ(\widetilde{\boldsymbol{\theta}})\right\}} > \mathfrak{z}\right) \leq \alpha\right\}$$

$$I\!P\left(\sqrt{2\{L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*)\}} > \mathfrak{z}_\alpha^\circ(1+\delta) + \Delta_1\right) - \alpha \leq \Delta_2,$$

$$I\!P\left(\sqrt{2\{L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*)\}} > \mathfrak{z}_\alpha^\circ(1-\delta) - \Delta_1\right) - \alpha \geq -\Delta_2,$$

$\delta, \Delta_{1,2} \asymp \{\log(n)\}^{a_k}/n^{b_k}$, $a_k \geq 0, b_k > 0$

deterministic, explicit up to a generic constant.

- For a linear w.r.t. u_i function in the bootstrap world its deterministic part has an “unbiased” $\boldsymbol{Y}$ -random equivalent. For instance, for the likelihood process:

$$\mathbb{E}^{\circ} L^{\circ}(\boldsymbol{\theta}) = L(\boldsymbol{\theta}), \qquad \mathbb{E}^{\circ}\bigl\{L^{\circ}(\boldsymbol{\theta}) - L^{\circ}(\widetilde{\boldsymbol{\theta}})\bigr\} = L(\boldsymbol{\theta}) - L(\widetilde{\boldsymbol{\theta}}),$$

  and for the Hessian matrix:

$$\mathfrak{D}^2(\boldsymbol{\theta}) \stackrel{\mathrm{def}}{=} -\nabla^2_{\boldsymbol{\theta}} \mathbb{E}^{\circ} L^{\circ}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \nabla^2_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}).$$

- The local geometry on Θ around target points is defined with matrices D_0^2 and $\mathfrak{D}_0^2 \stackrel{\mathrm{def}}{=} \mathfrak{D}^2(\widetilde{\boldsymbol{\theta}})$. The following bound holds with exponentially high probability:

$$\sup_{\boldsymbol{\theta} \in Loc(\boldsymbol{\theta}^*)} \bigl\| D_0^{-1} \mathfrak{D}^2(\boldsymbol{\theta}) D_0^{-1} - \boldsymbol{I}_p \bigr\|_{\infty} \leq \mathtt{C}(p + \mathtt{y})/\sqrt{n}.$$

  It allows to use the same (deterministic) Fisher information matrix in the quadratic expansion of the likelihood processes.

- Local quadratic approximations hold locally around the target points:

$\forall \boldsymbol{\theta} : \|D_0(\boldsymbol{\theta} - \boldsymbol{\theta}^*)\| \le \mathtt{r}_0$ and some $0 < \mathtt{r}_0 \le \mathtt{C}\sqrt{p + \mathtt{y}}$:

$$L(\boldsymbol{\theta}) - L(\boldsymbol{\theta}^*) \overset{\Delta_{\mathrm{sq}}(\mathtt{y})}{\approx} \nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta}^*)^\top (\boldsymbol{\theta} - \boldsymbol{\theta}^*) - \|D_0(\boldsymbol{\theta} - \boldsymbol{\theta}^*)\|^2/2$$
$$\overset{\boldsymbol{\theta} := \widetilde{\boldsymbol{\theta}}}{\approx} \|\boldsymbol{\xi}\|^2/2,$$

$$L^{\circledcirc}(\boldsymbol{\theta}) - L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}) \overset{\Delta_{\mathrm{sq}}(\mathtt{y})}{\approx} \nabla_{\boldsymbol{\theta}} L^{\circledcirc}(\widetilde{\boldsymbol{\theta}})^\top (\boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}}) - \|D_0(\boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}})\|^2/2$$
$$\overset{\boldsymbol{\theta} := \widetilde{\boldsymbol{\theta}}^{\circledcirc}}{\approx} \|\boldsymbol{\xi}^{\circledcirc}\|^2/2,$$

where $\boldsymbol{\xi} \overset{\mathrm{def}}{=} D_0^{-1} \nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta}^*)$, $\boldsymbol{\xi}^{\circledcirc} \overset{\mathrm{def}}{=} D_0^{-1} \nabla_{\boldsymbol{\theta}} L^{\circledcirc}(\widetilde{\boldsymbol{\theta}})$.

- The approximating term $\|\boldsymbol{\xi}^{\circledcirc}\|^2$ is of the 2-d order w.r.t. u_i :

  for $\ell_i(\boldsymbol{\theta}) \stackrel{\mathrm{def}}{=} \log \frac{dP_i(\boldsymbol{\theta})}{d\mu_0}(Y_i)$

$$\|\boldsymbol{\xi}^{\circledcirc}\|^2 = \sum_{i,j=1}^{n} \nabla_{\boldsymbol{\theta}} \ell_i(\widetilde{\boldsymbol{\theta}})^\top \nabla_{\boldsymbol{\theta}} \ell_j(\widetilde{\boldsymbol{\theta}})(u_i - 1)(u_j - 1),$$

$$\|\boldsymbol{\xi}\|^2 = \sum_{i,j=1}^{n} \bigl\{\nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) - \nabla_{\boldsymbol{\theta}} I\!\!E \ell_i(\boldsymbol{\theta}^*)\bigr\}^\top \bigl\{\nabla_{\boldsymbol{\theta}} \ell_j(\boldsymbol{\theta}^*) - \nabla_{\boldsymbol{\theta}} I\!\!E \ell_j(\boldsymbol{\theta}^*)\bigr\}.$$

Let $\breve{\boldsymbol{\xi}} \sim \mathcal{N}(0, \operatorname{Var} \boldsymbol{\xi})\,, \quad \breve{\boldsymbol{\xi}}^{\circ} \sim \mathcal{N}(0, \operatorname{Var} \boldsymbol{\xi}^{\circ})$

	Wilks approximation		Gaussian approximation	
$\sqrt{2\{L(\widetilde{\boldsymbol{\theta}}) - L(\boldsymbol{\theta}^*)\}}$	$\approx$	$\|\boldsymbol{\xi}\|$	$\overset{\mathcal{L}}{\approx}$	$\|\breve{\boldsymbol{\xi}}\|$
				$\wr$
$\sqrt{2\{L^{\circ}(\widetilde{\boldsymbol{\theta}}^{\circ}) - L^{\circ}(\widetilde{\boldsymbol{\theta}})\}}$	$\approx$	$\|\boldsymbol{\xi}^{\circ}\|$	$\overset{\mathcal{L}^{\circ}}{\approx}$	$\|\breve{\boldsymbol{\xi}}^{\circ}\|$

For $\breve{\xi} \sim \mathcal{N}(0, \operatorname{Var}\xi)$ $\mathcal{L}(\|\xi\|) \approx \mathcal{L}(\|\breve{\xi}\|)$.

- $\|\xi\| = \sup_{\gamma \in I\!R^p, \|\gamma\|=1} \gamma^\top \xi \approx \sup_{\gamma \in \mathrm{G}_\gamma} \gamma^\top \xi$

  for a finite grid G_γ on the sphere $\{\gamma \in I\!R^p : \|\gamma\| = 1\}$.

