Nonstationarity in Time Series of State Densities

Yoosoon Chang

Department of Economics

Indiana University

Joint with Changsik Kim and Joon Park

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Outline

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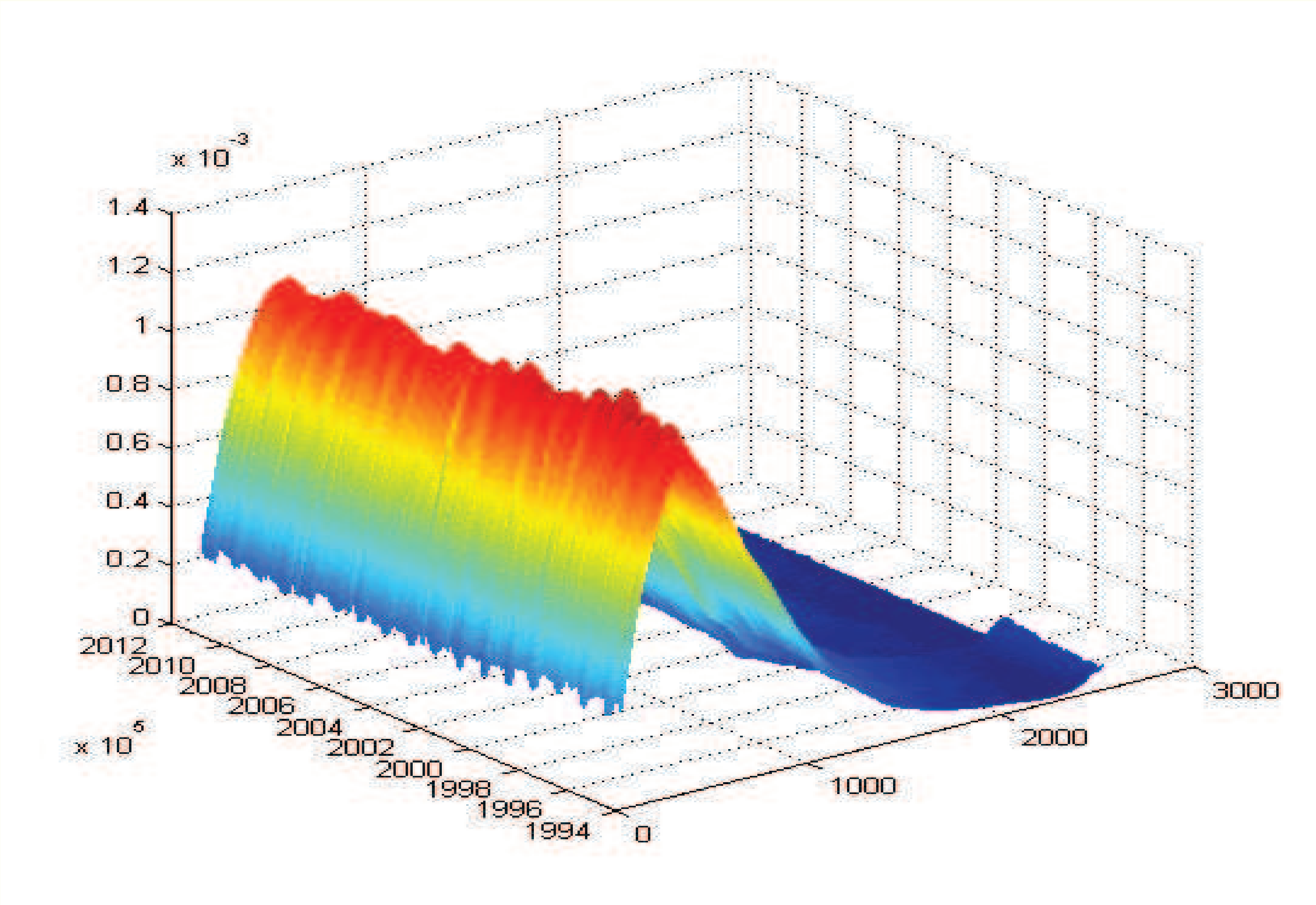
Overview

Objective

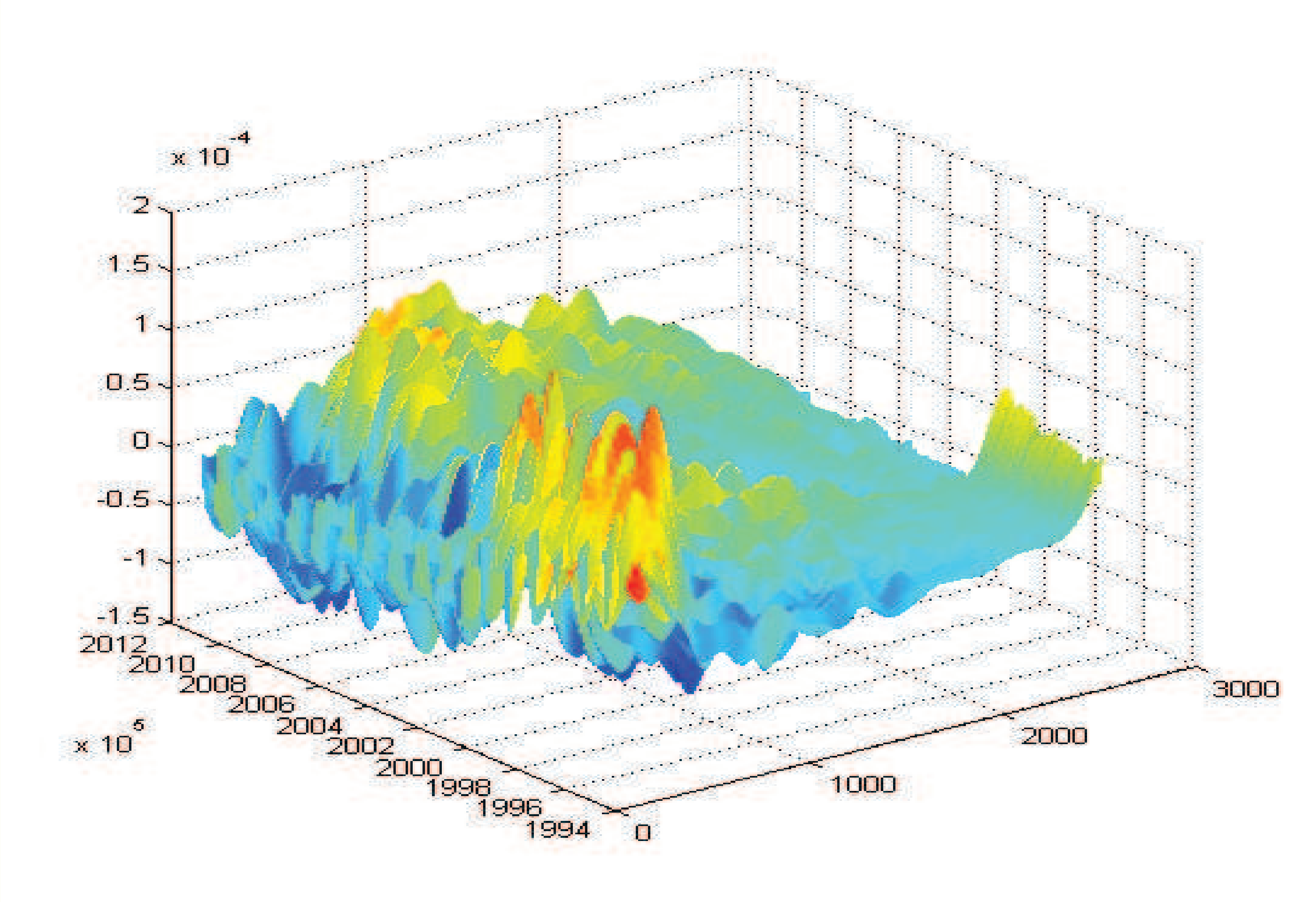
To analyze the nonstationarity in the time series of state densities, which may represent either cross-sectional or intra-period distributions of some economic state variables. Examples include

- Cross-sectional distributions of individual earnings
- Intra-month distributions of S&P 500 returns

