Risky Linear Approximations

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ÖKONOMISCHES
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Motivation & Contribution

What are the effects of small shock realizations in risky economies?

- Linear approx. around the deterministic s.s. insufficient for
  - Asset pricing, recursive utility, etc.
- Why?
  - linear approx. around the deterministic s.s. shuts down risk

My contribution: **Risky linear approximation**

- Construct linear approximation around
  - risky (stoch.) steady state
  - ergodic mean
- estimation of risk-sensitive parameters using Kalman filter
- uniformly improve Euler eq. accuracy of linear approx.

Related Studies

DSGE Perturbation

- Local approx. around det. s.s.
  - First order, ubiquitous, see, for example, Uhlig (1999)
  - Higher order, Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Kim et al. (2008), Lombardo (2010), Lan and Meyer-Gohde (2013c)
- Capturing risk means abandoning linearity

Risky Steady State Approximations and Risk Perturbations

- Coeurdacier et al. (2011), Juillard (2011), Kliem and Uhlig (2013), Evers (2012), de Groot (2013)
  - Approx. the equilibrium conditions in risk/perturbation parameter
  - Require iterative/fix-point procedures to approx. risky s.s.

My method works implicitly w/ unknown policy function

- Solves linear equations in output from stand. perturbation
- Can approx. ergodic mean as well as risky s.s.

Outline

Model Setup

I analyze a family of discrete-time rational expectations models

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \sigma\varepsilon_t)]$$

- $f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_y}$ equilibrium conditions
- $y_t \in \mathbb{R}^{n_y}$ endogenous variables
- $\varepsilon_t \in \mathbb{R}^{n_e}$ exogenous shocks
- $\sigma = 1$—stochastic model; $\sigma = 0$—deterministic model

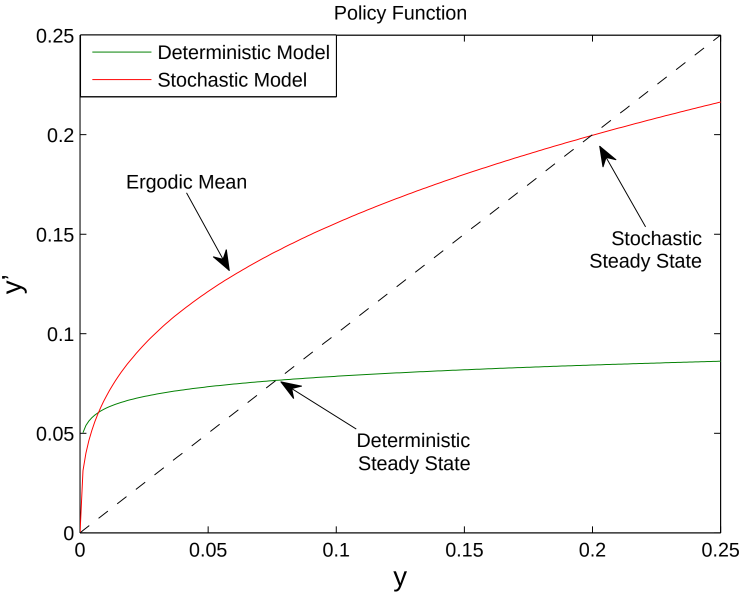
Solution (policy function) is a time-invariant function

$y_t = g(y_{t-1}, \sigma\varepsilon_t, \sigma), \quad g : \mathbb{R}^{ny} \times \mathbb{R}^{n_e} \times \mathbb{R} \to \mathbb{R}^{ny}$

Assumptions

- g and f locally analytic around det. s.s. in a domain containing stoch. s.s. and ergodic mean
- g locally stable around det. s.s.
- $\varepsilon_t \in \mathbb{R}^{n_e}$ iid $\sim \phi$ with $E[\varepsilon_t] = 0$ and $|E\left[\varepsilon_t^{\otimes[j]}\right]| < \infty$, for $j \in \mathbb{Z}^+$