- $I\!P(\|\xi\| > t) = I\!E\, \mathbb{1}(\|\xi\| > t)$.

- $\mathbb{1}\left(\sup_{\gamma \in \mathrm{G}_\gamma} \gamma^\top \xi > t\right) \approx f_\beta(\xi, t)$,

$$f_\beta(\xi, t) \stackrel{\mathrm{def}}{=} \frac{\sum_{\gamma \in \mathrm{G}_S(\varepsilon)} \exp\{\beta(\gamma^\top \xi - t)\}}{1 + \sum_{\gamma \in \mathrm{G}_S(\varepsilon)} \exp\{\beta(\gamma^\top \xi - t)\}}, \ \beta > 0.$$

$f_\beta(\xi, t)$ is a composition of logistic function and smooth max function:

$$\mathbb{1}(\|\xi\| > t) \approx \frac{\exp\{\beta(\|\xi\| - t)\}}{1 + \exp\{\beta(\|\xi\| - t)\}} \qquad \sup_{\gamma \in \mathrm{G}_\gamma} \gamma^\top \xi \approx \frac{1}{\beta} \log\Big\{ \sum_{\gamma \in \mathrm{G}_\gamma} \exp(\beta \gamma^\top \xi) \Big\}$$

For $\breve{\xi} \sim \mathcal{N}(0, \operatorname{Var} \xi)$ $\mathcal{L}(\|\xi\|) \approx \mathcal{L}(\|\breve{\xi}\|)$.

- $\|\xi\| = \sup_{\gamma \in I\!R^p, \|\gamma\|=1} \gamma^\top \xi \approx \sup_{\gamma \in \mathrm{G}_\gamma} \gamma^\top \xi$

  for a finite grid G_γ on the sphere $\{\gamma \in I\!R^p : \|\gamma\| = 1\}$.

- $I\!P(\|\xi\| > t) = I\!E\, \mathbb{1}(\|\xi\| > t)$.

- $\mathbb{1}\left(\sup_{\gamma \in \mathrm{G}_\gamma} \gamma^\top \xi > t\right) \approx f_\beta(\xi, t)$,

$$f_\beta(\xi, t) \stackrel{\mathrm{def}}{=} \frac{\sum_{\gamma \in \mathrm{G}_S(\varepsilon)} \exp\{\beta(\gamma^\top \xi - t)\}}{1 + \sum_{\gamma \in \mathrm{G}_S(\varepsilon)} \exp\{\beta(\gamma^\top \xi - t)\}}, \ \beta > 0.$$

For $\gamma \in I\!R^p$

$$\left| f_\beta(\gamma + \xi, t) - f_\beta(\gamma, t) - \xi^\top \nabla_\gamma f_\beta(\gamma, t) - \frac{1}{2} \xi^\top \nabla^2_\gamma f_\beta(\gamma, t) \xi \right| \leq 0.7 \beta^3 \|\xi\|^3$$

- By Lindeberg's telescopic sum device [Lindeberg, 1922]:

$$\left| I\!E f_\beta(\boldsymbol{\xi}, z) - I\!E f_\beta(\breve{\boldsymbol{\xi}}, z) \right| \leq \frac{\beta^3}{8} \sum_{i=1}^{n} I\!E \left(\|\boldsymbol{\xi}_i\|^3 + \|\breve{\boldsymbol{\xi}}_i\|^3 \right),$$

where

$$\begin{aligned} \boldsymbol{\xi}_i &\stackrel{\mathrm{def}}{=} D_0^{-1} \left\{ \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) - \nabla_{\boldsymbol{\theta}} I\!E \ell_i(\boldsymbol{\theta}^*) \right\}, \\ \breve{\boldsymbol{\xi}}_i &\sim \mathcal{N}(0, \operatorname{Var} \boldsymbol{\xi}_i). \end{aligned}$$

$$\begin{aligned}
I\!P\left(\|\boldsymbol{\xi}\| \geq z + \sqrt{p}\Delta\right) &\leq I\!P\left(\sup_{\boldsymbol{\gamma}\in \mathrm{G}_S(\varepsilon)} \boldsymbol{\gamma}^\top(\boldsymbol{\xi} - \overline{\Delta}) \geq (1-\varepsilon)z\right) \\
&\leq I\!P\left(\|\breve{\boldsymbol{\xi}}\| \geq z\left(1 - \frac{2\varepsilon}{1+\varepsilon}\right) - \sqrt{p}\Delta\right) \\
&\quad + \left[1 + \mathrm{card}\{\mathrm{G}_S(\varepsilon)\}\right] \mathrm{e}^{-\beta\Delta} + \mathtt{C}_3 \frac{\left(\beta \mathtt{C}_2 \|\Sigma^{1/2}\|\right)^3}{4\sqrt{n}}.
\end{aligned}$$

Similarly for the lower bound:

$$\begin{aligned}
I\!P\left(\|\boldsymbol{\xi}\| \geq z - \sqrt{p}\Delta\right) &\geq I\!P\left(\|\breve{\boldsymbol{\xi}}\| \geq z\left(1 + \frac{2\varepsilon}{1-\varepsilon}\right) + \sqrt{p}\Delta\right) \\
&\quad - \left[1 + \mathrm{card}\{\mathrm{G}_S(\varepsilon)\}\right] \mathrm{e}^{-\beta\Delta} - \mathtt{C}_3 \frac{\left(\beta \mathtt{C}_2 \|\Sigma^{1/2}\|\right)^3}{4\sqrt{n}}.
\end{aligned}$$

- Finally we compare the norms of two Gaussian vectors: $\|\breve{\boldsymbol{\xi}}\|$ and $\|\breve{\boldsymbol{\xi}}^{\circledcirc}\|$:

$$\begin{array}{ccc} \|\boldsymbol{\xi}\| & \overset{\mathcal{L}}{\approx} & \|\breve{\boldsymbol{\xi}}\| \\ & & \wr\wr \\ \|\boldsymbol{\xi}^{\circledcirc}\| & \overset{\mathcal{L}^{\circledcirc}}{\approx} & \|\breve{\boldsymbol{\xi}}^{\circledcirc}\| \end{array}$$

Gaussian approximation

i.e. their covariance matrices:

$$\mathrm{Var}(\breve{\boldsymbol{\xi}}) = D_0^{-1} \mathrm{Var}\left(\nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta}^*)\right) D_0^{-1},$$
$$\mathrm{Var}^{\circledcirc}(\breve{\boldsymbol{\xi}}^{\circledcirc}) = D_0^{-1} \mathrm{Var}^{\circledcirc}\left(\nabla_{\boldsymbol{\theta}} L^{\circledcirc}(\widetilde{\boldsymbol{\theta}})\right) D_0^{-1}.$$

Let $\Sigma \stackrel{\text{def}}{=} \operatorname{Var}(\boldsymbol{\xi}),\ \Sigma^{\circledcirc} \stackrel{\text{def}}{=} \operatorname{Var}^{\circledcirc}(\boldsymbol{\xi}^{\circledcirc})$.