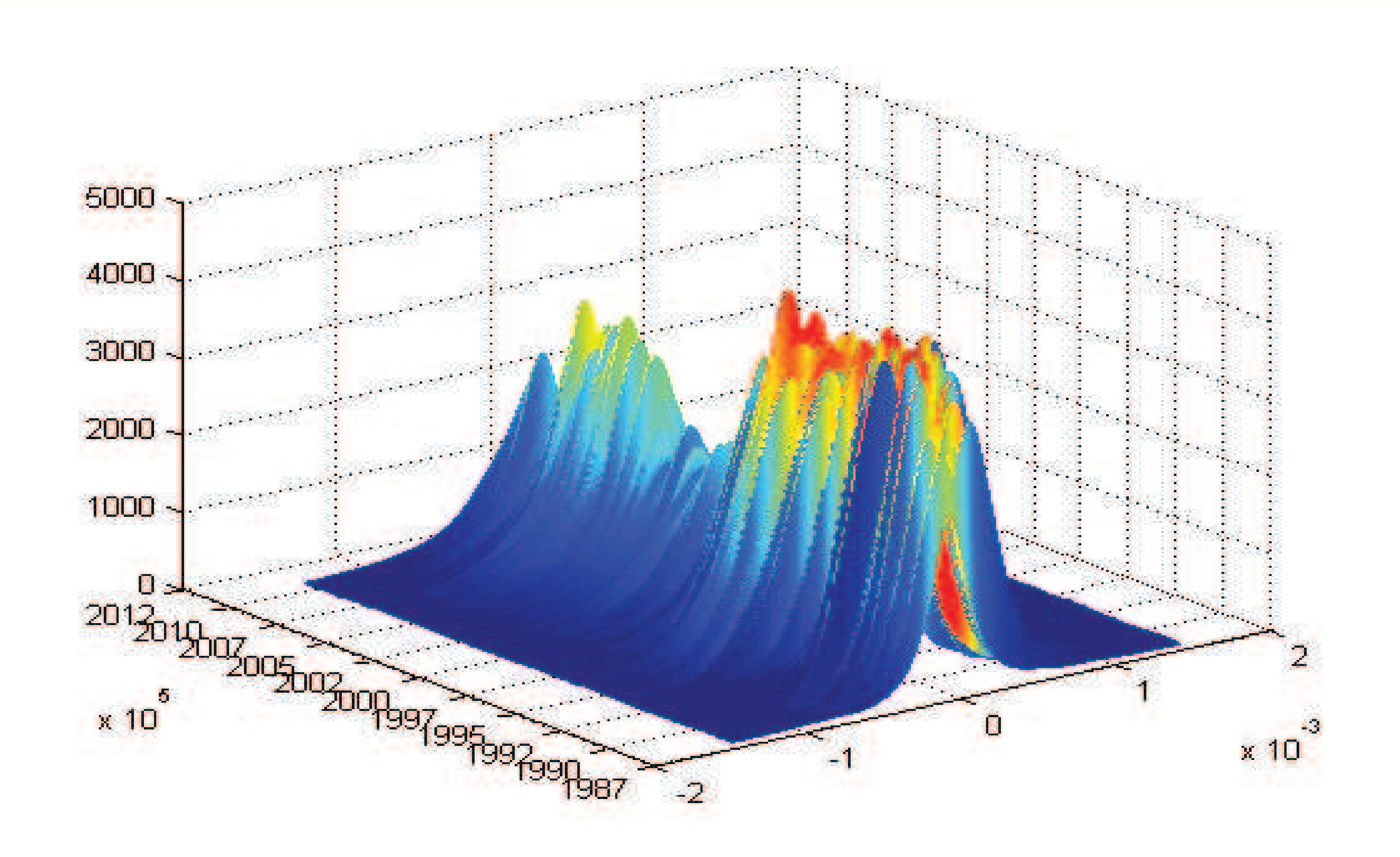
Densities of Weekly Individual Earnings



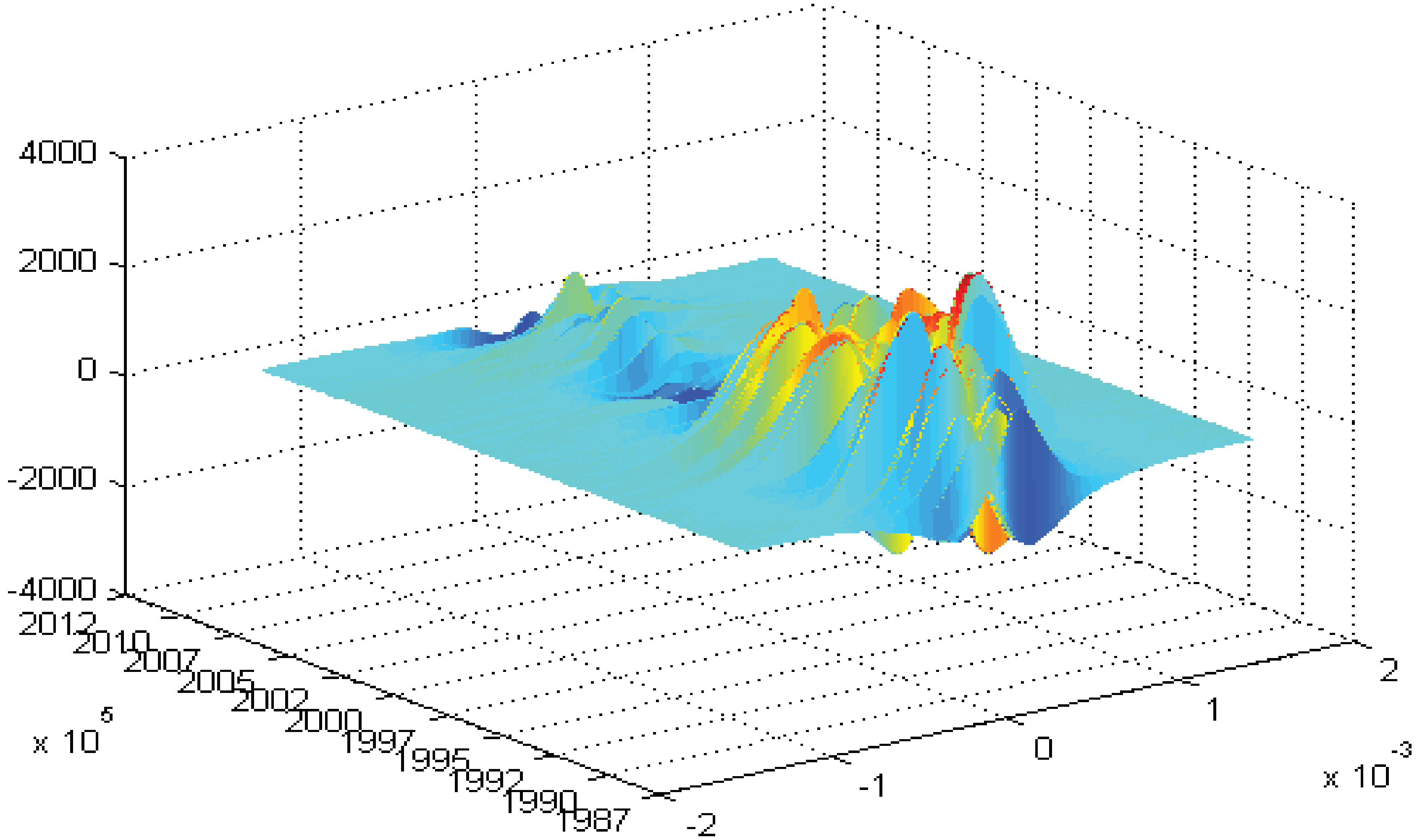
Demeaned Densities of Weekly Individual Earnings



Densities of Intra-Month S&P 500 Returns



Demeaned Densities of Intra-Month S&P 500 Returns



Motivations

Stylized Facts

- Fat tail and high peak
- Substantially skewed
- Time-varying and stochastic moments
- Serial dependence across moments

Direct Implications

- Normal distribution inadequate
- The first two moments not enough
- The dependence structure of moments may be complicated

Parametric Way Out

Use of more complicated parametric families

- Stable Paretian
- Skewed Student-t
- Normal Inverse Gaussian
- Many others

Limitations

- No "perfect" parametrization
- Not easy to handle any realistic family of distributions

Related Literature

Models of conditional distributions

- ARCH (Engle 1982), ARCH-M (Engle, Lilien and Robins 1987)
- Semi-parametric conditional distributions (Gallant and Tauchen 1989)
- ARCD (Auto-Regressive Conditional Density) (Hansen 1994)

Fully Nonparametric Approach

Advantages

- Data-adaptive shape with no parametric restrictions
- No limits to how many dimensions on which the distribution may change

Problems

- Difficult to estimate
- Hard to introduce dynamics into infinite-dimensional nonparametric distributions

Related Literature

Stationary functional time series models

- General statistical theory of stationary functional time series (Bosq 1998, 2000)
- Autoregressive modeling of time varying densities in function space (Park and Qian, 2007)
- Functional regression of continuous state distributions (Park and Qian, 2009)

In This Paper

- A new framework is proposed to analyze the nonstationarity in the time series of state densities, which may represent either cross-sectional or intra-period distributions.
- Each state density is regarded as a realization of Hilbertian random variable, and a functional time series model is used to fit a given time series of state variables.
- This allows us to explore various sources of nonstationarity in state distribution, including higher moments such as variance, skewness and kurtosis. In contrast, the conventional unit root tests are applied to cross-sectional aggregates or intra-period averages, thereby examining only the nonstationarity in the mean of state distribution.
- Potential unit roots are identified through functional principal component analysis, and subsequently tested by generalized eigenvalues of leading components of the normalized estimated variance operator.

Technical Background

- Model and Preliminaries
- Statistical Procedure and Asymptotic Theory
- Models with Estimated Densities

Hilbert-Valued Random Variables

Let

$$g: \ \Omega \to H$$

where H is a Hilbert space.

Hilbert-valued random variables include

- Real random variables: $H = \mathbb{R}$
- Vector-valued random variables: $H = \mathbb{R}^N$
- Function-valued random variables: $H = L^2(\mathbb{R})$

Expectation and Variance Operators

Expectation $\mathbb{E}g$ of $g \in H$ is defined as a vector in H satisfying

$$\langle v, \mathbb{E}g \rangle = \mathbb{E}\langle v, g \rangle$$

for all $v \in H$. $\mathbb{E}g$ exists iff $\mathbb{E}\|g\| < \infty$.

Variance Σ of g is given by an operator for which

$$\mathbb{E}\langle v_i, g - \mathbb{E}g \rangle \langle v_j, g - \mathbb{E}g \rangle = \langle v_i, \Sigma v_j \rangle$$

for all v_i, $v_j \in H$.

- For $w \in H$ with $\mathbb{E}w = 0$, $\mathbb{E}\langle v_i, w \rangle \langle v_j, w \rangle = \langle v_i, \mathbb{E}(w \otimes w) v_j \rangle$, and the variance operator Σ is given simply by $\mathbb{E}(w \otimes w)$.
- For a finite dimensional w, $\mathbb{E}(w \otimes w)$ reduces to $\mathbb{E}ww'$.

Model and Preliminaries

Model for Functional Data

For each time $t = 1, 2, \ldots$, suppose there is a distribution represented by a probability density f_t, whose value at ordinate $s \in \mathbb{R}$ is denoted by $f_t(s)$. Denote by

$$w_t = f_t - \mathbb{E}f_t$$

a centered density function and treat w_t as functional data taking values in Hilbert space H, where we define H to be the set of functions on a compact subset K of $\mathbb{R}$ that have vanishing integrals and are square integrable, i.e.,

$$H = \left\{ w \,\middle|\, \int_K w(s)ds = 0, \int_K w^2(s)ds < \infty \right\}$$

with inner product $\langle v, w \rangle = \int v(s)w(s)ds$ for $v, w \in H$.

Coordinate Process

We assume that there exists an orthonormal basis (v_i) of H such that the i-*th coordinate process*

$$\langle v_i, w_t \rangle$$

has a unit root for $i = 1, \ldots, M$, while it is stationary for all $i \geq M + 1$.

By convention, we set $M = 0$ if all the coordinate processes are stationary.

Unit Root and Stationarity Subspaces

Using the symbol $\bigvee$ to denote span, we let

$$H_N = \bigvee_{i=1}^{M} v_i \quad \text{and} \quad H_S = \bigvee_{i=M+1}^{\infty} v_i$$

so that $H = H_N \oplus H_S$. In what follows, H_N and H_S will respectively be referred to as the unit root and stationarity subspaces of H. We also let Π_N and Π_S be the projections on H_N and H_S, respectively. Moreover, we define

$$w_t^N = \Pi_N w_t \quad \text{and} \quad w_t^S = \Pi_S w_t$$

Note that $\Pi_N + \Pi_S = 1$ (the identity operator on H), so in particular we have

$$w_t = w_t^N + w_t^S$$

Generalized Linear Process

Assumption 2.1 For $u_t = \Delta w_t$, we let

$$u_t = \Phi(L)\varepsilon_t = \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t-i}$$

where we assume that (a) $\sum_{i=1}^{\infty} i\|\Phi_i\| < \infty$, (b) $\Pi_N \Phi(1)$ is of rank M and $\Pi_S \Phi(1) = 0$, and (c) (ε_t) is an iid sequence with mean zero and variance $\Sigma > 0$, for which we have $\mathbb{E}\|\varepsilon_t\|^p < \infty$ with some $p \geq 4$.

- The process (u_t) is a generalization of the finite dimensional linear process, and studied extensively in Bosq (2000) under stationarity assumption.
- The coefficients (Φ_i) used to define the process are linear operators in H and the innovation (ε_t) is a sequence of random elements in H.

Decomposition of Generalized Linear Process

We may decompose the process (u_t) introduced in Assumption 2.1 as

$$u_t = \Phi(1)\varepsilon_t + (\bar{u}_{t-1} - \bar{u}_t)$$

where

$$\bar{u}_t = \bar{\Phi}(L)\varepsilon_t = \sum_{i=0}^{\infty} \bar{\Phi}_i \varepsilon_{t-i}$$

with $\bar{\Phi}_i = \sum_{j=i+1}^{\infty} \Phi_j$. This representation is widely known as the Beveridge-Nelson decomposition (Phillips and Solo, 1992) for finite-dimensional linear processes. Due to Assumption 2.1 (a), we have $\sum_{i=0}^{\infty} \|\bar{\Phi}_i\| < \infty$, and therefore the process $(\bar{u}_t)$ is well defined.

Unit Root and Stationary Processes

It follows from Assumption 2.1 (b) that

$$w_t^N = \Pi_N w_t = \Pi_N \Phi(1) \sum_{i=1}^{t} \varepsilon_i - \Pi_N \bar{u}_t$$

and

$$w_t^S = \Pi_S w_t = -\Pi_S \bar{u}_t$$

ignoring the initial values w_0 and $\bar{u}_0$ that are unimportant. Clearly, (w_t^N) is an integrated process, while (w_t^S) is stationary.

The unit root dimension M is unknown in practical applications. We will explain how to statistically determine M, as well as how to estimate the subspaces H_S and H_N.

Brownian Motion on Unit Root Subspace

B is Brownian motion on the unit root subspace H_N with variance operator Ω, if B takes values on H_N and if for any $v \in H_N$, $< v, B >$ is BM with variance $< v, \Omega v >$. Naturally for the random sequence (B_T) taking values on H_N, we let

$$B_T \to_d B$$

if

$$< v, B_T > \to_d < v, B >$$

for any $v \in H_N$.

Vector Brownian Motion on Unit Root Subspace

It can be shown that if B is BM on H_N, then for any $(v_i), i = 1, \ldots, M$, in H_N,

$$(\langle v_1, B \rangle, \ldots, \langle v_M, B \rangle)'$$

becomes an M-dimensional vector Brownian motion with covariance matrix having (i, j)-th entry given by $\langle v_i, \Omega v_j \rangle$ for $i, j = 1, \ldots, M$.

Furthermore, if $B_T \to_d B$, then

$$(\langle v_1, B_T \rangle, \ldots, \langle v_M, B_T \rangle)' \to_d (\langle v_1, B \rangle, \ldots, \langle v_M, B \rangle)'$$

for any $(v_i), i = 1, \ldots, M$, in H_N.

Statistical Procedure and Asymptotic Theory

Functional Principal Component Analysis (FPCA)

Our procedure to estimate H_N and test for its dimension M is based on the FPCA on the unnormalized sample variance operator of (w_t)

$$Q_T = \sum_{t=1}^{T} w_t \otimes w_t$$

where T is the sample size.

Denote the pairs of eigenvalues and eigenvectors of Q_T by

$$(\lambda_i(Q_T), v_i(Q_T)), \quad i = 1, \ldots, T$$

and order $(\lambda_i(Q_T))$ so that $\lambda_1(Q_T) \geq \cdots \geq \lambda_T(Q_T)$.

Sample Unit Root and Stationarity Subspaces

Assuming $T > M$, we define sample unit root space as the subspace

$$H_N^T = \bigvee_{i=1}^{M} v_i(Q_T)$$

spanned by the eigenvectors corresponding to M largest eigenvalues of Q_T. Denote by Π_N^T the projection on H_N^T.

The sample stationarity subspace is defined by $\Pi_S^T = 1 - \Pi_N^T$, so that we have $\Pi_N^T + \Pi_S^T = 1$ analogously as the relationship $\Pi_N + \Pi_S = 1$.

Proposition 3.2 Under Assumption 2.1, we have

$$\Pi_N^T = \Pi_N + O_p(T^{-1}) \quad \text{and} \quad \Pi_S^T = \Pi_S + O_p(T^{-1})$$

for all large T.

Decomposition of Sample Variance Operator

To develop our asymptotics, we decompose Q_T as

$$Q_T = T^2 Q_{NN}^T + T Q_{NS}^T + T Q_{SN}^T + T Q_{SS}^T$$

where

$$Q_{NN}^T = \frac{1}{T^2} \Pi_N \left(\sum_{t=1}^T w_t \otimes w_t \right) \Pi_N = \frac{1}{T^2} \sum_{t=1}^T w_t^N \otimes w_t^N$$

$$Q_{NS}^T = \frac{1}{T} \Pi_N \left(\sum_{t=1}^T w_t \otimes w_t \right) \Pi_S = \frac{1}{T} \sum_{t=1}^T w_t^N \otimes w_t^S$$

$$Q_{SS}^T = \frac{1}{T} \Pi_S \left(\sum_{t=1}^T w_t \otimes w_t \right) \Pi_S = \frac{1}{T} \sum_{t=1}^T w_t^S \otimes w_t^S$$

and Q_{SN}^T is the adjoint of Q_{NS}^T, i.e., $Q_{SN}^T = Q_{NS}^{T\prime}$.