Policy Function Illustration



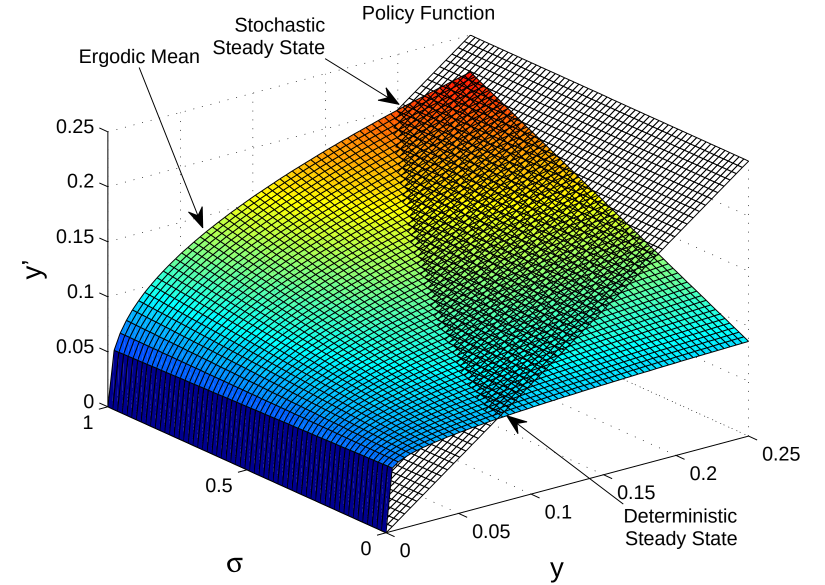
Locally Analytic Policy Functions

- If $g(y_{t-1}, \sigma\varepsilon_t, \sigma)$ is locally analytic around $y_{t-1} = \bar{y}$, $\varepsilon_t = 0$, $\sigma = 0$
  - in a domain that contains the stochastic steady state and ergodic mean
- Then y_t can be written within this domain as

$$
\begin{aligned}
y_t &= \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{i=0}^{\infty} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i \right] \begin{bmatrix} y_{t-1}' - \bar{y} & \sigma\varepsilon_t' \end{bmatrix}^{\prime \otimes [j]} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{n!} y_{\sigma^n i_1 \cdots i_m} \sigma^n \right] (\sigma\varepsilon_{t-i_1} \otimes \cdots \otimes \sigma\varepsilon_{t-i_m})
\end{aligned}
$$

- Both the stoch. steady state: $0 = -\bar{y}^{\text{stoch}} + \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{i=0}^{\infty} \frac{1}{i!} y_{z^j \sigma^i} \right] \begin{bmatrix} \bar{y}^{\text{stoch}\prime} - \bar{y} & 0' \end{bmatrix}^{\prime \otimes [j]}$
- and ergodic mean: $E[y_t] = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{n!} y_{\sigma^n i_1 \cdots i_m} \right] E\left[\varepsilon_{t-i_1} \otimes \cdots \otimes \varepsilon_{t-i_m} \right]$
- are recoverable from the IFT, requiring only
  - derivatives of g evaluated at the deterministic steady state
  - moments of ε_t
- y_t analytic at the respective points

Policy Function Illustration



Preliminaries

Risky Linear Approximation

Risk-Sensitive Business Cycle Model

Calibrated Results

Estimated Results

Wrap Up

Risky Linear Approximation

1. Calculate derivatives of g at the deterministic steady state
2. Construct approximations (in σ) of either
   - stochastic steady
     $$\overline{y}^{\text{stoch}}(\sigma) = g(\overline{y}^{\text{stoch}}(\sigma), 0, \sigma)$$
   - ergodic mean
     $$\overline{y}^{\text{mean}}(\sigma) \equiv E[y_t] = E[g(y_{t-1}, \sigma \varepsilon_t, \sigma)]$$
3. Construct approx (in σ) of first derivatives of g at desired point
   $$\overline{y_y}(\sigma) \equiv g_y(\overline{y}(\sigma), 0, \sigma), \quad \overline{y_\varepsilon}(\sigma) \equiv g_\varepsilon(\overline{y}(\sigma), 0, \sigma)$$
4. Assemble a σ-dependant linear approximation at desired point
   $$y_t \simeq \overline{y}(\sigma) + \overline{y_y}(\sigma)(y_{t-1} - \overline{y}(\sigma)) + \overline{y_\varepsilon}(\sigma)\varepsilon_t$$
5. Set σ to one

The Stochastic Steady State

Define a σ-dependant steady state implicitly

$$\bar{y}(\sigma) = g(\bar{y}(\sigma), 0, \sigma)$$

The point where, in the absence of shocks this period, agents decide to stay while expecting shocks in the future and knowing their probability distribution. Juillard (2011)

- Stoch. s.s.: $\bar{y}^{\text{stoch}} \equiv \bar{y}(1)$ Det. s.s.: $\bar{y}^{\text{det}} = \bar{y}(0)$

Taylor approx for σ steady states around $\sigma = 0$

$$\bar{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{y}^{(i)}(0) \sigma^i$$

- Differentiate $\bar{y}(\sigma) = g(\bar{y}(\sigma), 0, \sigma)$ at $\sigma = 0$
- To second order in σ, the stochastic steady state is

$$\bar{y}^{\text{stoch}} \approx \bar{y} + \frac{1}{2}\left(I - g_y\right)^{-1} g_{\sigma^2}$$