$$D_0 \Sigma D_0 = \sum_{i=1}^{n} I\!E \left\{ \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*)^\top \right\} - \sum_{i=1}^{n} I\!E \left\{ \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) \right\} I\!E \left\{ \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) \right\}^\top,$$

$$D_0 \Sigma^{\circledcirc} D_0 = \sum_{i=1}^{n} \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*)^\top.$$

It holds:

$$\left\| \Sigma^{-1/2} \Sigma^{\circledcirc} \Sigma^{-1/2} - \boldsymbol{I}_p \right\|_\infty \leq \mathtt{C}_{\mathcal{I}}^2 \delta_\xi^2 \asymp (p + \mathtt{y})/n^{1/2},$$

the terms $\mathtt{C}_{\mathcal{I}}^2, \delta_\xi^2$ are responsible for the modelling bias, they are proportional to the error terms δ and Δ_1 in the statement on the bootstrap validity above.

Δ_1, Δ_2 are also determined by $\mathtt{C}_{\mathcal{I}}^2$ and the errors induced by $\mathbb{I}\big(\sup_{\mathrm{G}_\gamma} \boldsymbol{\gamma}^\top \boldsymbol{\xi} > t\big) \approx f_\beta(\boldsymbol{\xi}, t)$.

Conditions include

- Independency of the observations Y_i .
- Bounded exponential moments of gradient of $L(\boldsymbol{\theta}) - I\!\!E L(\boldsymbol{\theta})$ up to 3-d order.
- Small modelling bias:

$$\sum_{i=1}^{n} \| D_0^{-1} I\!\!E \nabla_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}^*) \|^2 \leq \delta_\xi^2 = \mathtt{C}(p + \mathtt{y})/n^{1/2}$$

  for $\ell_i(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \log \frac{dP_i(\boldsymbol{\theta})}{d\mu_0}(Y_i)$, $D_0^2 \stackrel{\text{def}}{=} -\nabla_{\boldsymbol{\theta}}^2 I\!\!E L(\boldsymbol{\theta}^*)$.

- $\| D_0 \operatorname{Var}\left(\nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta}^*)\right)^{-1} D_0 \|_\infty \leq \mathtt{C}_{\mathcal{I}}^2$.

  ($\delta_\xi^2 = 0, \mathtt{C}_{\mathcal{I}}^2 = 1$ if the parametric model is true)

I.i.d. errors, misspecified distribution

$$Y_i = 2 + \varepsilon_i, \;\; \varepsilon_i \sim Lap(0, 2^{-1/2}), \text{ i.i.d.}, \mathrm{Var}(\varepsilon_i) = 1,$$

$$L(\theta) = -\sum_{i=1}^{n} (Y_i - \theta)^2/2 \quad \text{corresponds to } \varepsilon_i \sim \mathcal{N}(0,1).$$

Quantiles of $\sqrt{2}\{L(\widetilde{\theta}) - L(\theta^*)\}^{1/2}$ are estimated from 5000 $\boldsymbol{Y}$ samples of length 50.

Quantiles of $\sqrt{2}\{L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})\}^{1/2}$ are estimated from 5000 $\{u_i\} \sim exp(1)$ samples and 50 $\boldsymbol{Y}$ samples of length 50.

Confidence level	0.99	0.95	0.90	0.85	0.80	0.75
$\sqrt{2}\{L(\widetilde{\theta}) - L(\theta^*)\}^{1/2}$	2.55	1.99	1.64	1.44	1.29	1.16
$\sqrt{2}\{L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})\}^{1/2}$, `Mean`	2.63	1.92	1.58	1.38	1.22	1.09
$\sqrt{2}\{L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})\}^{1/2}$, `St.dev.`	0.39	0.27	0.22	0.19	0.17	0.15

Comparison of the empirical quantiles for the sample size $n = 50$

Heteroscedastic errors, misspecified distribution

$$Y_i = 2 + \sigma_i \varepsilon_i, \quad \varepsilon_i \sim Lap(0, 2^{-1/2}), \quad \sigma_i = 2 - (i \mod 4)/2,$$

$$L(\theta) = -\sum_{i=1}^{n} (Y_i - \theta)^2/2 \quad \text{corresponds to} \quad \sigma_i \varepsilon_i \sim \mathcal{N}(0,1).$$

Quantiles of $\sqrt{2}\{L(\widetilde{\theta}) - L(\theta^*)\}^{1/2}$ are estimated from 5000 $\boldsymbol{Y}$ samples of length 50.

Quantiles of $\sqrt{2}\{L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})\}^{1/2}$ are estimated from 5000 $\{u_i\} \sim exp(1)$ samples and 50 $\boldsymbol{Y}$ samples of length 50.

Confidence level	0.99	0.95	0.90	0.85	0.80	0.75
$\sqrt{2}\{L(\widetilde{\theta}) - L(\theta^*)\}^{1/2}$	5.06	3.86	3.87	2.85	2.52	2.27
$\sqrt{2}\{L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})\}^{1/2}$, Mean	5.26	3.87	3.19	2.78	2.46	2.20
$\sqrt{2}\{L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})\}^{1/2}$, St.dev.	0.88	0.65	0.54	0.46	0.41	0.37

Comparison of the empirical quantiles for the sample size $n = 50$

Biased model

$$Y_i = \beta \sin(X_i) + \varepsilon_i, \quad \varepsilon_i \sim Lap(0, 2^{-1/2}), \text{ i.i.d.},$$

X_i are equidistant in $[0, 2\pi]$,

$$L(\theta) = -\sum_{i=1}^{n} (Y_i - \theta)^2/2 \;\Rightarrow\; \theta^* = 0,$$

$\beta > 0$ is an amplitude.

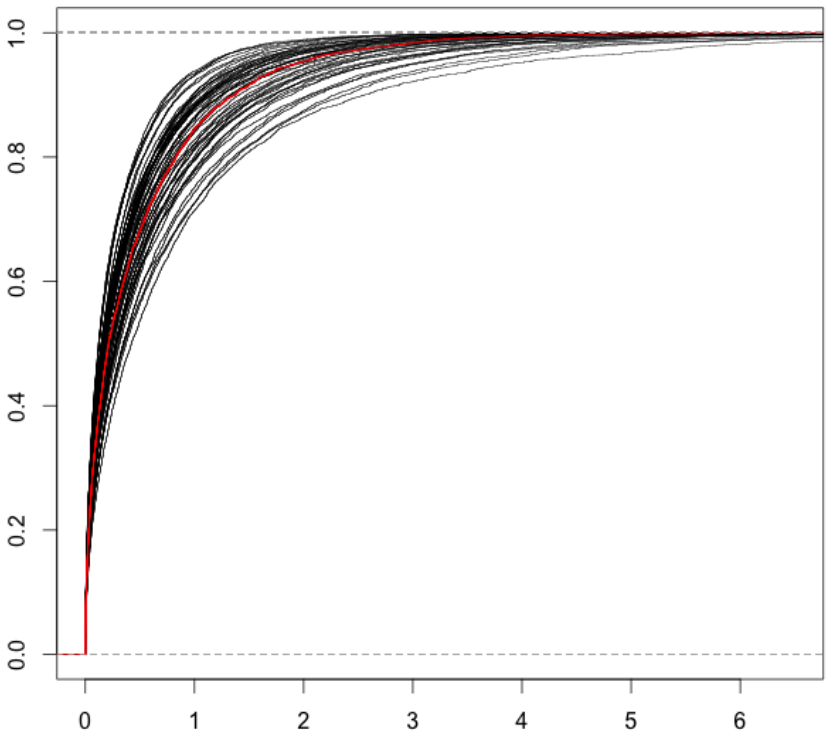
Let us check how the confidence set for θ^* changes with β.