Asymptotics for Sample Variance Operators 1

Lemma 3.1 Let Assumption 2.1 hold. We have

$$Q_{NN}^T \to_d Q_{NN} = \int_0^1 (W \otimes W)(r)dr$$

where W is Brownian motion on H_N with variance operator $\Pi_N \Phi(1)\Sigma\Phi(1)'\Pi_N$. Also, it follows that

$$Q_{SS}^T \to_p Q_{SS} = \Pi_S \left(\sum_{i=0}^{\infty} \bar{\Phi}_i \Sigma \bar{\Phi}_i' \right) \Pi_S$$

Moreover, we have

$$Q_{NS}^T, Q_{SN}^T = O_p(1)$$

for all large T.

Asymptotics for Sample Variance Operators 2

- Lemma 3.1 establishes the limits and stochastic orders for each component appearing in the decomposition of the unnormalized sample variance operator Q_T of the centered state density (w_t).
- The normalized sample operators Q_{NN}^T and Q_{SS}^T have well defined limits. Q_{NN}^T converges in distribution to a random operator represented by a functional of BM on H_N, while Q_{SS}^T converges in probability to its population counterpart $Q_{SS} = \Pi_S\Big(\mathbb{E}(w_t \otimes w_t)\Big)\Pi_S$ on H_S.
- The normalized sample covariance operators Q_{NS}^T and Q_{SN}^T become negligible asymptotically.

Limiting Eigenvalues and Eigenvectors 1

Denote the nonzero eigenvalues and their associated eigenvectors of the distributional limit Q_{NN} of the normalized sample variance Q_{NN}^T in the unit root space by

$$(\lambda_i(Q_{NN}), v_i(Q_{NN})), \quad i = 1, \ldots, M$$

which we order

$$\lambda_1(Q_{NN}) \geq \cdots \geq \lambda_M(Q_{NN})$$

Note that Q_{NN} is stochastic, and therefore, so are $(\lambda_i(Q_{NN}), v_i(Q_{NN}))$ for $i = 1, \ldots, M$.

Clearly, $(v_i(Q_{NN}))$, $i = 1, \ldots, M$, span H_N. Though the set of vectors $(v_i(Q_{NN}))$ are given randomly by the realization of Brownian motion W, the space spanned by them is nonrandom and uniquely determined.

Limiting Eigenvalues and Eigenvectors 2

Let

$$(\lambda_i(Q_{SS}), v_i(Q_{SS}))\,, \quad i = 1, 2, \ldots$$

be the nonzero eigenvalues and their associated eigenvectors of Q_{SS}, for which we assume

$$\lambda_1(Q_{SS}) \geq \lambda_2(Q_{SS}) \geq \cdots$$

Since Q_{SS} is the variance operator of the stationary process $(\Pi_S w_t)$, it is positive semi-definite and nuclear, i.e., $\lambda_i(Q_{SS}) \geq 0$ and

$$\sum_{i=1}^{\infty} \lambda_i(Q_{SS}) < \infty$$

In particular, $\lambda_i(Q_{SS}) \to 0$ as $i \to \infty$, and the origin is the limit point of the spectrum of Q_{SS}. See Bosq (2000, Theorem 1.7).

Theorem 3.3 Under Assumption 2.1, we have

$$\left(T^{-2}\lambda_i(Q_T), v_i(Q_T)\right) \to_d \left(\lambda_i(Q_{NN}), v_i(Q_{NN})\right)$$

jointly for $i = 1, \ldots, M$, and

$$\left(T^{-1}\lambda_{M+i}(Q_T), v_{M+i}(Q_T)\right) \to_p \left(\lambda_i(Q_{SS}), v_i(Q_{SS})\right)$$

for $i = 1, 2, \ldots$.

- In the stationarity subspace H_S, the eigenvectors and appropriately normalized eigenvalues of the sample variance operator Q_T of (w_t) converge in probability to their population counterparts.
- In the unit root subspace H_N, on the other hand, they converge in distribution, and their distributional limits are given by the distributions of eigenvalues and eigenvectors of the random operator Q_{NN} on H_N.

Onto Testing for Unit Root Dimension

We may use the criterion

$$\Lambda_T = -T^{-2}\sum_{i=1}^{M}\lambda_i(Q_T) + c_T T^{-1}$$

to determine the dimension M of unit roots in (w_t), where (c_T) is a numerical sequence such that $c_T \to \infty$ and $c_T T^{-1} \to 0$. In fact, if we set

$$\hat{M}_T = \text{argmin}_{0\le M\le M_{\max}}\Lambda_T \qquad (1)$$

with some fixed number $M_{\max}$ large enough to ensure $M \le M_{\max}$, then we may easily show following, e.g., Cheng and Phillips (2009), that $\hat{M}_T$ is weakly consistent for M and $\mathbb{P}\{\hat{M}_T = M\} \to 1$.

In the paper, however, we follow a more conventional approach based on a successive testing procedure similar to the testing procedure by Johansen (1988). Note that any of the existing procedures developed to analyze cointegrating rank are not directly applicable for our model, since it is infinite dimensional.

Testing for Unit Root Dimension

To determine the dimension M of the unit root subspace H_N, the null hypothesis

$$\mathrm{H}_0 : \ \dim(H_N) = M$$

is tested against the alternative hypothesis

$$\mathrm{H}_1 : \ \dim(H_N) \leq M - 1$$

successively downward starting from $M = M_{\max}$, where $M_{\max}$ is some fixed number large enough to ensure $M \leq M_{\max}$. By convention, $M = 0$ implies that there is no unit root, and H_N consists only of the origin.

Our estimate for M is given by $M_{\min} - 1$, where $M_{\min}$ is the smallest value of M for which H_0 is rejected in favor of H_1. Obviously, we may find the true value of M with asymptotic probability one, if we apply any consistent test in the successive manner as suggested here.

Intuitive But Infeasible Test Statistic

$T^{-2}\lambda_i(Q_T)$ has a non-degenerate asymptotic distribution for $i = 1, \ldots, M$, whereas it converges to zero in probability for all $i \geq M + 1$. Therefore, we may use $\lambda_i(Q_T)$ to determine the unit root dimension M in (w_t).

In particular, we may consider

$$\sigma_M^T = T^{-2}\lambda_M(Q_T)$$

to test the null against the alternative for $M = 1, 2, \ldots$.

The test would be consistent, if we reject the null in favor of the alternative, when the test σ_M^T takes a small value.

Unfortunately, limit distribution of σ_M^T is generally dependent upon various nuisance parameters, and cannot be used directly to test for M.

Construction of Nuisance Parameter Free Test

Let (v_i), $i = 1, \ldots, M$, be an *arbitrary* set of vectors generating H_N and

$$z_t = (\langle v_1, w_t \rangle, \ldots, \langle v_M, w_t \rangle)'$$

for $t = 1, \ldots, T$. The choice of (v_i) is unimportant. Define the sample variance in the unit root subspace by

$$Q_M^T = Z_T' Z_T, \quad \text{with} \quad Z_T = (z_1, \ldots, z_T)'$$

Then we have

$$T^{-2} Q_M^T \ \to_d \ Q_M = \int_0^1 W_M(r) W_M(r)' dr$$

where W_M is M-dimensional vector BM with variance Ω_M.

Asymptotics for Generalized Eigenvalues and Vectors 1

Now we define

$$\left(\lambda_i(Q_M^T, \Omega_M^T), v_i(Q_M^T, \Omega_M^T)\right), \quad i = 1, \ldots, M$$

to be the pairs of generalized eigenvalues and eigenvectors of Q_M^T with respect to consistent estimate Ω_M^T of Ω_M.

It follows immediately that

$$\left(T^{-2}\lambda_i(Q_M^T, \Omega_M^T), v_i(Q_M^T, \Omega_M^T)\right) \to_d (\lambda_i(Q_M, \Omega_M), v_i(Q_M, \Omega_M))$$

jointly for $i = 1, \ldots, M$, where

$$(\lambda_i(Q_M, \Omega_M), v_i(Q_M, \Omega_M)), \quad i = 1, \ldots, M$$

are the pairs of generalized eigenvalues and eigenvectors of Q_M with respect to Ω_M.

Asymptotics for Generalized Eigenvalues and Vectors 2

Let $W_M^* = \Omega_M^{-1/2} W_M$ denote the M-dimensional standard vector BM, and denote by

$$(\lambda_i(Q_M^*), v_i(Q_M^*)), \quad i = 1, \ldots, M$$

the eigenvalue and eigenvector pairs of the integrated product moment

$$Q_M^* = \int_0^1 W_M^*(r) W_M^*(r)' dr$$

Then we have for $i = 1, \ldots, M$

$$\lambda_i(Q_M, \Omega_M) = \lambda_i(Q_M^*), \quad \Omega_M^{1/2} v_i(Q_M, \Omega_M) = v_i(Q_M^*)$$

and in particular, we have

$$T^{-2} \lambda_i(Q_M^T, \Omega_M^T) \;\to_d\; \lambda_i(Q_M^*)$$

jointly for $i = 1, \ldots, M$. The distributions of $(\lambda_i(Q_M^*))$ are free of nuisance parameters, and thus can be tabulated by simulations.

Onto A Feasible Test

The generalized eigenvalues $(\lambda_i(Q_M^T, \Omega_M^T))$, $i = 1, \ldots, M$, are based on (z_t), which are not observable since it consists of coordinate processes given by an arbitrary set of vectors (v_i) spanning H_N.

Therefore, we consider instead

$$\tilde{z}_t = (\langle v_1 (Q_T), w_t \rangle, \ldots, \langle v_M (Q_T), w_t \rangle)'$$

for $t = 1, \ldots, T$. Moreover, we define $\tilde{Q}_M^T = \tilde{Z}_T' \tilde{Z}_T$ with $\tilde{Z}_T = (\tilde{z}_1, \ldots, \tilde{z}_T)'$, and

$$\tilde{\Omega}_M^T = \sum_{|i| \leq \ell} \varpi_\ell(i) \tilde{\Gamma}_T(i)$$

accordingly as Ω_M^T.

Nuisance Parameter Free Test

To test $\mathrm{H}_0 : \dim(H_N) = M$ against $\mathrm{H}_1 : \dim(H_N) \leq M - 1$, we propose

$$\tau_M^T = T^{-2} \lambda_M \left(\tilde{Q}_M^T, \tilde{\Omega}_M^T \right)$$

Theorem 3.5 Let Assumption 2.1 hold. Under the null H_0, we have

$$\tau_M^T \to_d \lambda_M (Q_M^*)$$

Moreover, we have under the alternative H_1

$$\tau_M^T \to_p 0$$

- The limit null distribution of τ_M^T is given by the distribution of the smallest eigenvalue of the integrated product moment Q_M^*, which is free of nuisance parameters, and depends only upon M. Therefore, the asymptotic critical values of τ_M^T can be tabulated for each M.
- The test based on τ_M^T is consistent.

Unit Root Moment Decomposition

By convention, we identify any square integrable function μ on a compact subset K of $\mathbb{R}$ with $\mu - \int_K \mu(s)ds$, so that we may regard it as an element in H. Note that

$$\langle \mu, v \rangle = \left\langle \mu - \int_K \mu(s)ds, v \right\rangle$$

for any $v \in H$, if we define the inner product $\langle \cdot, \cdot \rangle$ in an extended Hilbert space including all square integrable functions on K. For any μ in the extended Hilbert space, we define the norm $\|\cdot\|$ as

$$\|\mu\|^2 = \left\langle \mu - \int_K \mu(s)ds, \mu - \int_K \mu(s)ds \right\rangle = \sum_{i=1}^{\infty} \langle \mu, v_i \rangle^2$$

for an orthonormal basis (v_i) of H.

Decomposition of Moments

Consider

$$\mu_i(s) = s^i$$

for $i = 1, 2, \ldots$. Note that

$$\langle \mu_i, w_t \rangle = \langle \mu_i, f_t \rangle - \langle \mu_i, \mathbb{E} f_t \rangle = \langle \mu_i, f_t \rangle - \mathbb{E} \langle \mu_i, f_t \rangle$$

represents the random fluctuation of the i-th moment of the distribution given by probability density (f_t). Decompose $\mu_i = \Pi_N \mu_i + \Pi_S \mu_i$, and this gives

$$\|\mu_i\|^2 = \|\Pi_N \mu_i\|^2 + \|\Pi_S \mu_i\|^2 = \sum_{j=1}^{M} \langle \mu_i, v_j \rangle^2 + \sum_{j=M+1}^{\infty} \langle \mu_i, v_j \rangle^2$$

where (v_j), $j = 1, 2, \ldots$, is an orthonormal basis of H such that $(v_j)_{1 \leq j \leq M}$ and $(v_j)_{j \geq M+1}$ span H_N and H_S, respectively.

Unit Root Proportion of i-th Moment

Unit roots in state densities implies that their moments are generally persistent and nonstationary. We propose a measure to represent the proportion of unit root component in each moment.

The proportion of the i-th moment μ_i lying in H_N is measured by

$$\pi_i = \frac{\|\Pi_N \mu_i\|}{\|\mu_i\|} = \sqrt{\frac{\sum_{j=1}^{M} \langle \mu_i, v_j \rangle^2}{\sum_{j=1}^{\infty} \langle \mu_i, v_j \rangle^2}}$$

π_i represents the proportion of unit root component in the i-th moment μ_i of densities (f_t). The i-th moment of (f_t) has more dominant unit root component as $\pi_i \to 1$, while it becomes more stationary as $\pi_i \to 0$. If μ_i is entirely in H_N, $\pi_i = 1$. On the other hand, $\pi_i = 0$ if μ_i is entirely in H_S.