- Identical to Lan and Meyer-Gohde (2013b)

The Ergodic Mean

Define the σ-dependant mean

$$\overline{y}(\sigma) = E\left[g(y_{t-1}, \sigma\varepsilon_t, \sigma)\right]$$

- Ergodic mean: $\overline{y}^{\text{mean}} \equiv \overline{y}(1) = E[y_t]$ Det. s.s.: $\overline{y}^{\text{det}} = \overline{y}(0)$

Taylor approx for σ means around $\sigma = 0$

$$\overline{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \overline{y}^{(i)}(0)\sigma^i$$

- Differentiate $\overline{y}(\sigma) = E\left[g(y_{t-1}, \sigma\varepsilon, \sigma)\right]$ at $\sigma = 0$
- To second order in σ, the ergodic mean is

$$\overline{y}^{\text{mean}} \approx \overline{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_{\varepsilon}^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

- Same as Lan and Meyer-Gohde (2013a) and Andreasen et al. (2013)

Risk Adjusted First Derivatives

The linear approximation around a risk-adjusted point needs

$$\overline{y_y}(\sigma) \equiv g_y(\overline{y}(\sigma), 0, \sigma), \qquad \overline{y_\varepsilon}(\sigma) \equiv g_\varepsilon(\overline{y}(\sigma), 0, \sigma)$$

- At the det. s.s., $\overline{y_y}(0) = g_y$ $\qquad \overline{y_\varepsilon}(0) = g_\varepsilon$

Taylor approx for σ derivative ($\overline{y_y}(\sigma)$) around $\sigma = 0$

$$\overline{y_y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \overline{y_y}^{(i)}(0)\, \sigma^i$$

- Differentiate $\overline{y_y}(\sigma) = g_y(\overline{y}(\sigma), 0, \sigma)$ at $\sigma = 0$
- To second order in σ, $\overline{y_y}(1)$ is

$$\overline{y_y}(1) \approx g_y + \frac{1}{2}\left(g_{y^2}\left(\overline{y}_{\sigma^2}(0) \otimes I_{n_y}\right) + g_{\sigma^2 y}\right)$$

- Analogous derivations follow for $\overline{y_\varepsilon}(\sigma)$

Standard Perturbation First Derivatives

The Taylor series of y_t, with $z_t \doteq \begin{bmatrix} y'_{t-1} & \sigma \varepsilon'_t \end{bmatrix}'$, is

$$y_t \approx \sum_{j=0}^{M} \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^j \sigma^i} \sigma^i \right] (z_t - \bar{z})^{\otimes[j]}$$

The derivative of which with respect to z_t is

$$\mathscr{D}_{z_t} y_t \approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1} \sigma^i} \sigma^i \right] \left[(z_t - \bar{z})^{\otimes[j]} \otimes I_{n_z} \right]$$

To second order in σ and evaluated at the deterministic steady state gives

- $\mathscr{D}_{z_t} \{y_t\} \big|_{y_t = \bar{y}} \approx g_z + \frac{1}{2} g_{z\sigma^2}$

but at a risk-adjusted point $\bar{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{y}_{\sigma^i}(0) \sigma^i$

- $\mathscr{D}_{z_t} \{y_t\} \big|_{y_t = \bar{y}(\sigma)} \approx g_z + \frac{1}{2} \left[g_{z^2} \left(\begin{bmatrix} \bar{y}_{\sigma^2}(0) \\ 0 \end{bmatrix} \otimes I_{n_z} \right) + g_{z\sigma^2} \right]$

Identical to risk adjusted derivatives above!

RBC with Risk-Sensitivity and Long-Run Risk

A planner's maximization problem

$$V_t = \max_{C_t, L_t} \left[(1-\beta) \left(C_t^{\nu} (1-L_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[V_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}}$$

- Where $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$ with γ: Risk aversion, ψ: IES

subject to resource constraint

$$C_t + K_t = K_{t-1}^{\alpha} \left(e^{Z_t} L_t \right)^{1-\alpha} + (1-\delta) K_{t-1}$$

Productivity is a random walk with drift

$$a_t \equiv Z_t - Z_{t-1} = \overline{a} + \overline{\sigma} \varepsilon_t, \;\; \varepsilon_t \sim \mathcal{N}(0,1)$$

Asset Pricing and Calibration

- The stochastic discount factor $\equiv \frac{\partial V_t / \partial C_{t+1}}{\partial V_t / \partial C_t}$