Numerical results (III). Biased model

$n = 50$

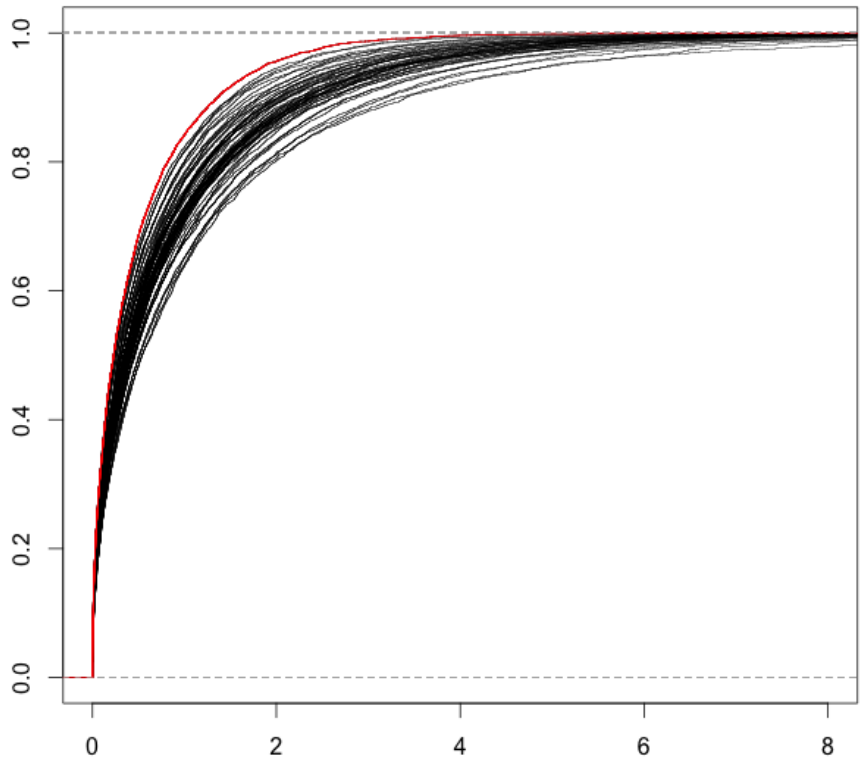
— empirical distribution function of $L(\widetilde{\theta}) - L(\theta^*)$ estimated with 5000 $\mathbf{Y}$ samples

— 50 empirical distribution functions of $L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})$ estimated with 5000 $u_1, \ldots, u_n \sim exp(1)$ samples



$Y_i = 0.25 \sin(X_i) + Lap(0, 2^{-1/2})$

the "small modelling bias" case

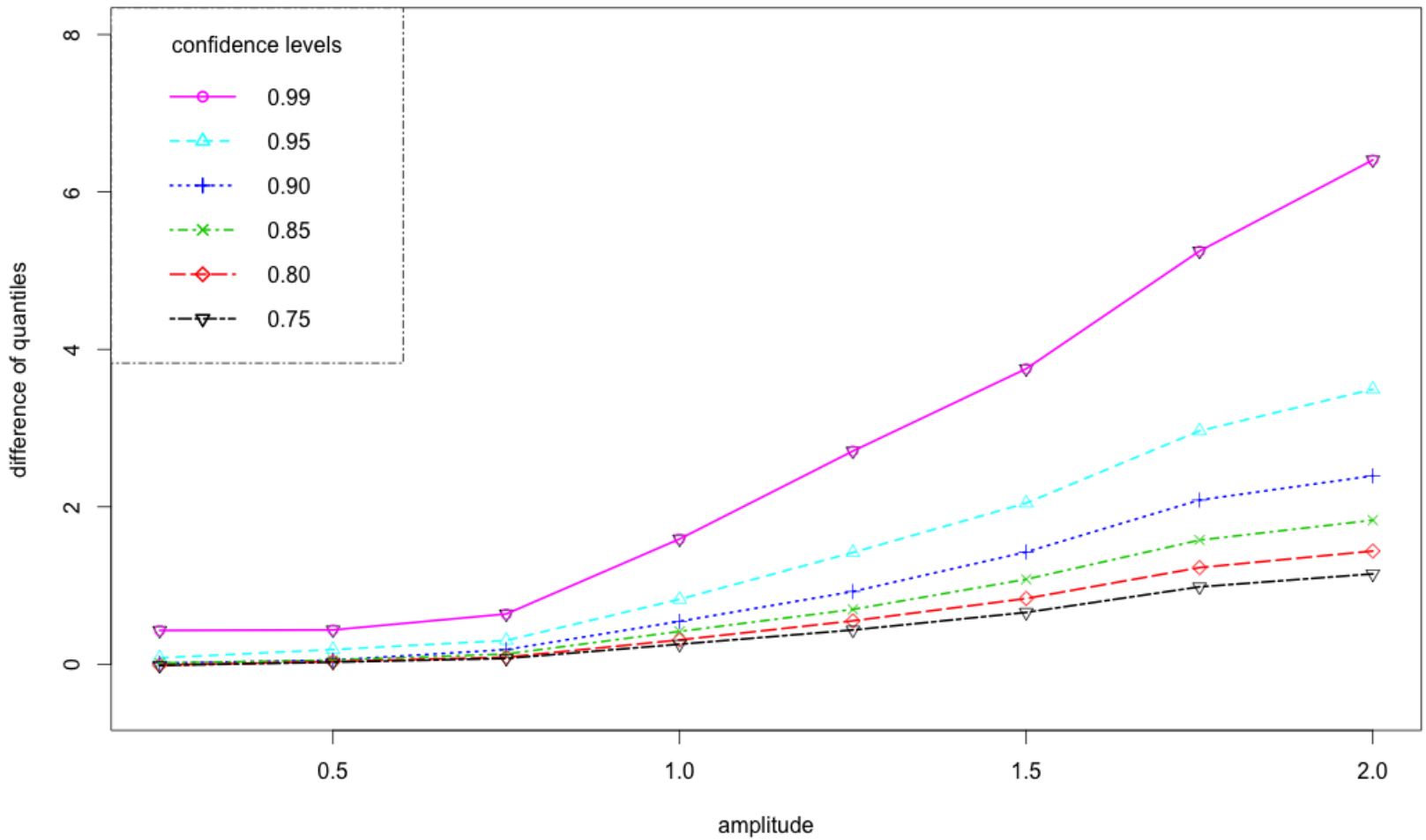


$Y_i = 1.25 \sin(X_i) + Lap(0, 2^{-1/2})\,,$

"small modelling bias" condition is not fulfilled

Numerical results (III). Biased model

The difference $\big($"Bootstrap quantile" $-$ "$\boldsymbol{Y}$-quantile"$\big)$ growing with modelling bias:



Logistic regression with bias

$$Y_i \sim Bernoulli(\beta X_i),$$

X_i are equidistant in $[0, 2]$,

$$\beta \in (0, 1/2],$$

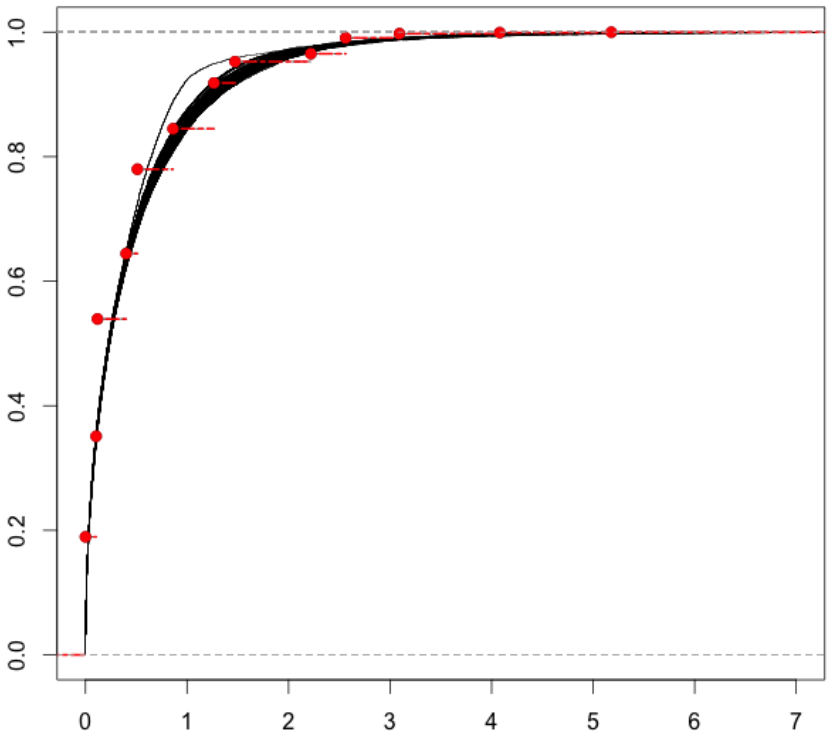
$$L(\theta) = \sum_{i=1}^{n} \left\{ Y_i \theta - \log(1 + \mathrm{e}^{\theta}) \right\},$$

$$\Rightarrow \quad \theta^* = \log\{\beta/(1-\beta)\}.$$

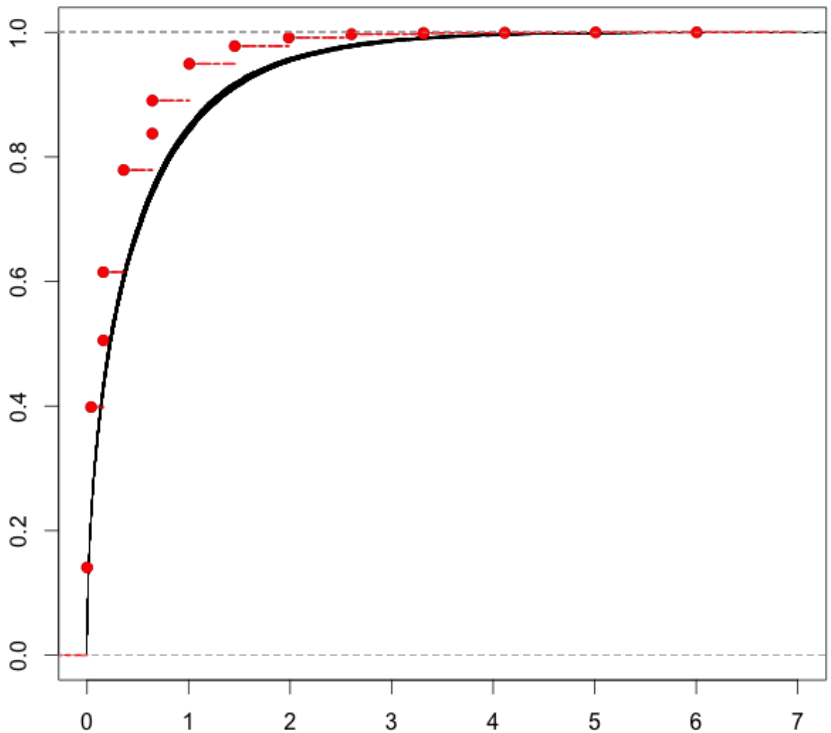
$n = 50$

— empirical distribution function of $L(\widetilde{\theta}) - L(\theta^*)$ estimated with 10000 $\boldsymbol{Y}$ samples

— 50 empirical distribution functions of $L^{\circledcirc}(\widetilde{\theta}^{\circledcirc}) - L^{\circledcirc}(\widetilde{\theta})$ estimated with 10000 $u_1, \ldots, u_n \sim exp(1)$ samples