Sample Unit Root Proportion of i-th Moment

The unit root proportion π_i of the i-th moment is not directly applicable, since H_N and H_S are unknown. However, we may use its sample version

$$\pi_i^T = \sqrt{\frac{\sum_{j=1}^{M} \langle \mu_i, v_j(Q_T) \rangle^2}{\sum_{j=1}^{T} \langle \mu_i, v_j(Q_T) \rangle^2}}$$

We may readily show that

$$\pi_i^T = \pi_i + O_p(T^{-1})$$

for all $i = 1, 2, \ldots$. Hence, the sample unit root proportion π_i^T is a consistent estimator for the original unit root proportion π_i.

Models with Estimated Densities

Overview

- State densities (f_t) are not directly observed and thus need to be estimated using the data, either cross-sectional or high frequency observations. Denote by $(\hat{f}_t)$ the estimated density functions and let

$$\hat{w}_t = \hat{f}_t - \frac{1}{T}\sum_{t=1}^{T}\hat{f}_t$$

be the demeaned density estimate for $t = 1, \ldots, T$.

- It is well expected that the replacement of (w_t) with $(\hat{w}_t)$ does not affect asymptotics as long as the number of cross-sectional or high frequency observations available in each time period to estimate $(\hat{f}_t)$ is large enough relative to the number T of time series observations.
- We show that our earlier asymptotic theories continue to hold even when we use $(\hat{w}_t)$ in the place of (w_t).

Assumptions

Assumption 4.1 Let $\Delta_t = \hat{f}_t - f_t$ for $t = 1, \ldots, T$ and assume (a) $\sup_{t \geq 1} \|\Delta_t\| = O_p(1)$, and (b) $T^{-1} \sum_{t=1}^{T} \|\Delta_t\| \to_p 0$.

- Condition (a) would hold in general, since both (f_t) and $(\hat{f}_t)$ are proper density functions.
- Condition (b) is expected to hold whenever the number N of observations used to estimate f_t at each t is sufficiently large relative to the sample size T of (f_t).
- Under general regularity conditions, $\mathbb{E}\|\Delta_t\| = O(N^{-2/5})$ for each t, if state distributions are defined as cross-sectional distributions and iid observations are available to estimate them for each period. The same holds if the state distributions are given by intra-period distributions, as long as within each period the underlying economic variables can be regarded as stationary processes satisfying some general mixing conditions, as shown in Bosq (1998) and Hansen (2008).

FPCA for Estimated Variance Operator

To construct a feasible version of the test τ_M^T for unit roots in the state density, define all sample statistics using $(\hat{w}_t)$ in place of (w_t). Our testing procedure is then based on FPCA for the estimated variance operator

$$\hat{Q}_T = \sum_{t=1}^{T} \hat{w}_t \otimes \hat{w}_t$$

As before, decompose $\hat{Q}_T$ to develop asymptotics as

$$\hat{Q}_T = T^2 \hat{Q}_{NN}^T + T\hat{Q}_{NS}^T + T\hat{Q}_{SN}^T + T\hat{Q}_{SS}^T$$

where $\hat{Q}_{NN}^T$, $\hat{Q}_{NS}^T, \hat{Q}_{SN}^T$ and $\hat{Q}_{SS}^T$ are defined in the same way as their counterpart components of Q_T, except that they are all based on the demeaned state density functions $(\hat{w}_t)$.

Estimated Unit Root Subspace

Denote the pairs of the eigenvalues and eigenvectors of $\hat{Q}_T$ by

$$\Big(\lambda_i(\hat{Q}_T), v_i(\hat{Q}_T)\Big), \quad i = 1, \ldots, T$$

where we order $\lambda_1(\hat{Q}_T) \geq \cdots \geq \lambda_T(\hat{Q}_T)$.

For $T > M$, we use the eigenvectors $v_i(\hat{Q}_T)$ corresponding to the M largest eigenvalues $\lambda_i(\hat{Q}_T)$ of $\hat{Q}_T$ to define the estimated unit root subspace $\hat{H}_N^T$ as

$$\hat{H}_N^T = \bigvee_{i=1}^{M} v_i(\hat{Q}_T)$$

Denote by $\hat{\Pi}_N^T$ the projection on $\hat{H}_N^T$, and let $\hat{\Pi}_S = 1 - \hat{\Pi}_N$, so that we have $\hat{\Pi}_N^T + \hat{\Pi}_S^T = 1$ analogously with the relationship $\Pi_N + \Pi_S = 1$.

Asymptotics for Estimated Variance Operator

Lemma 4.1 Let Assumptions 2.1 and 4.1 hold. We have

$$\hat{Q}^T_{NN} \to_d \underline{Q}_{NN} = \int_0^1 (\underline{W} \otimes \underline{W})(r)dr$$

with $\underline{W} = W(r) - \int_0^1 W(s)ds$ and

$$\hat{Q}^T_{SS} \to_p Q_{SS}$$

where the limit Brownian motion W and operator Q_{SS} are defined in Lemma 3.1. Moreover, we have

$$\hat{Q}^T_{NS}, \hat{Q}^T_{SN} = O_p(1)$$

for all large T.

- The basic asymptotics for the estimated variance operator $\hat{Q}_T$ of (w_t) corresponds to those for the sample variance operator Q_T of (w_t).
- $\hat{Q}_T$ differs from Q_T in two aspects. First, (f_t) used to define Q_T is replaced by $(\hat{f}_t)$ for $\hat{Q}_T$. Second, $\hat{Q}_T$ is defined with the sample mean $T^{-1}\sum_{t=1}^T \hat{f}_t$ instead of the expectation of state density $\mathbb{E}f_t$ used in Q_T. The replacement of (f_t) by $(\hat{f}_t)$ does not affect any of our asymptotic theory. This is true regardless of the stationarity/nonstationarity of the time series of state density.
- However, the use of the sample mean of the state density in place of its expectation has no asymptotic effect only for the stationary component of state density. For the nonstationary component, it yields different asymptotics. Note that the limit Brownian motion W appeared in the limit of Q_T is replaced by the demeaned Brownian motion $\underline{W}$ in the limit of $\hat{Q}_T$.

Asymptotics for Eigenvalues and Eigenvectors

Theorem 4.2 Let Assumptions 2.1 and 4.1 hold. We have

$$\left(T^{-2}\lambda_i(\hat{Q}_T), v_i(\hat{Q}_T)\right) \to_d \left(\lambda_i(\underline{Q}_{NN}), v_i(\underline{Q}_{NN})\right)$$

jointly for $i = 1, \ldots, M$, and

$$\left(T^{-1}\lambda_{M+i}(\hat{Q}_T), v_{M+i}(\hat{Q}_T)\right) \to_p (\lambda_i(Q_{SS}), v_i(Q_{SS}))$$

for $i = 1, 2, \ldots$.

- The results are completely analogous to those for the eigenvalues and eigenvectors of the sample variance operator Q_T.
- The only difference is that we now have $\underline{Q}_{NN}$ defined with the demeaned Brownian motion $\underline{W}$ instead of Q_{NN} defined with the undemeaned Brownian motion W. The appearance of $\underline{W}$ is due to the use of the sample mean of state density in lieu of its expectation.

A Feasible Unit Root Test

We construct a feasible version of the test τ_M^T using demeaned state density estimate $(\hat{w}_t)$ and estimated variance operator $\hat{Q}_T$ of (w_t). Let

$$\hat{z}_t = \Big(\langle v_1(\hat{Q}_T), \hat{w}_t \rangle, \ldots, \langle v_M(\hat{Q}_T), \hat{w}_t \rangle \Big)'$$

for $t = 1, \ldots, T$, and define $\hat{Z}_T = (\hat{z}_1, \ldots, \hat{z}_T)'$ and

$$\hat{Q}_M^T = \hat{Z}_T' \hat{Z}_T, \quad \hat{\Omega}_M^T = \sum_{|i| \leq \ell} \varpi_\ell(i) \hat{\Gamma}_T(i)$$

Our feasible version $\hat{\tau}_M^T$ of the test τ_M^T is given by

$$\hat{\tau}_M^T = T^{-2} \lambda_M(\hat{Q}_M^T, \hat{\Omega}_M^T)$$

where $\lambda_M(\hat{Q}_M^T, \hat{\Omega}_M^T)$ is the smallest generalized eigenvalues of $\hat{Q}_M^T$ with respect to $\hat{\Omega}_M^T$.

Notation

To effectively present the asymptotics of the feasible test $\hat{\tau}_M^T$, we introduce some additional notations. In parallel with Q_M^*, we define

$$\underline{Q}_M^* = \int_0^1 \underline{W}_M^*(r)\underline{W}_M^*(r)'dr$$

with

$$\underline{W}_M^*(r) = W_M^*(r) - \int_0^1 W_M^*(s)ds$$

where W_M^* is the M-dimensional standard vector Brownian motion, used earlier to represent the limit null distribution of the original infeasible test τ_M^T.

Asymptotics for Feasible Test

Theorem 4.3 Let Assumptions 2.1 and 4.1 hold. Under the null H_0,

$$\hat{\tau}_M^T \;\to_d\; \lambda_M(\underline{Q}_M^*)$$

and under the alternative H_1

$$\hat{\tau}_M^T \;\to_p\; 0$$

- The limit null distribution of the feasible test $\hat{\tau}_M^T$ is given by the distribution of the smallest eigenvalue of the integrated product moment of the demeaned M-dimensional standard vector BM on the unit interval. It is nuisance parameter free, and therefore can be obtained through simulation for each M.
- The test based on the feasible statistic $\hat{\tau}_M^T$ is also consistent. The consistency is therefore unaffected by using estimated densities and demeaned density estimates of state distributions.

Critical Values of Feasible Test

We obtain critical values for the tests based on $\hat{\tau}_M^T$ for $M = 1, \ldots, 5$, by simulating the integrated product moment $\underline{Q}_M^*$ of the demeaned M-dimensional standard vector BM, $\underline{W}_M^*$. For simulations, BM is approximated by standardized partial sum of mean zero i.i.d. normal random variates with sample size 10,000, and actual critical values are computed using 100,000 iterations.

M	1	2	3	4	5
1%	0.0248	0.0163	0.0123	0.0100	0.0084
5%	0.0365	0.0215	0.0156	0.0122	0.0101
10%	0.0459	0.0254	0.0177	0.0136	0.0111

As expected, the critical values of $\hat{\tau}_M^T$ decrease as M increases. Recall the limit null distribution of $\hat{\tau}_M^T$ is given by the smallest eigenvalue of $\underline{Q}_M^*$.

Estimated Unit Root Proportion of Moments

Once the unit root dimension M is determined at $\hat{M}_T$, we may obtain the estimated unit root subspace $\hat{H}_N^T$, which is generated by $\hat{M}_T$-eigenvectors given by $v_i(\hat{Q}_T)$, $i = 1, \ldots, \hat{M}_T$.

Then the unit root proportion π_i of the i-th moment can be consistently estimated by

$$\hat{\pi}_i^T = \sqrt{\frac{\sum_{j=1}^{\hat{M}_T} \langle \mu_i, v_j(\hat{Q}_T) \rangle^2}{\sum_{j=1}^{T} \langle \mu_i, v_j(\hat{Q}_T) \rangle^2}}$$

It can indeed be readily deduced from our earlier results that $\hat{\pi}_i^T = \pi_i + o_p(1)$ for all $i = 1, 2, \ldots$.

Empirical Applications

Overview

- Empirical applications on two types of state densities representing cross-sectional distributions and intra-period distributions. We demonstrate how to define and estimate the state densities, and test for unit roots in the time series of state densities.
- State densities are estimated using standard Gaussian kernel on cross-sectional or intra-period observations, and their nonstationarities are analyzed using the test $\hat{\tau}_T^M$.
- Unit root dimension M of state densities is determined by applying $\hat{\tau}_T^M$ successively downward starting from $M = M_{\max}$ with $M_{\max} = 5$. Unit root space H_N is then estimated and the unit root proportion (π_i) is computed for the first four moments. π_i provides the proportion of nonstationary fluctuation in the i-th moment of state distribution.