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} e^{a_{t+1}} \right)^{\nu \frac{1-\gamma}{\theta} - 1} \left(\frac{1 - L_{t+1}}{1 - L_t} \right)^{\frac{(1-\nu)(1-\gamma)}{\theta}} \left(\frac{(v_{t+1} e^{\nu a_{t+1}})^{1-\gamma}}{E_t \left[(v_{t+1} e^{\nu a_{t+1}})^{1-\gamma} \right]} \right)^{1 - \frac{1}{\theta}}$$

- The (squared) conditional market price of risk

$$cmpr_t^2 = \frac{E_t \left[(m_{t+1} - E_t [m_{t+1}])^2 \right]}{E_t [m_{t+1}]^2}$$

- The expected (ex ante) risk premium $erp_t = E_t \left[r_{t+1} - r_t^f \right]$
- The realized (ex post) risk premium $rp_t = r_t - r_{t-1}^f$

Calibration

- Follow Caldara et al. (2012); Baseline: $\gamma = 5$, Extreme $\gamma = 40$
- $\overline{a}$, $\overline{\sigma}$, and ψ set to match 1948:3-2013:2 output and consumption growth mean and std. dev.

Calibration

Common Calibration

Parameter	$\overline{a}$	δ	ν	β	ξ
Value	0.46%	0.0196	0.357	0.991	0.3

- $\overline{a}$—1948:3-2013:2 average output growth
- Remaining values from Caldara et al. (2012)

Baseline Calibration

Parameter	γ	ψ	$\overline{\sigma}$
Value	5	1.008	1.12625%

- γ and η—Caldara et al.'s (2012) baseline values
- $\overline{\sigma}$ and ψ—1948:3-2013:2 average output and consumption growth volatilities

Extreme Calibration

Parameter	γ	ψ	$\overline{\sigma}$
Value	40	1.0085	1.1269%

- γ and η—Caldara et al.'s (2012) extreme values
- $\overline{\sigma}$ and ψ—1948:3-2013:2 average output and consumption growth volatilities

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Euler Equation Errors

- Judd and Guu (1997) and Judd (1998) advocate unit-free residuals of Euler equations as a measure of accuracy
- Applied by, e.g., Aruoba et al. (2006) and Caldara et al. (2012) to assess the accuracy of varying solution methods

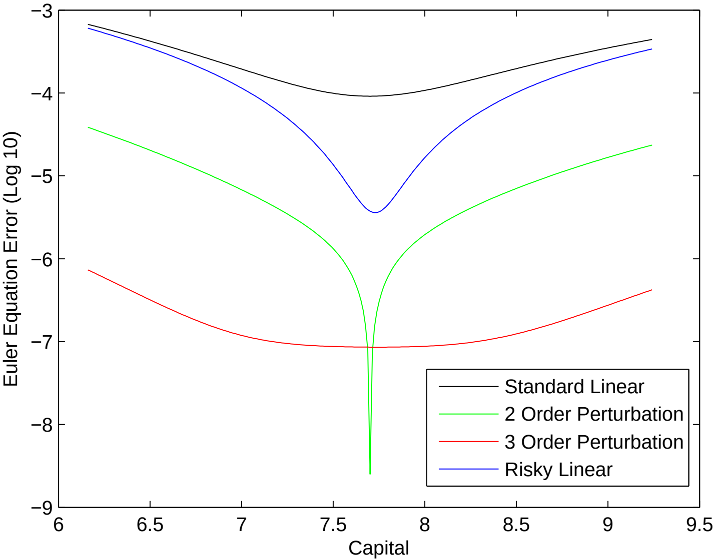
The Euler Equation Error expressed as a fraction of time t consumption

$$E_t\left[\beta\left(c_{t+1}e^{a_{t+1}}\right)^{\nu\frac{1-\gamma}{\theta}-1}\left(\frac{1-L_{t+1}}{1-L_t}\right)^{\frac{(1-\nu)(1-\gamma)}{\theta}}\left(\frac{\left(v_{t+1}e^{\nu a_{t+1}}\right)^{1-\gamma}}{E_t\left[\left(v_{t+1}e^{\nu a_{t+1}}\right)^{1-\gamma}\right]}\right)^{1-\frac{1}{\theta}}\left(\alpha k_t^{\alpha-1}\left(e^{a_{t+1}}L_{t+1}\right)^{1-\alpha}+1-\delta\right)\right]^{\frac{1}{\nu\frac{1-\gamma}{\theta}-1}}/c_t-1$$