$Y_i \sim Bernoulli(0.1X_i)$



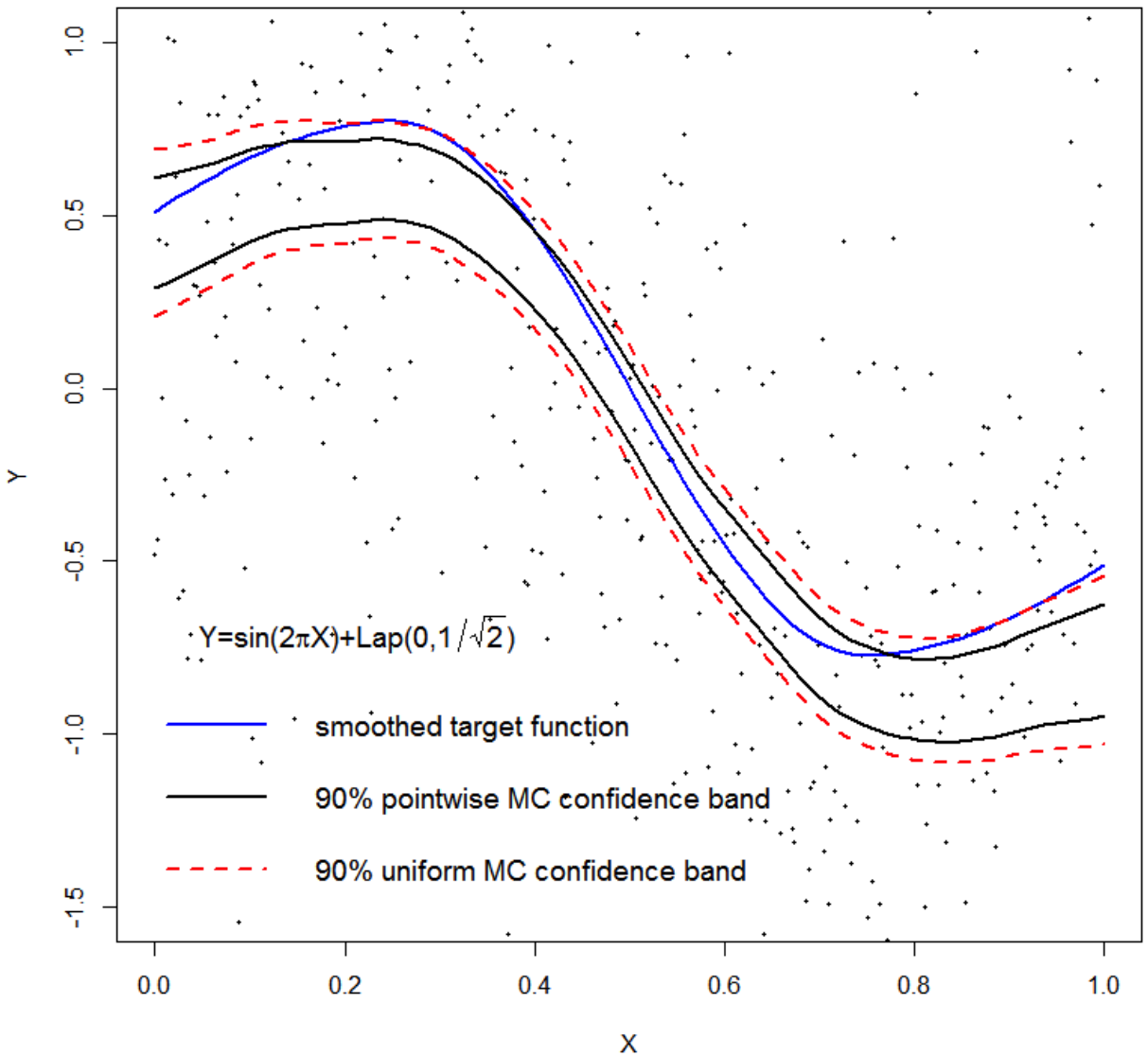
$Y_i \sim Bernoulli(0.5X_i)$

Outline

For a fixed sample size n construct a simultaneous confidence set for $f(x)$, when $x \in \mathcal{X} \subset \mathbb{R}^d$, with a given confidence level $1-\alpha$ and a minimal possible diameter.

$$\mathbb{P}\left\{\bigcap_{x\in\mathcal{X}} \{f(x) \in \mathcal{E}_x\}\right\} \geq 1-\alpha.$$

Uniform vs. pointwise confidence corridors



Simulated data:

$Y_i = \sin(2\pi X_i) + \varepsilon_i$,

$\varepsilon_i \sim Lap(0, 1/\sqrt{2})$ i.i.d.,

X_i are equidistant on $[0, 1]$,

sample size $= 500$,

confidence sets are based on the Nadaraya-Watson estimator with parabolic weights and bandwidth $= 0.25$.

Local likelihood function for $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^p$, $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^d$:

$$L(\boldsymbol{\theta}, \boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \ell(Y_i, f(X_i, \boldsymbol{\theta})) w_i(\boldsymbol{x}),$$

$$\ell(Y_i, f(X_i, \boldsymbol{\theta})) \stackrel{\text{def}}{=} \frac{dP_i(f(X_i, \boldsymbol{\theta}))}{d\mu_0}(Y_i),$$

$f(\boldsymbol{x}, \boldsymbol{\theta})$ is a known smooth function $\mathbb{R}^d \times \mathbb{R}^p \mapsto \mathbb{R}$,

$w_i(\boldsymbol{x}) \stackrel{\text{def}}{=} K\left(\frac{\boldsymbol{x} - X_i}{h}\right)$ are localising weights,

$K(\boldsymbol{x}) \geq 0$ is a symmetric kernel function,

$h > 0$ is a fixed bandwidth (however, the results here are easily extended to the uniform in bandwidth bounds, when h lies on a fine finite grid).

- Local maximum likelihood estimate:

$$\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} \operatorname*{argmax}_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta}, \boldsymbol{x}),$$

$$\widetilde{f}(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} f(\boldsymbol{x}, \widetilde{\boldsymbol{\theta}}(\boldsymbol{x})).$$

- Target parameters (smoothed versions):

$$\boldsymbol{\theta}^*(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} \operatorname*{argmax}_{\boldsymbol{\theta}\in\Theta} I\!\!E L(\boldsymbol{\theta}, \boldsymbol{x}),$$

$$f^*(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} f(\boldsymbol{x}, \boldsymbol{\theta}^*(\boldsymbol{x})).$$

- Pointwise confidence sets

$$\mathcal{E}(\mathfrak{z}, \boldsymbol{x}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{\theta} : L(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x}) - L(\boldsymbol{\theta}, \boldsymbol{x}) \le \mathfrak{z} \right\} \subset I\!R^p,$$

$$\mathcal{E}_f(\mathfrak{z}, \boldsymbol{x}) \stackrel{\text{def}}{=} \{ f(\boldsymbol{x}, \boldsymbol{\theta}) : \boldsymbol{\theta} \in \mathcal{E}(\mathfrak{z}, \boldsymbol{x}) \} \subset I\!R.$$

- Uniform (simultaneous) confidence corridor:

$$\mathcal{E}_f(\mathfrak{z}) \stackrel{\text{def}}{=} \bigcup_{\boldsymbol{x} \in \mathcal{X}} \{ \boldsymbol{x} \times \mathcal{E}_f(\mathfrak{z}, \boldsymbol{x}) \} \subset I\!R^d \times I\!R.$$

- Aim: to build for each $\boldsymbol{x} \in \mathcal{X}$ a quantile function $\mathfrak{z}(\alpha, \boldsymbol{x})$ s.t.

$$I\!P \left(\bigcap_{\boldsymbol{x} \in \mathcal{X}} \left\{ f^*(\boldsymbol{x}) \in \mathcal{E}_f(\mathfrak{z}(\alpha, \boldsymbol{x}), \boldsymbol{x}) \right\} \right) \ge 1 - \alpha.$$

For a smooth $f(\cdot, \boldsymbol{\theta})$

$$f^*(\boldsymbol{x}) \in \mathcal{E}_f(\mathfrak{z}, \boldsymbol{x}) \;\Leftrightarrow\; \boldsymbol{\theta}^*(\boldsymbol{x}) \in \mathcal{E}(\mathfrak{z}, \boldsymbol{x})$$
$$\Leftrightarrow\; L(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x}) - L(\boldsymbol{\theta}^*(\boldsymbol{x}), \boldsymbol{x}) \le \mathfrak{z}.$$