Representation of Functions as Numerical Vectors

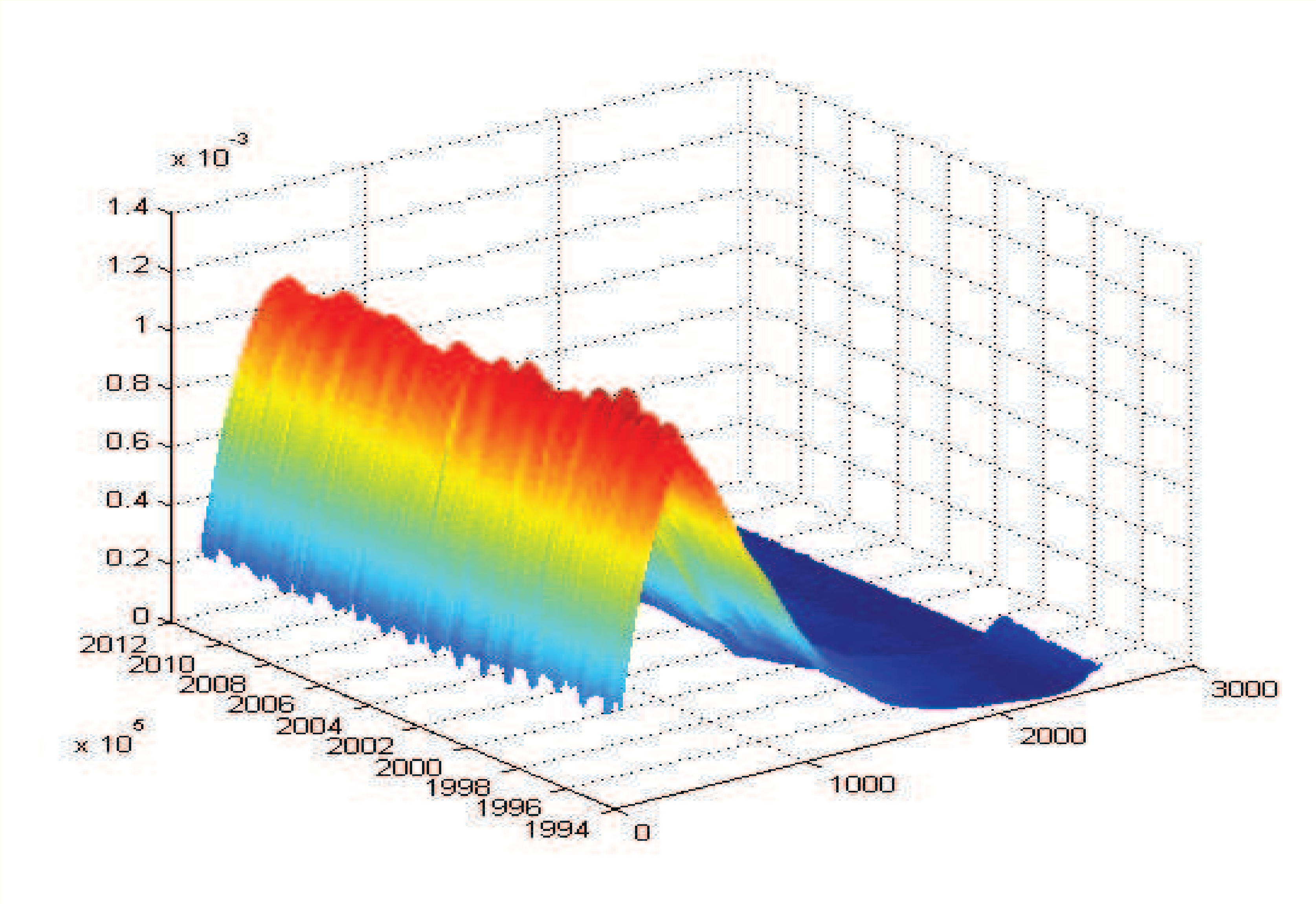
- For the representation of functions in Hilbert space as numerical vectors, we use a Daubechies wavelet basis.
- Wavelets are spatially varying orthonormal bases with two parameters, i.e., scale and translation, and hence they provide more flexibilities in fitting the state densities in our applications, some of which have severe asymmetry and time-varying support. The wavelet basis in general yields a much better fit than the trigonometric basis.
- The Daubechies wavelet is implemented with 1037 basis functions.

Cross-Sectional Distributions of Individual Earnings

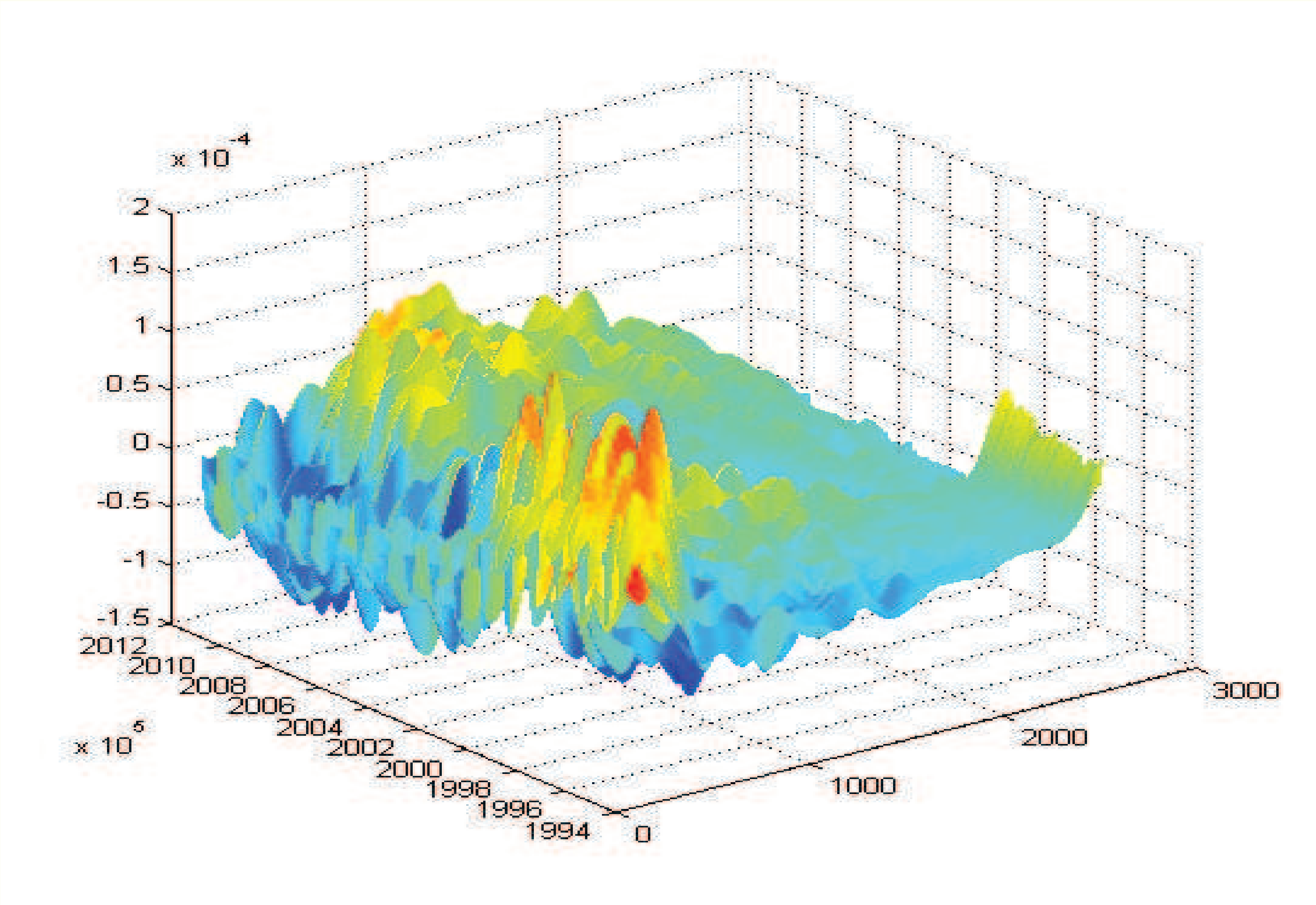
Cross-Sectional Distributions of Individual Earnings

- For the first empirical application, we consider the time series of cross-sectional distributions of individual earnings.
- The cross-sectional observations of individual weekly earnings are obtained at monthly frequency from the Current Population Survey (CPS) data set. The individual weekly earnings are deflated by consumer price index with base year 2005.
- The data set provides 204 time series observations spanning from January 1994 to December 2010 at monthly frequency, and the number of cross-sectional observations in the data set for each month ranges from 12,323 to 15,700.
- For confidentiality reasons, individual earnings are topcoded above a certain level. In our empirical analysis, we drop all topcoded individual earnings as well as zero earnings as in Liu (2011) and Shin and Solon (2011).

Densities of Weekly Individual Earnings



Demeaned Densities of Weekly Individual Earnings



Densities of Individual Earnings

We may see clearly from the time series of the estimated densities for cross-sectional distributions of individual earnings that

- The distributions change over time.
- There exists some evidence of nonstationarity in the time series of cross-sectional distributions of individual earnings.

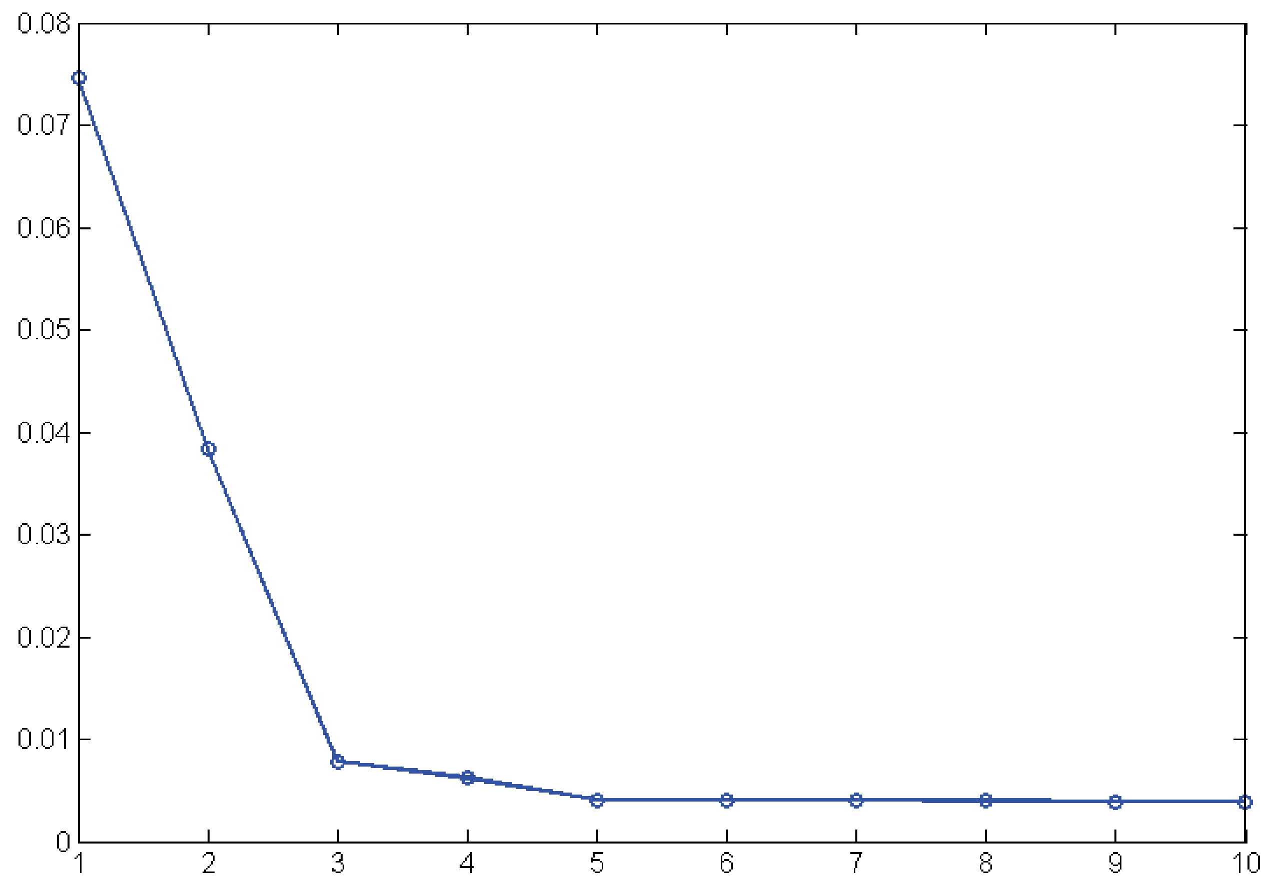
Testing for Unit Root Dimension

To determine the unit root dimension M in the time series of cross-sectional distributions of individual earnings, we use the feasible statistic $\hat{\tau}_M^T$ to test for the null hypothesis $\mathrm{H}_0 : \dim(H_N) = M$ against the alternative $\mathrm{H}_1 : \dim(H_N) \leq M - 1$ with $M = 1, \ldots, 5$.

M	1	2	3	4	5
$\hat{\tau}_M^T$	0.0746	0.0383	0.0079	0.0062	0.0040

- Our test, strongly and unambiguously, rejects H_0 against H_1 successively for $M = 5, 4, 3$. Clearly, however, the test cannot reject H_0 in favor of H_1 for $M = 2$.
- We conclude that there exists two-dimensional unit root, and set $\hat{M}_T = 2$.

Scree Plot of Eigenvalues - Individual Earnings



Unit Root Proportions in Moments

We compute the estimates $\hat{\pi}_i^T$ of the unit root proportions π_i^T with $\hat{M}_T = 2$ for the first four moments.