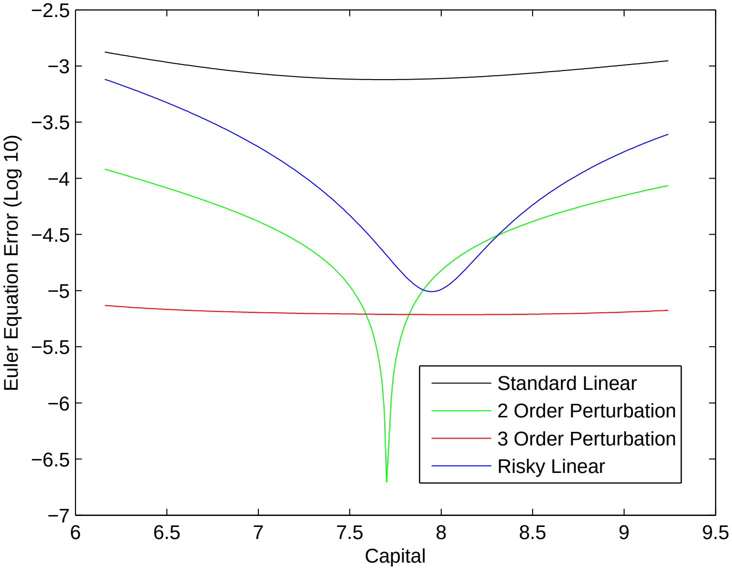
Here, the value of, say, 0.1 implies a 1 € mistake for each 10 € spent. This is a function of all the states and shocks

- I will set current shocks to zero
- exogenous states to their steady state values
- and examine how this error depends on the endogenous state, k_{t-1}

Euler Equation Errors, Baseline Calibration



Euler Equation Errors, Extreme Calibration



Impulse Response Functions

For nonlinear perturbations

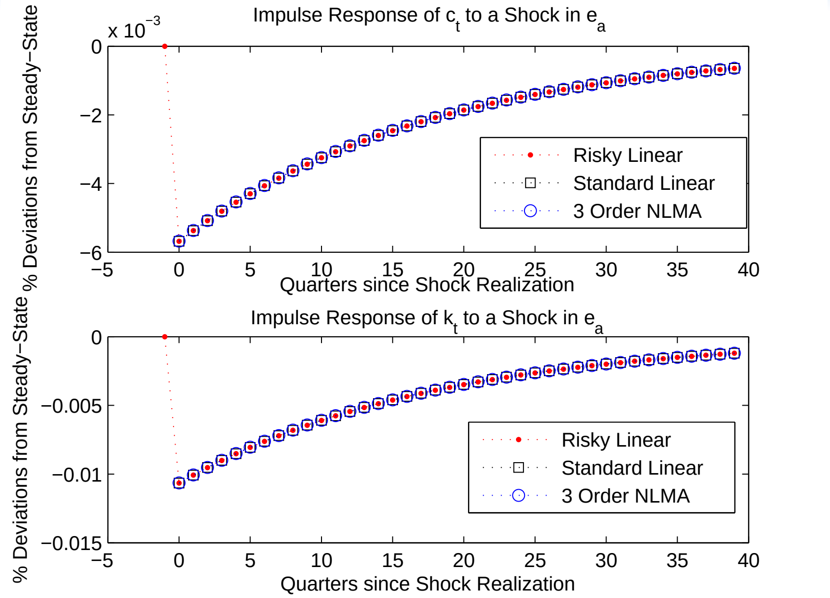
- IRFs become state-dependent
- Divergent views on how to best represent them
  - Fernández-Villaverde et al. (2011) simulate to put model in vicinity of ergodic mean and then calculate response to impulse
  - Lan and Meyer-Gohde (2013c) and Andreasen et al. (2013) use specific Generalized IRFs following Koop et al. (1996)
  - but differ in their assumptions (risky s.s. history vs. pruned state space history)

With the risky linear approximations

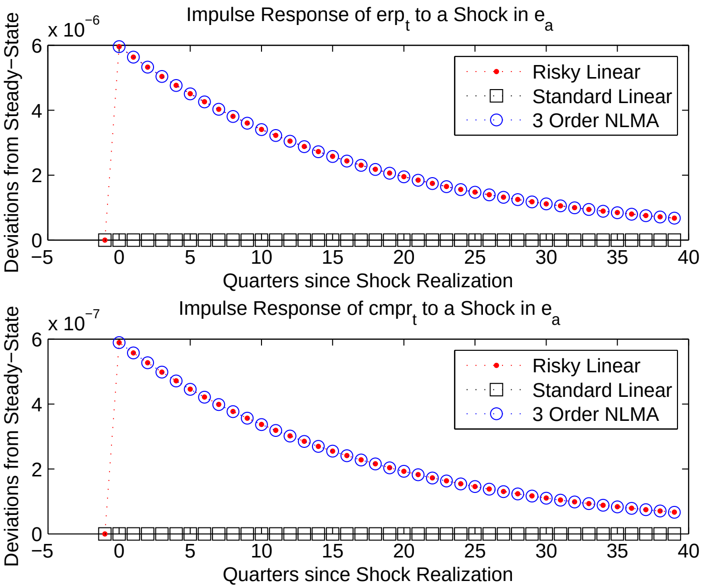
- impulse responses straight forward (i.e., standard) due to linearity
- identical (for approx. at stoch. s.s.) to Lan and Meyer-Gohde (2013c)