Therefore, the initial problem translates into finding $\mathfrak{z}(\alpha, \boldsymbol{x})$ s.t.

$$\mathbb{P}\left(\sup_{\boldsymbol{x} \in \mathcal{X}} \left\{ L(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x}) - L(\boldsymbol{\theta}^*(\boldsymbol{x}), \boldsymbol{x}) - \mathfrak{z}(\alpha, \boldsymbol{x}) \right\} \le 0 \right) \ge 1 - \alpha.$$

Similarly to the parametric approach define the bootstrap objects for the local likelihood conditioned on $\boldsymbol{Y}$:

$$L^{\circledcirc}(\boldsymbol{\theta}, \boldsymbol{x}) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} \ell(Y_i, f(X_i, \boldsymbol{\theta})) w_i(\boldsymbol{x}) u_i.$$

The target parameter

$$\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}) = \operatorname*{argmax}_{\boldsymbol{\theta} \in \Theta} I\!\!E^{\circledcirc} L^{\circledcirc}(\boldsymbol{\theta}, \boldsymbol{x}).$$

Maximum likelihood ratio for the bootstrap case:

$$L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}^{\circledcirc}, \boldsymbol{x}) - L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x}) \stackrel{\mathrm{def}}{=} \sup_{\boldsymbol{\theta} \in \Theta} L^{\circledcirc}(\boldsymbol{\theta}, \boldsymbol{x}) - L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x}).$$

$$\sqrt{2\{L(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}),\boldsymbol{x}) - L(\boldsymbol{\theta}^*(\boldsymbol{x}),\boldsymbol{x})\}} \overset{\bigcap_{x\in\mathcal{X}}}{\approx} \|\boldsymbol{\xi}(\boldsymbol{x})\|$$

$$\wr\wr\, w_X$$

$$\sqrt{2\{L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}^{\circledcirc}(\boldsymbol{x}),\boldsymbol{x}) - L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}),\boldsymbol{x})\}} \overset{\bigcap_{x\in\mathcal{X}}}{\approx} \|\boldsymbol{\xi}^{\circledcirc}(\boldsymbol{x})\|$$

- $\overset{\bigcap_{x\in\mathcal{X}}}{\approx}$ denotes the uniform Wilks-type approximation,
- $\overset{w_X}{\approx}$ means approximation in distribution jointly in $\boldsymbol{x}\in\mathcal{X}$.
- The normalized scores and full Fisher Information matrices are:

$$\boldsymbol{\xi}(\boldsymbol{x}) \overset{\text{def}}{=} D_0^{-1}(\boldsymbol{x}_0)\nabla_{\boldsymbol{\theta}} L(\boldsymbol{\theta}^*(\boldsymbol{x}),\boldsymbol{x}) \qquad \boldsymbol{\xi}^{\circledcirc}(\boldsymbol{x}) \overset{\text{def}}{=} \mathfrak{D}_0^{-1}(\boldsymbol{x})\nabla_{\boldsymbol{\theta}} L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}),\boldsymbol{x})$$

$$D_0^2(\boldsymbol{x}) \overset{\text{def}}{=} -\nabla^2_{\boldsymbol{\theta}} I\!\!E L(\boldsymbol{\theta}^*(\boldsymbol{x}),\boldsymbol{x}) \qquad \mathfrak{D}_0^2(\boldsymbol{x}) \overset{\text{def}}{=} -\nabla^2_{\boldsymbol{\theta}} I\!\!E^{\circledcirc} L^{\circledcirc}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}),\boldsymbol{x})$$

Let $\mathrm{G}_{\mathcal{X}}(\varepsilon)$ denote a finite ε-net on $\mathcal{X}$: $\forall \boldsymbol{x} \in \mathcal{X}\ \exists \boldsymbol{x}_0 \in \mathrm{G}_{\mathcal{X}}(\varepsilon) : \|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \varepsilon$.

1. Pointwise step

For each $\boldsymbol{x} \in \mathrm{G}_{\mathcal{X}}(\varepsilon)$ compute an empirical distribution of the ratio $\sqrt{2\{L^{\circ}(\widetilde{\boldsymbol{\theta}}^{\circ}(\boldsymbol{x}), \boldsymbol{x}) - L^{\circ}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x})\}}$, which yields the pointwise quantile function $\mathfrak{z}^{\circ}(\alpha, \boldsymbol{x})$ for each $\alpha \in (0,1)$ and $\boldsymbol{x} \in \mathrm{G}_{\mathcal{X}}(\varepsilon)$.

2. Correction for the uniformity

By an iterative procedure find a maximum value $\alpha(\varepsilon)$ s.t.

$$\mathbb{P}\left(\sup_{\boldsymbol{x} \in \mathrm{G}_{\mathcal{X}}(\varepsilon)} \left[\sqrt{2\{L^{\circ}(\widetilde{\boldsymbol{\theta}}^{\circ}(\boldsymbol{x}), \boldsymbol{x}) - L^{\circ}(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x})\}} - \mathfrak{z}^{\circ}(\alpha(\varepsilon), \boldsymbol{x}) \right] \leq 0 \right) \geq 1 - \alpha.$$

The value $\alpha(\varepsilon)$ is a corrected confidence level for $\sup_{\boldsymbol{x} \in \mathrm{G}_{\mathcal{X}}(\varepsilon)}$.

It holds $\alpha(\varepsilon)/\alpha \geq \mathtt{c}\,|\mathrm{G}_{\mathcal{X}}(\varepsilon)|^{-1}$.

Local constant regression:

$$Y_i = \sin(2\pi X_i) + \varepsilon_i, \quad \varepsilon_i \sim Lap(0, 2^{-1/2}), \text{ i.i.d.},$$

X_i are equidistant in $[0,1]$,

$$L(\theta) = -\frac{1}{2}\sum_{i=1}^{n}(Y_i - \theta)^2 w_i(x),$$

$$w_i(x) = \begin{cases} 1 - \{(x - X_i)/h\}^2, & \text{if } |(x - X_i)/h| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Nadaraya-Watson estimate:

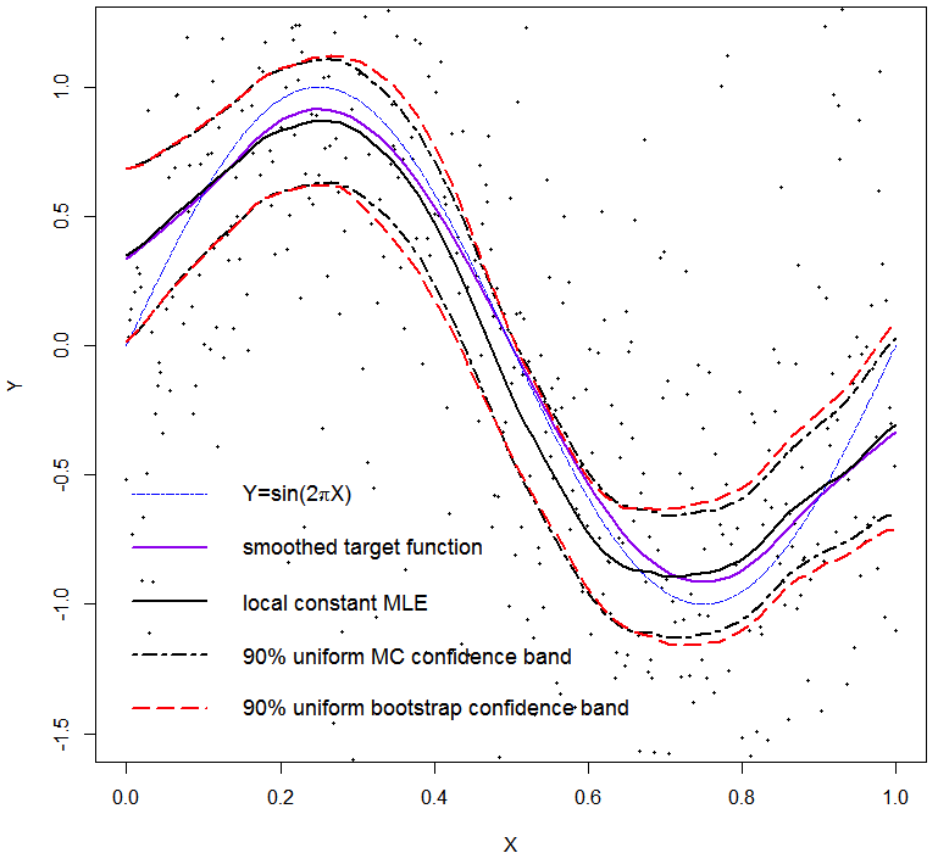
$$\widetilde{\theta}(x) = \sum_{i=1}^{n} Y_i \frac{w_i(x)}{\sum_{i=1}^{n} w_i(x)},$$

Smoothed target function:

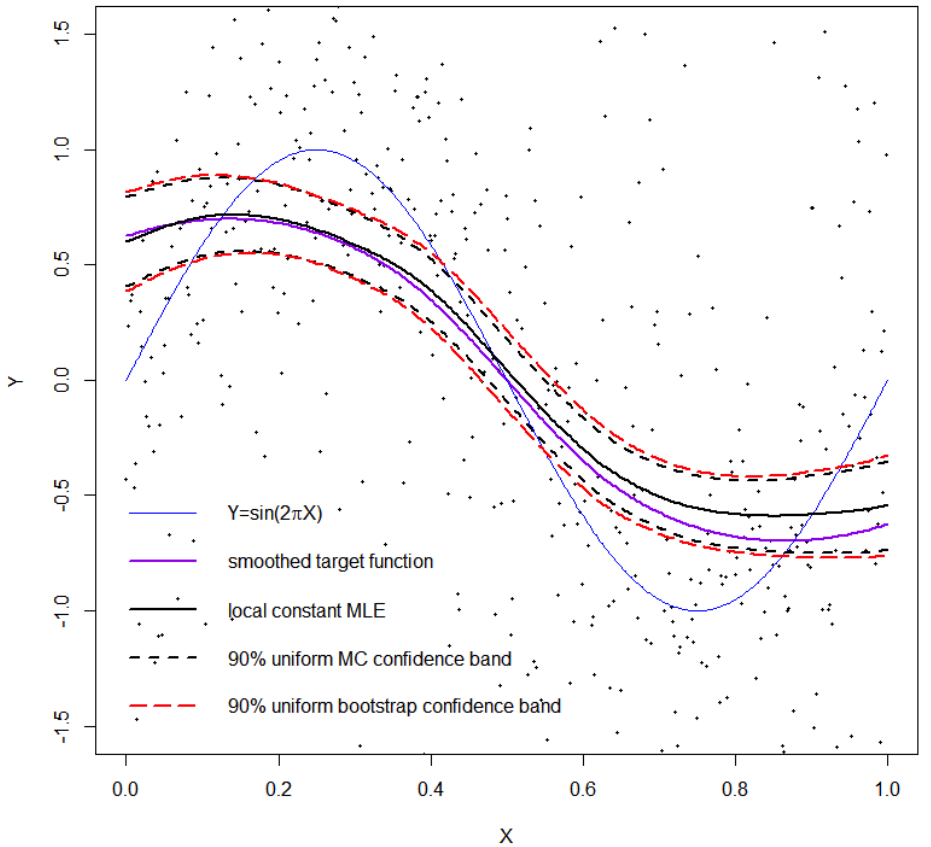
$$\theta^*(x) = \sum_{i=1}^{n} I\!\!E Y_i \frac{w_i(x)}{\sum_{i=1}^{n} w_i(x)}.$$

Numerical results. Uniform confidence corridors

$$Y_i = \sin(2\pi X_i) + \varepsilon_i, \quad \varepsilon_i \sim Lap(0, 2^{-1/2}), \text{ i.i.d.}, \quad n = 500\,.$$



$h = 0.15$



$h = 0.35$

Let $\boldsymbol{x}_G$ denote the projection of a point $\boldsymbol{x} \in \mathcal{X}$ on the grid $\mathrm{G}_{\mathcal{X}}(\varepsilon)$:

$$\boldsymbol{x}_G \stackrel{\mathrm{def}}{=} \operatorname{argmin}_{\boldsymbol{x}_1 \in \mathrm{G}_{\mathcal{X}}(\varepsilon)} \|\boldsymbol{x} - \boldsymbol{x}_1\| .$$

It holds with probability $\geq 1 - \mathtt{C} \left|\mathrm{G}_{\mathcal{X}}(\varepsilon)\right| \mathrm{e}^{-\mathtt{y}} = 1 - \mathtt{C} \exp\left\{-\mathtt{y} + \log \left|\mathrm{G}_{\mathcal{X}}(\varepsilon)\right|\right\}$

$$\sup_{\boldsymbol{x}:\|\boldsymbol{x}-\boldsymbol{x}_G\| \leq \varepsilon} \left| \sqrt{2\bigl\{L(\widetilde{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{x}) - L(\boldsymbol{\theta}^*(\boldsymbol{x}), \boldsymbol{x})\bigr\}} - \|\boldsymbol{\xi}(\boldsymbol{x}_G)\| \right| \leq \Delta(\varepsilon, \mathtt{y}),$$

where $\Delta(\varepsilon, \mathtt{y}) = \mathtt{C}(p + \mathtt{y})/\sqrt{(nh^d)}$.

Some further problems

- Adaptive choice of optimal bandwidth using a multiscale correction.
- Non-asymptotic bounds for Monte Carlo simulations (joint with Christian Bayer).
- Estimation of dynamic quantile/expectile curves (joint with project B1).
- Application to the multiple hypothesis testing: choice of a correction for multiplicity.

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Thank you for your attention!