$\hat{\pi}_1^T$	$\hat{\pi}_2^T$	$\hat{\pi}_3^T$	$\hat{\pi}_4^T$
0.5261	0.3420	0.2462	0.2013

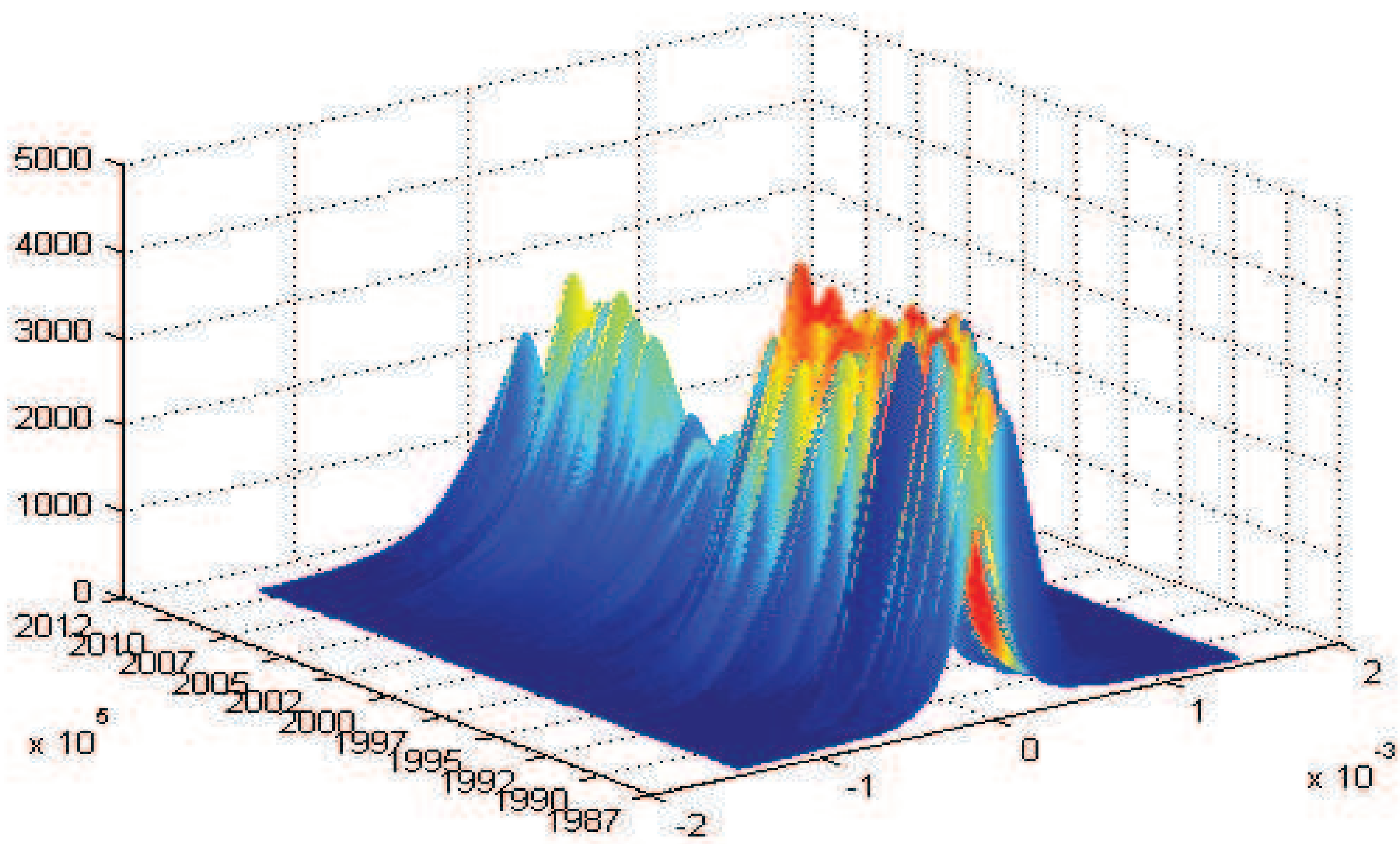
- The unit root proportions for the first four moments are all nonnegligibly large. In particular, the unit root proportions for the first two moments are quite substantial.
- The presence of a substantial unit root proportion in the second moment explains the recent empirical findings on changes in volatilities of individual earnings. Dynan *et al* (2008) and others.
- Nonstationarity in time series of individual earnings distributions would certainly make their volatilities more persistent.

Intra-Month Distributions of Stock Returns

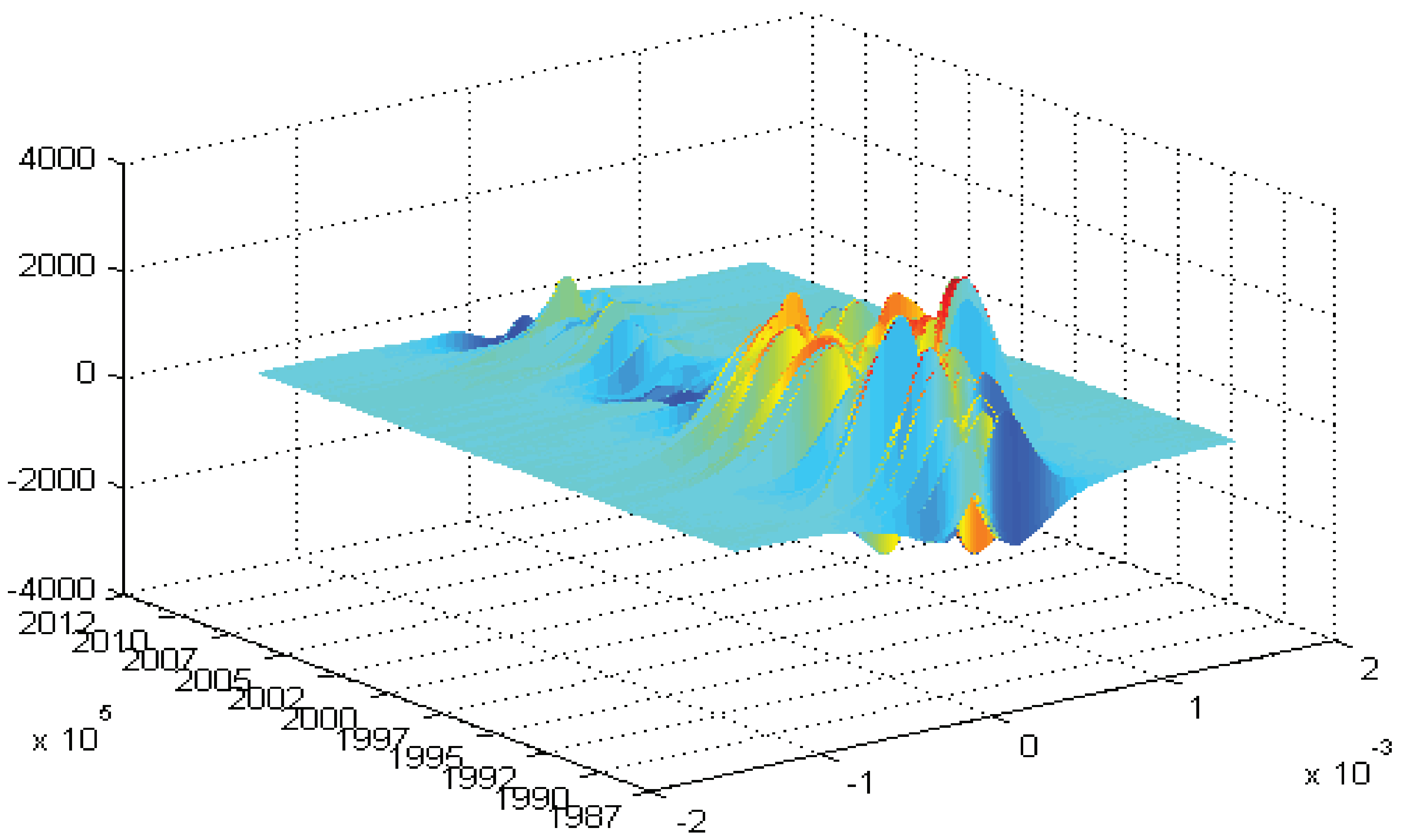
Intra-Month Stock Return Distributions

- For the second empirical application, we consider the time series of intra-month distributions of stock returns.
- For each month during the period from January 1992 to June 2010, we use the S&P 500 index returns at one-minute frequency to estimate 222 densities for the intra-month distributions. The one-minute returns of S&P 500 index are obtained from Tick Data Inc. The number of intra-month observations available for each month varies from 7211 to 9177, except for September 2001, for which we only have 5982 observations.
- The intra-month observations are truncated at 0.50% and 99.5% percentiles before we estimate the state densities.
- To avoid the micro-structure noise, we also use the five-minute observations to estimate the intra-month observations. Our empirical results are, however, virtually unchanged.

Densities of Intra-Month S&P 500 Returns



Densities of Intra-Month S&P 500 Returns



Densities of Intra-Month S&P 500 Returns

From the time series of the estimated densities, both undemeaned and demeaned, it can be clearly seen that

- The mean locations and volatility levels, in particular, of intra-month return distributions vary with time.
- There is some evidence of nonstationarity in the time series of intra-month return distributions.

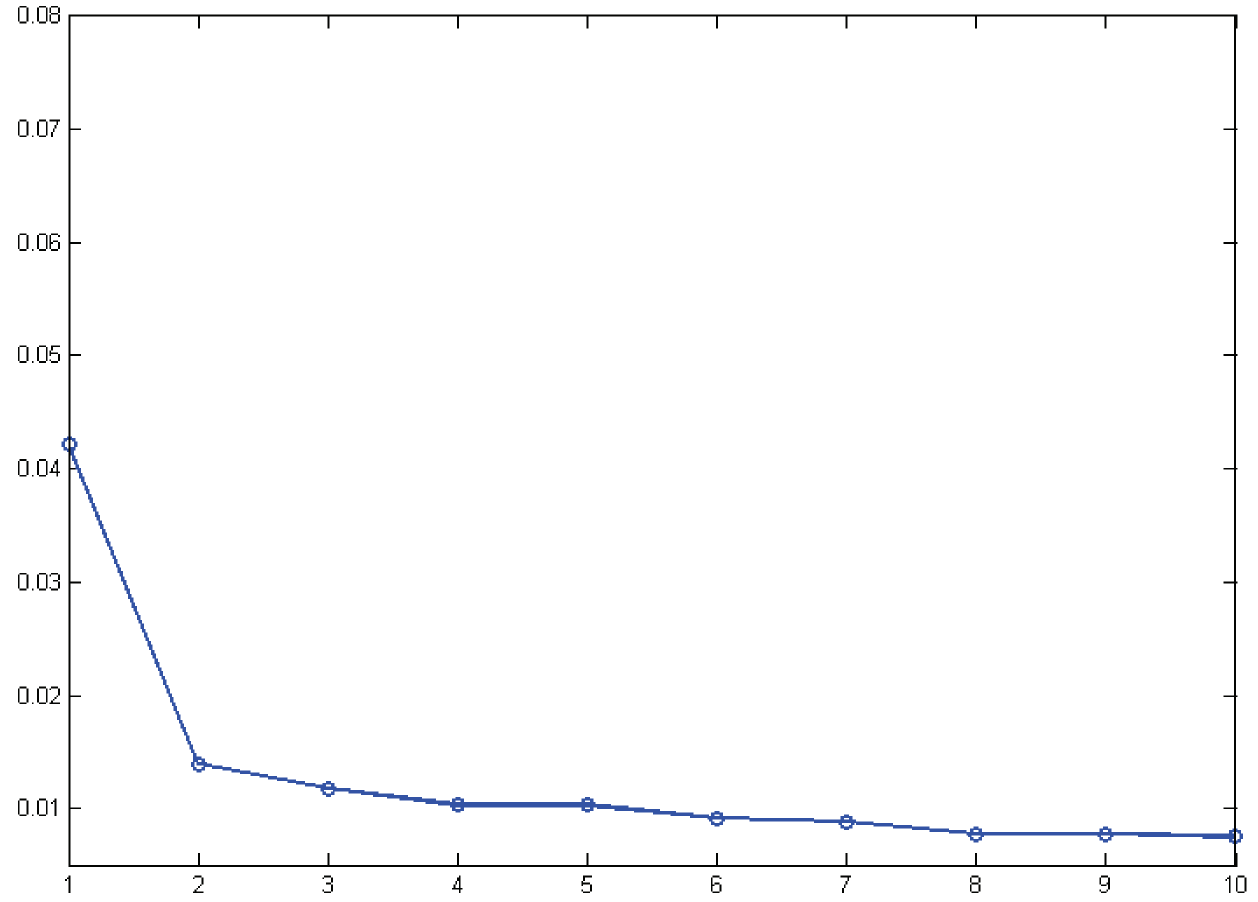
Testing for Unit Root Dimension

To test for existence of nonstationarity in time series of intra-month S&P 500 return distributions, we use the feasible statistic $\hat{\tau}_M^T$ to test the null $\mathrm{H}_0 : \dim(H_N) = M$ against the alternative $\mathrm{H}_1 : \dim(H_N) \leq M - 1$ with $M = 1, \ldots, 5$.

M	1	2	3	4	5
$\hat{\tau}_M^T$	0.0421	0.0139	0.0118	0.0103	0.0095

- Our test successively rejects the null against the alternative for $M = 5, 4, 3, 2$.
- However, at 5% level, the test cannot reject H_0 in favor of H_1 for $M = 1$. Our test result implies that there exists one-dimensional unit root, i.e., $\hat{M}_T = 1$.

Scree Plot of Eigenvalues - Intra-Month Returns



Unit Root Proportions in Moments

Compute the estimates $\hat{\pi}_i^T$ of the unit root proportions π_i^T for the first four moments, with $\hat{M}_T = 1$.

$\hat{\pi}_1^T$	$\hat{\pi}_2^T$	$\hat{\pi}_3^T$	$\hat{\pi}_4^T$
0.0066	0.0826	0.0008	0.0269

- The unit root proportions are in general small for all of the first four moments, implying the nonstationarity in the time series of return distributions is not concentrated in the first four moments.
- However, the nonstationarity is relatively more concentrated in the second and fourth moments, with the unit root proportion of the second moment being the largest.
- The unit root proportion of the first and third moments are almost negligible. This is well expected, since for many financial time series strong persistency is observed mainly in volatility and kurtosis.