Capture precautionary responses to stoch. vol. and responses to risk sensitive variables (conditional asset prices)

Macro IRFs, Baseline Calibration



Conditional Asset Pricing IRFs, Baseline Calibration



Estimation

With linear policy functions and normally distributed errors

- endogenous variables are also normally distributed
- standard Kalman filter methods can be used to evaluate likelihood

Monte Carlo Study

- 10,000 periods of data from a third-order perturbation of model
- produce likelihood cuts using
  - standard linear approx
  - third order perturbation (particle filter with 40,000 particles)

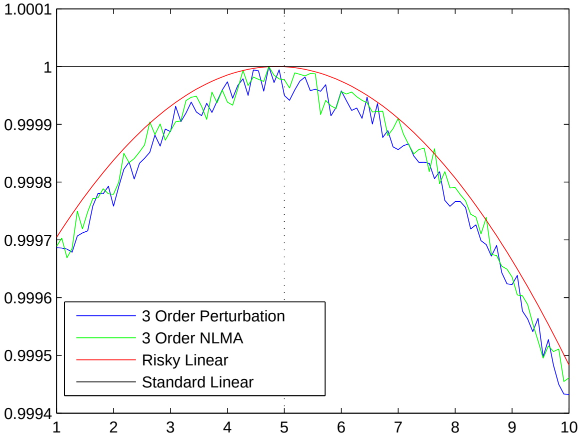
Evaluation Time (in seconds, per likelihood evaluation)

Linear	Risky Linear	3rd Order Pert.	3rd Order Pert. (pruned)
0.44	0.47	430	690

Likelihood Cut: γ, Baseline Calibration

Expressed relative to maximum along cut

All other parameters at pseudo-true vals

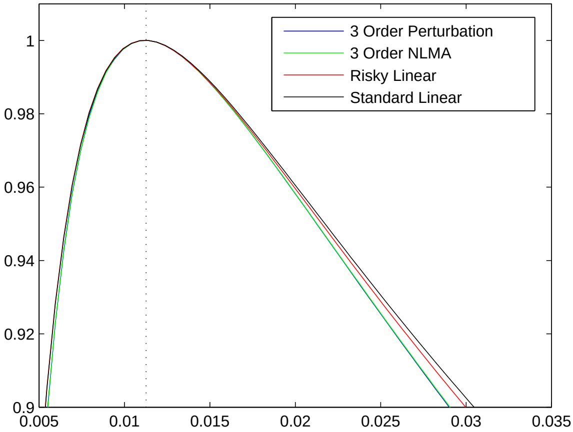


Dotted: Pseudo-true value

Likelihood Cut: $\overline{\sigma}$, Baseline Calibration

Expressed relative to maximum along cut

All other parameters at pseudo-true vals

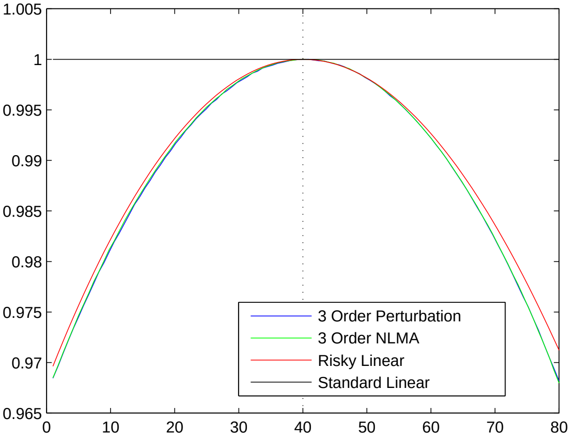


Dotted: Pseudo-true value

Likelihood Cut: γ, Extreme Calibration

Expressed relative to maximum along cut

All other parameters at pseudo-true vals



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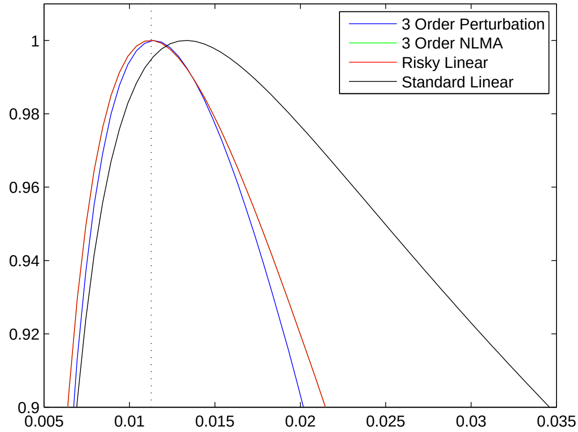

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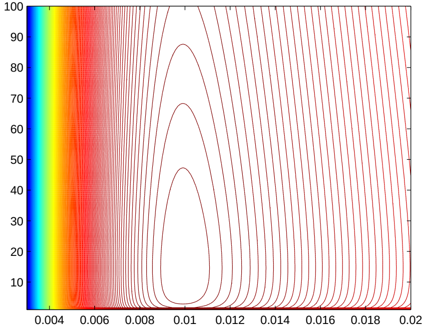
Data, Priors, and Posteriors

Data, 1948:2-2013:2

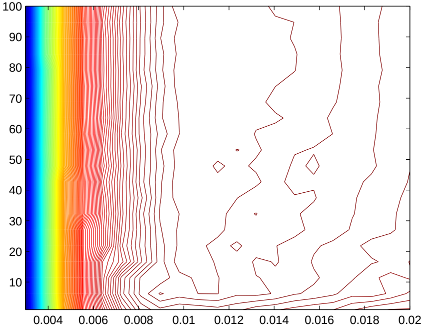
- output and consumption growth
- spread of NYSE value weighted portfolio over secondary market rate for the three month Treasury bill

	γ	$\overline{\sigma}$
Priors		
Type	Shifted Gamma	Inverse Gamma
Mean	20	0.22%
Mode	14.737	0.11%
Standard Deviation	10	0.6%
Domain	$(1, \infty)$	$(0, \infty)$
Posteriors		
Risky Linear Mode	29.296	1.0032%
Standard Linear Mode	14.737	0.9911%
3 Order Perturbation	N.A.	N.A.

Conventional Posteriors

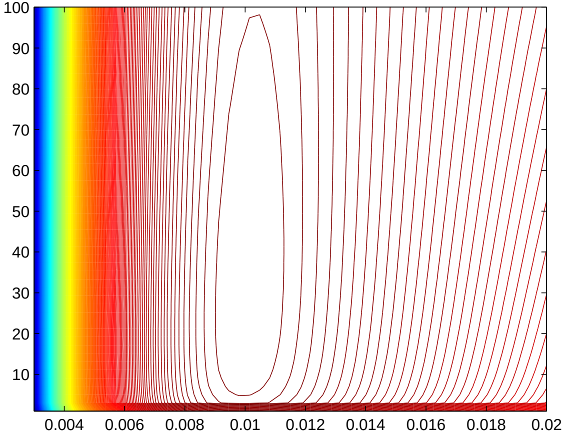


Conventional Linear



3 Order Perturbation

Risky Linear Posterior



Data and Posterior Asset Return Properties

Data, 1948:2-2013:2

- All returns are measured as real quarterly percentage returns
- NYSE value weighted portfolio
- secondary market rate for the three month Treasury bill

	Empirical		Risky Linear		Standard Linear	
Return	Mean	Std. Dev.	Mean	Std. Dev.		
r^k	2.14	8.25	0.5003	0.0801	0.5502	0.0758
r^f	0.26	0.62	0.4980	0.0767	0.5502	0.0726
$r^k - r^f$	1.88	8.25	0.0023	0.0212	0.000	0.0217
MPR	0.2283		0.2004		0.1049	
Sharpe Ratio			0.1072		0.0000	

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Conclusion

Risky Linear Approximations

- requires solving linear equations in standard perturbation coefficients
- uniformly improves the accuracy of det. linear approximations
- can estimate nonlinearities like risk aversion with the Kalman filter

Thank you for your attention!

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The Stochastic Steady State

Define a σ-dependant steady state implicitly

$$\bar{y}(\sigma) = g(\bar{y}(\sigma), 0, \sigma)$$

The point where, in the absence of shocks this period, agents decide to stay while expecting shocks in the future and knowing their probability distribution. Juillard (2011)

- Stochastic steady state: $\bar{y}^{\text{stoch}} \equiv \bar{y}(1)$
- Deterministic steady state: $\bar{y}^{\text{det}} = \bar{y}(0)$

For a second-order (in σ) approximation

- Taylor approx for σ steady states around $\sigma = 0$

$$\bar{y}(\sigma) = \sum_{i=0}^{\infty} \tfrac{1}{i!}\bar{y}^{(i)}(0)\sigma^i \approx \bar{y} + \bar{y}'(0)\sigma + \tfrac{1}{2}\bar{y}''(0)\sigma^2$$