Simulation

Overview

- We generate the data using a model that approximates as closely as possible the estimated models from our empirical applications. This is to make our simulation more realistic and practically relevant.
- We assume that state densities are observable, and therefore, our simulation would not provide information on the effect of estimating unobserved state densities. The number of observations used to estimate state densities in our empirical applications is quite large compared to the sample size. The approximation error due to the estimation of state densities should therefore be small and unimportant.
- We directly generate the centered state density (w_t), instead of the original state density (f_t). This causes no loss in generality, since we use demeaned state density $f_t - T^{-1}\sum_{t=1}^{T} f_t$ that is identical to the demeaned centered state density $w_t - T^{-1}\sum_{t=1}^{T} w_t$.

Data Generating Process 1

- To generate our simulation sample (w_t), we let

$$w_t = \sum_{i=1}^{I} c_{it} v_i$$

  where (c_{it}), $i = 1, \ldots, I$ and $t = 1, \ldots, T$, are scalar stochastic processes and (v_i), $i = 1, \ldots, I$, are nonrandom orthonormal vectors in H, which are specified more specifically below.
- Note that the simulation sample (w_t) is generated from the I-dimensional subspace of H spanned by (v_i), $i = 1, \ldots, I$, and that $c_{it} = \langle v_i, w_t \rangle$, and (c_{it}) becomes the i-th coordinate process for $i = 1, \ldots, I$.

Data Generating Process 2

- We set
$$(c_{jt} - \alpha c_{j,t-1}) = \beta_i (c_{j,t-1} - \alpha c_{j,t-2}) + \eta_{jt}$$
for $j = 1, \ldots, M_0$, where M_0 be the number of unit roots in (w_t), and
$$c_{it} = \alpha_i c_{i,t-1} + \eta_{it}$$
for $i = M_0 + 1, \ldots, I$, where (η_{it}) are independent normal random variates with mean zero and variance (σ_i^2) for $i = 1, \ldots, I$.
- The above specifications are obtained from the estimated coordinate processes in our empirical applications using AIC. We set $\alpha = 1$ to introduce a unit root in the first M_0 coordinate process. All other parameters, β_i, $i = 1, \ldots, M_0$, (α_i), $i = M_0 + 1, \ldots, I$, and (σ_i^2), $i = 1, \ldots, I$, are set to be our estimates for the estimated coordinate processes of our empirical applications. In particular, we have $|\beta_i| < 1$, $i = 1, \ldots, M_0$, and $|\alpha_i| < 1$ for $i = M_0 + 1, \ldots, I$.

Cross-Sectional Distributions of Individual Earnings

Simulation Setup

- We set $M_0 = 2$.
- Recall that we have $T = 204$ in our empirical application on the time series of cross-sectional individual earnings distributions.
- Therefore, we set $I = 204$, and let (v_i) used to define (w_t) be the orthonormal eigenvectors associated with nonzero eigenvalues of the estimated variance operator of (w_t). Note that for a sample (w_t) of size T, we only have T eigenvectors (v_i) associated with nonzero eigenvalues.
- Various choices of T between 100 and 500 are considered.

Finite Sample Size and Power

Table : Rejection Probabilities for 5% Test, $M_0 = 2$

	M				
T	1	2	3	4	5
100	0.0076	0.0264	0.7812	0.4804	0.2192
200	0.0152	0.0440	0.9996	1.0000	0.9992
300	0.0172	0.0476	1.0000	1.0000	1.0000

Finite Sample Power 1

Table : Rejection Probabilities for 5% Test, $M_0 = 1$ and $M = 2$

	α			
T	0.80	0.85	0.90	0.95
100	0.3190	0.1482	0.0452	0.0148
300	1.0000	0.9992	0.8912	0.2320
500	1.0000	1.0000	0.9996	0.7430

Finite Sample Power 2

Table : Rejection Probabilities for 5% Test, $M_0 = 0$ and $M = 1$

	α			
T	0.80	0.85	0.90	0.95
100	0.7964	0.4986	0.1800	0.0194
300	1.0000	1.0000	0.9980	0.6118
500	1.0000	1.0000	1.0000	0.9846

Finite Sample Power 3

Table : Rejection Probabilities for 5% Test, $M_0 = 0$ and $M = 2$

	α			
T	0.80	0.85	0.90	0.95
100	0.5016	0.2662	0.0882	0.0192
300	0.9998	1.0000	0.9864	0.4262
500	1.0000	1.0000	1.0000	0.9380

Intra-Month Distributions of Stock Returns

Simulation Setup

- We set $M_0 = 1$.
- We have $T = 222$ in our empirical analysis on the time series of intra-month distributions of S&P 500 returns.
- Therefore, we set $I = 222$, and let (v_i) used to define (w_t) be the orthonormal eigenvectors associated with nonzero eigenvalues of the estimated variance operator of (w_t).
- Various choices of T between 100 and 500 are considered.

Finite Sample Size and Power

Table : Rejection Probabilities for 5% Test, $M_0 = 1$

	M				
T	1	2	3	4	5
100	0.0480	1.0000	1.0000	0.1540	0.0204
200	0.0484	1.0000	1.0000	1.0000	0.9836

Finite Sample Power

Table : Rejection Probabilities for 5% Test, $M_0 = 0$ and $M = 1$

T	α 0.80	0.85	0.90	0.95
100	0.4812	0.3336	0.1680	0.0456
300	0.9988	0.9976	0.9952	0.7120
500	1.0000	1.0000	1.0000	0.9884

Simulation Summary

- The test performs quite well in finite samples. Both finite sample size and power are satisfactory.
- The test behaves badly only when M is excessively large compared to T. The problem, however, quickly and completely disappears as T increases.

Summary

- We consider testing for nonstationarity in the time series of state distributions, which can be either cross-sectional or intra-period distributions of some underlying economic variable.
- We regard the state densities as Hilbertian random variables, and employ the FPCA to construct a nuisance parameter free statistic for testing for unit roots in the time series of state densities.
- With the estimated unit root subspace, we may compute the unit root proportions in the moments of state distributions.
- We apply our methodology to analyze nonstationarity in the time series of cross-sectional distributions of individual earnings and intra-month distributions of stock returns. In both cases we find some clear evidence for the presence of nonstationarity.
- The presence of nonstationarity in the time series of state distributions yields interesting and important implications, both economic and statistical, which we will further explore in our future work.

Functional Unit Root Test: Climate Change

July 1, 2013

Table of Contents

1. Data
2. Empirical Results

Data

- compiled by the Climatic Research Unit at the University of East Anglia and the Hadley Centre of the UK Met office.
- global average of combined land and sea surface temperatures over widely dispersed locations, in a time series from 1850 to date (From 1,652 to 55,576 stations)
- expressed as the deviation from the average of the period 1961-1990 and theses deviations are called 'temperature anomalies'.
- temperature anomalies on a 5° by 5° grid-box basis
  (number of Monthly grids : 36*72 = 2,592, number of Annual grids: 2,592*12 = 31,104)

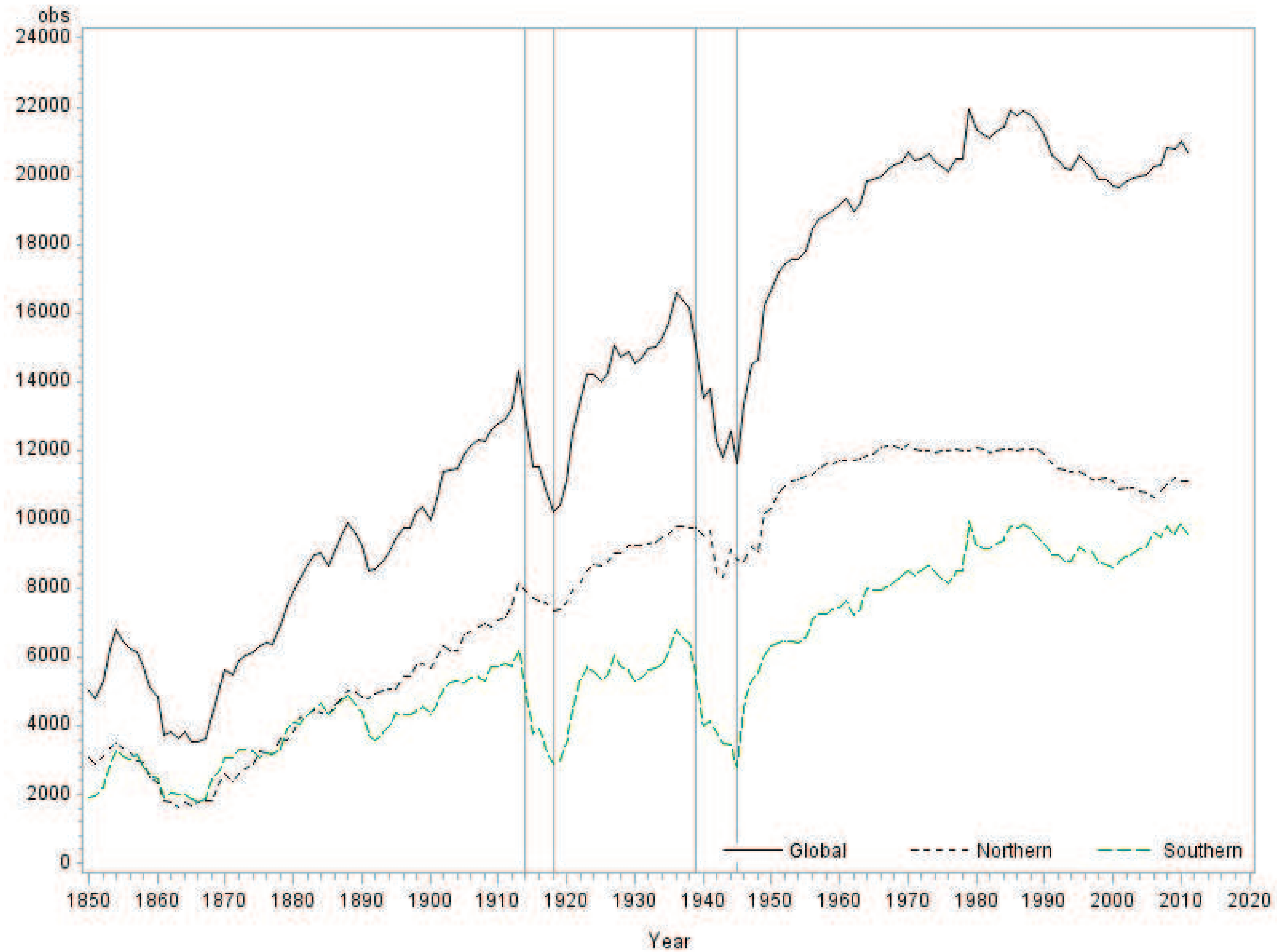


Figure : Number of observations for the globe, the northern hemisphere and the southern hemisphere; Total number of 5° by 5° grid-boxes is 31,104(=36*72*12) for the globe and 15,552(=18*72*12) for the northern and southern hemisphere.

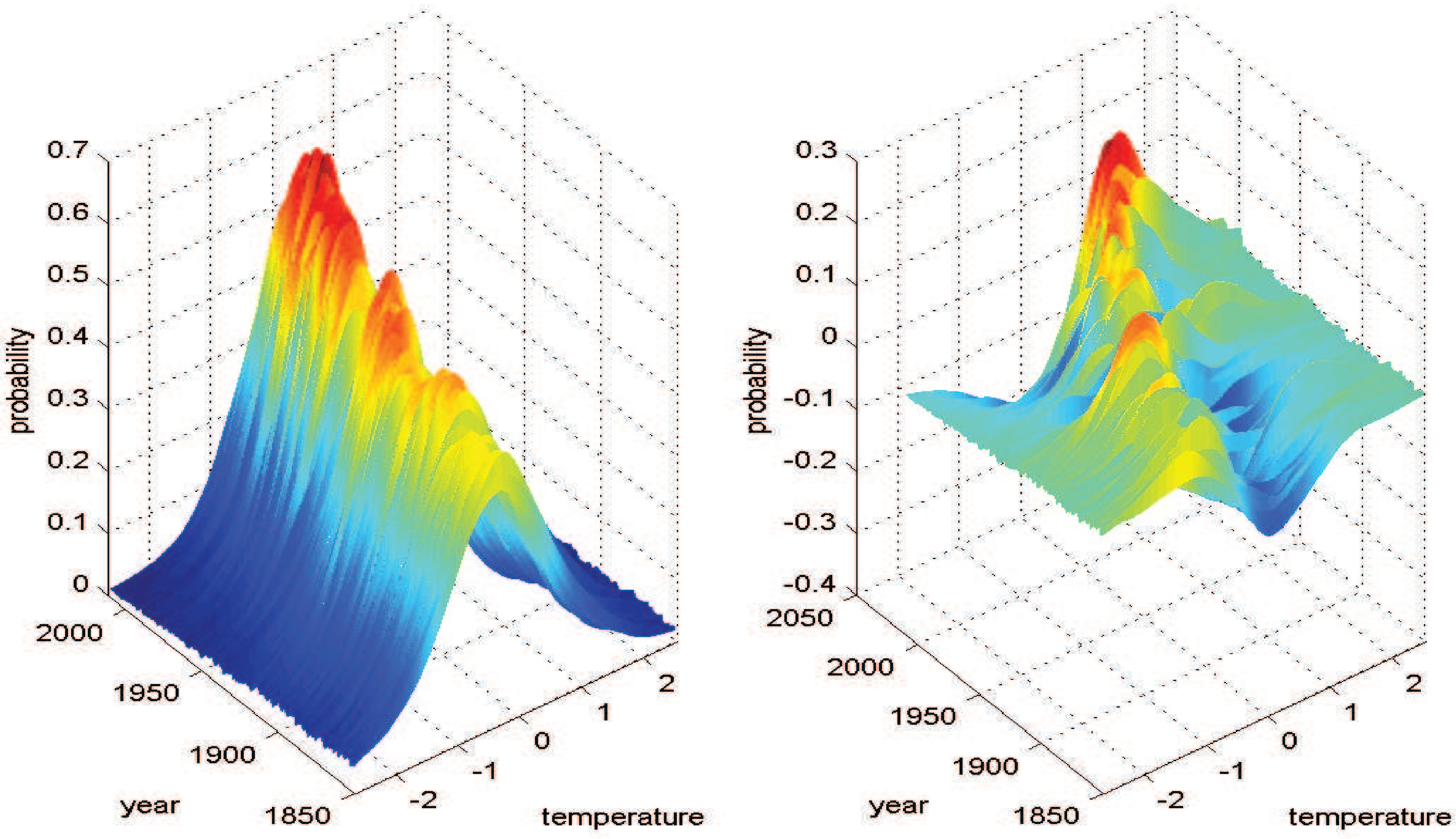


Figure : Annual temperature anomalies distributions (undemeaned and demeaned densities); Temperature anomalies on a 5° by 5° grid-box basis are used. A normal kernel with optimal fixed bandwidth is used for the estimation of the density functions.