- Differentiate $\bar{y}(\sigma) = g(\bar{y}(\sigma), 0, \sigma)$ at $\sigma = 0$
  - $\bar{y}'(0) = g_y \bar{y}'(0) + g_\sigma = \left(I - g_y\right)^{-1} g_\sigma = 0$
  - $\bar{y}''(0) = g_{y^2}\bar{y}'(0)^{\otimes[2]} + 2g_{y\sigma}\bar{y}_\sigma + g_y\bar{y}''(0) + g_{\sigma^2} = \left(I - g_y\right)^{-1} g_{\sigma^2}$

Thus, up to second order in σ, the stochastic steady state is

$$\bar{y}^{\text{stoch}} \approx \bar{y} + \frac{1}{2}\left(I - g_y\right)^{-1} g_{\sigma^2}$$

Identical to Lan and Meyer-Gohde (2013b)!

The Ergodic Mean

Define the σ-dependant mean

$$\bar{y}(\sigma) = E[y_t] = E[g(y_{t-1}, \sigma\varepsilon, \sigma)]$$

Deterministic steady state: $\bar{y}^{\text{det}} = \bar{y}(0)$

For a second-order (in σ) approximation

- Taylor approx for σ means around $\sigma = 0$

$$\bar{y}(\sigma) = \sum_{i=0}^{\infty} \tfrac{1}{i!}\bar{y}^{(i)}(0)\sigma^i \approx \bar{y} + \bar{y}'(0)\sigma + \tfrac{1}{2}\bar{y}''(0)\sigma^2$$

- Derivatives of $\bar{y}(\sigma) = E[g(y_{t-1}, \sigma\varepsilon_t, \sigma)]$ at $y_{t-1} = \bar{y}(0)$ and $\sigma = 0$
  - $\bar{y}'(0) = E\left[g_y \mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon \varepsilon_t + g_\sigma\right] = \left(I - g_y\right)^{-1} g_\sigma = 0$
  - $\bar{y}''(0) = \left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$

Thus, up to second order in σ, the ($\sigma = 1$) ergodic mean is

$$\bar{y}^{\text{mean}} \approx \bar{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

Identical to Lan and Meyer-Gohde (2013a) and Andreasen et al. (2013)!

The Ergodic Mean

For a second-order (in σ) approximation ($\bar{y}^{\text{mean}} \approx \bar{y} + \bar{y}_\sigma + \frac{1}{2}\bar{y}_{\sigma^2}$)

Differentiate $\bar{y}(\sigma) = E[g(y_{t-1}, \sigma\varepsilon_t, \sigma)]$ at $y_{t-1} = \bar{y}(0)$, $\varepsilon_t = 0$, $\sigma = 0$

- once for $\bar{y}_\sigma \equiv \mathscr{D}_\sigma\{E[y_t]\} = E\left[g_y \mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon \varepsilon_t + g_\sigma\right]$

$$= g_y \mathscr{D}_\sigma\{E[y_{t-1}]\} + g_\varepsilon E[\varepsilon_t] + g_\sigma = \left(I - g_y\right)^{-1} g_\sigma = 0$$

- twice for $\bar{y}_{\sigma^2} \equiv \mathscr{D}_{\sigma^2}\{E[y_t]\}$

$$= E\left[g_y \mathscr{D}_{\sigma^2}\{y_{t-1}\} + g_{y^2} \mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]} + 2g_{y\varepsilon}\varepsilon_t \otimes \mathscr{D}_\sigma\{y_{t-1}\}\right.$$
$$\left. + 2g_{y\sigma}\mathscr{D}_\sigma\{y_{t-1}\} + 2g_{\varepsilon\sigma}\varepsilon_t + g_{\varepsilon^2}\varepsilon_t^{\otimes[2]} + g_{\sigma^2}\right]$$
$$= \left(I - g_y\right)^{-1}\left(g_{y^2} E\left[\mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]}\right] + g_{\varepsilon^2} E\left[\varepsilon_t^{\otimes[2]}\right] + g_{\sigma^2}\right)$$
$$= \left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

as $E\left[\mathscr{D}_\sigma\{y_t\}^{\otimes[2]}\right] = E\left[\left(g_y \mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon\varepsilon_t + g_\sigma\right)^{\otimes[2]}\right] = g_y^{\otimes[2]} E\left[\mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]}\right] + g_\varepsilon^{\otimes[2]} E\left[\varepsilon_t^{\otimes[2]}\right]$

Thus $\bar{y}^{\text{stoch}} \approx \bar{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$

This is identical to the value reported in Lan and Meyer-Gohde (2013a) and Andreasen et al. (2013)!