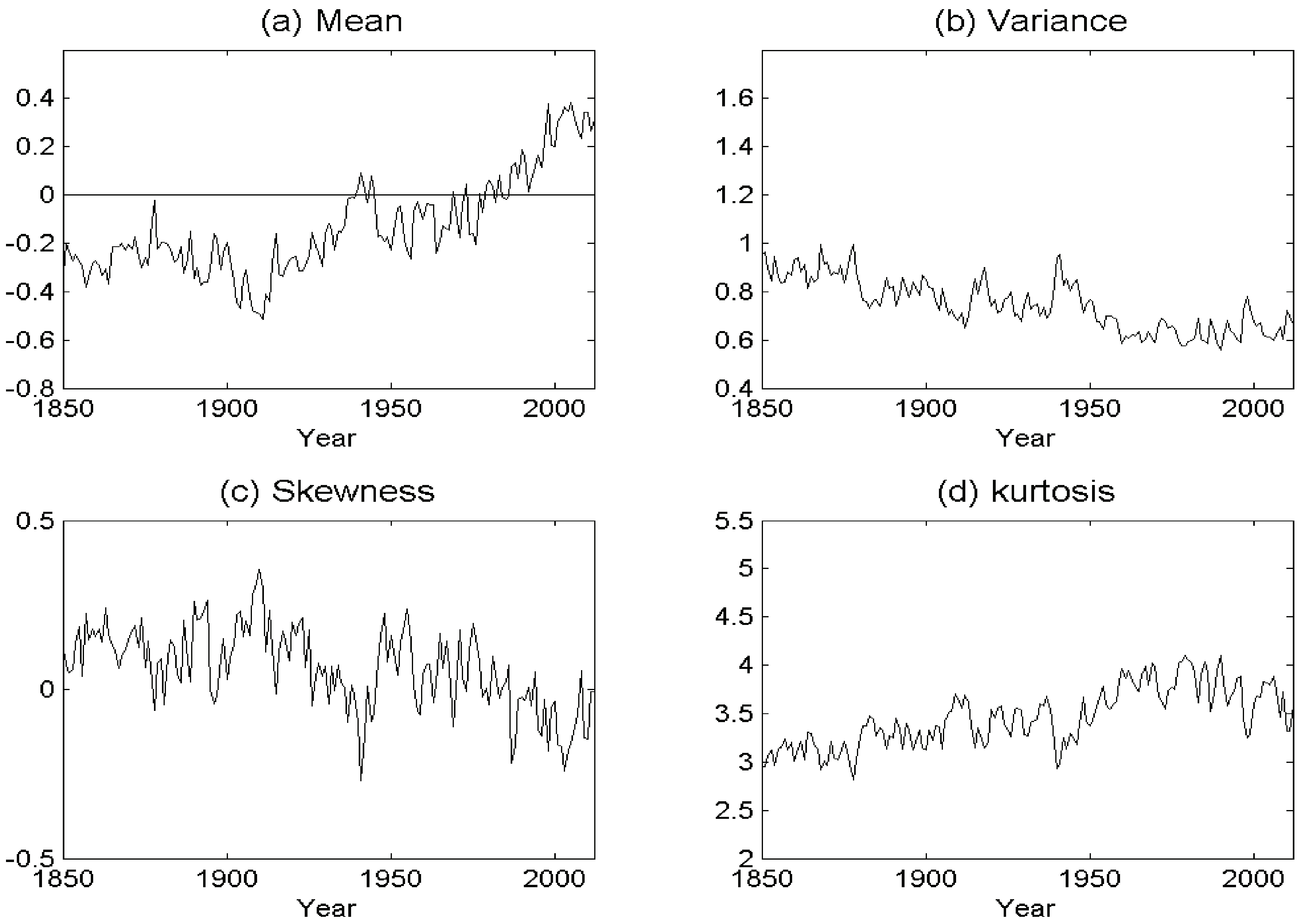


Figure : Mean, Variance, Skewness and Kurtosis for the globe

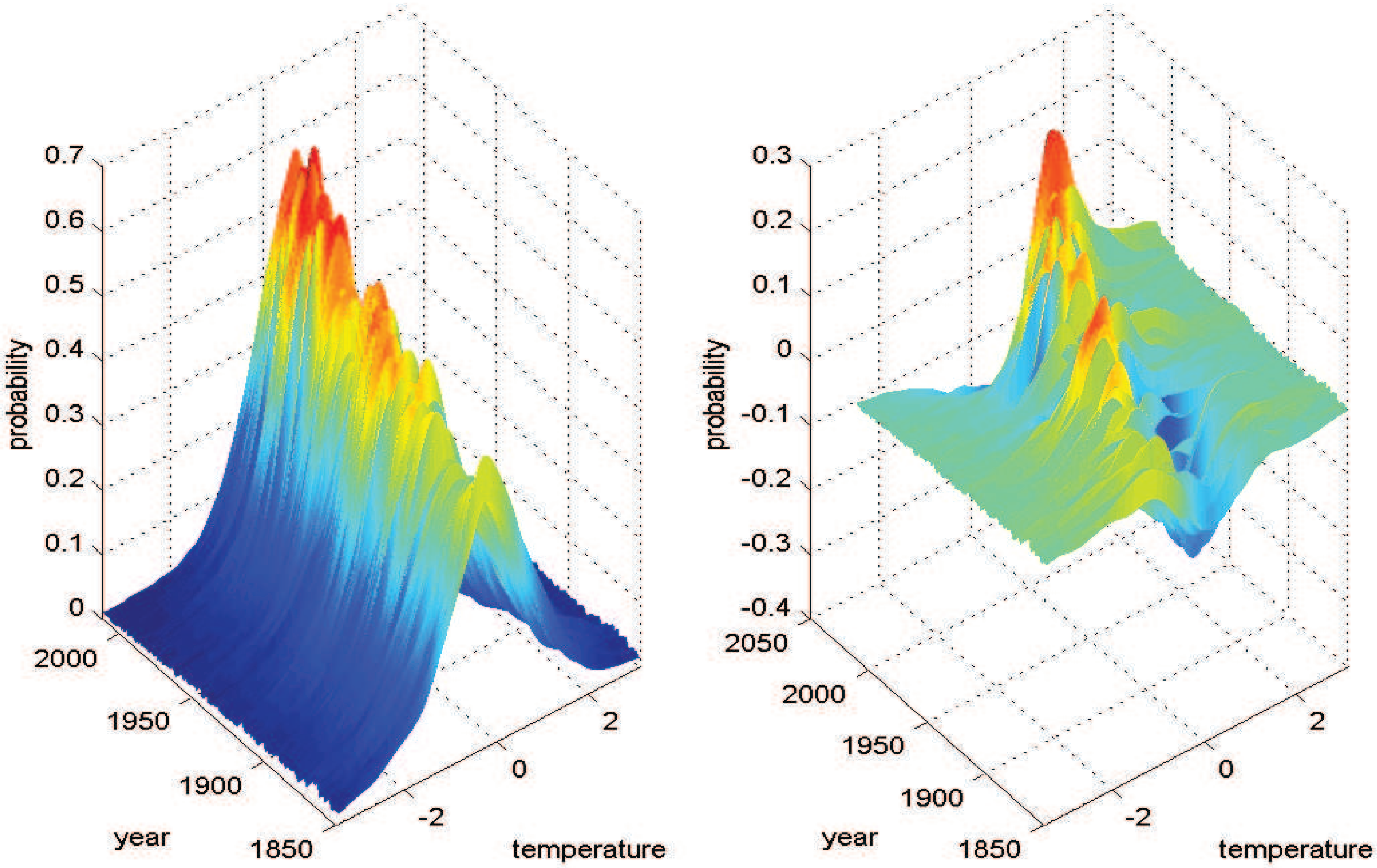


Figure : Annual temperature anomalies distributions for the northern hemisphere (undemeaned and demeaned densities)

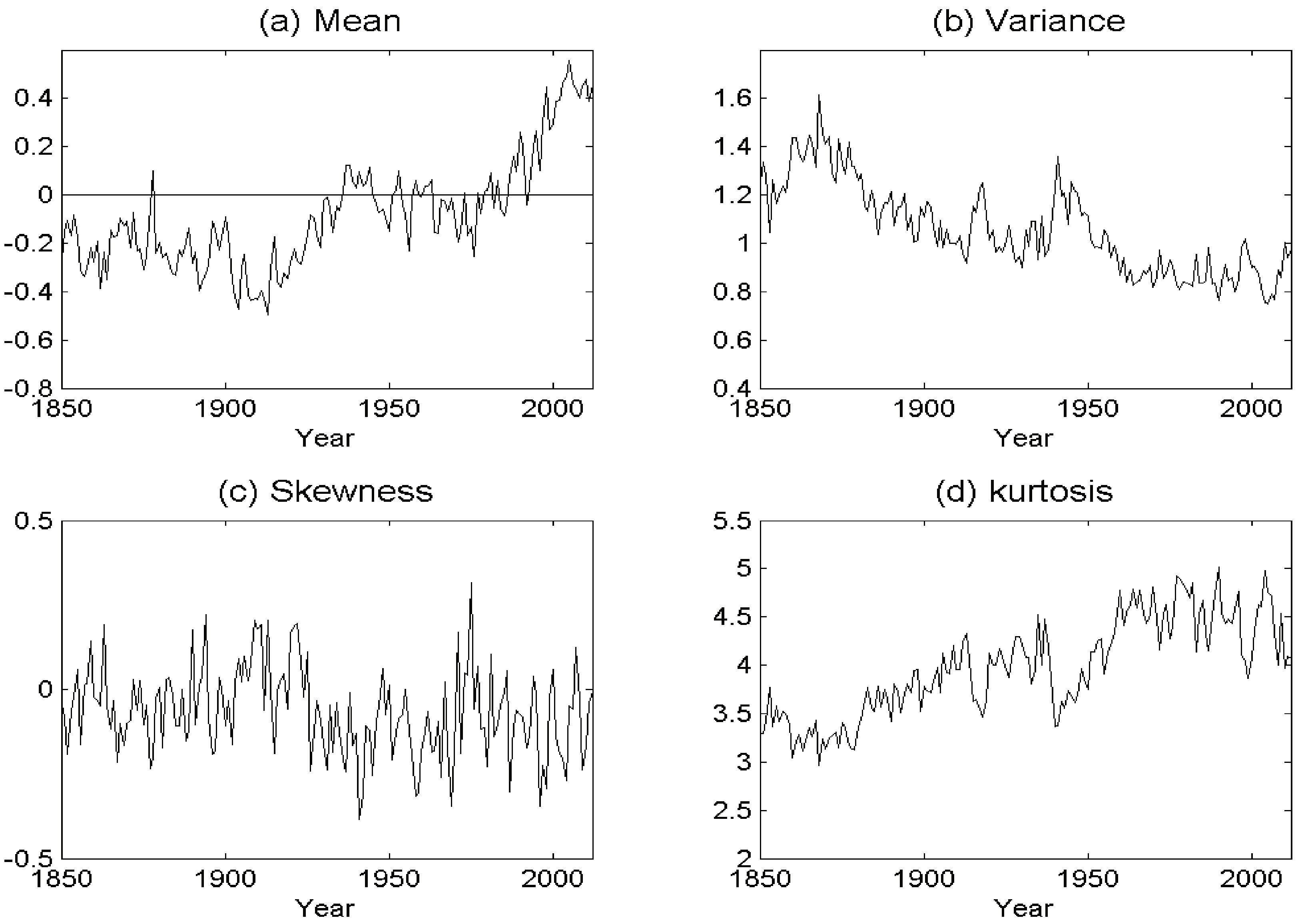


Figure : Mean, Variance, Skewness and Kurtosis for the northern hemisphere (Estimated annual temperature anomalies distributions are used.)

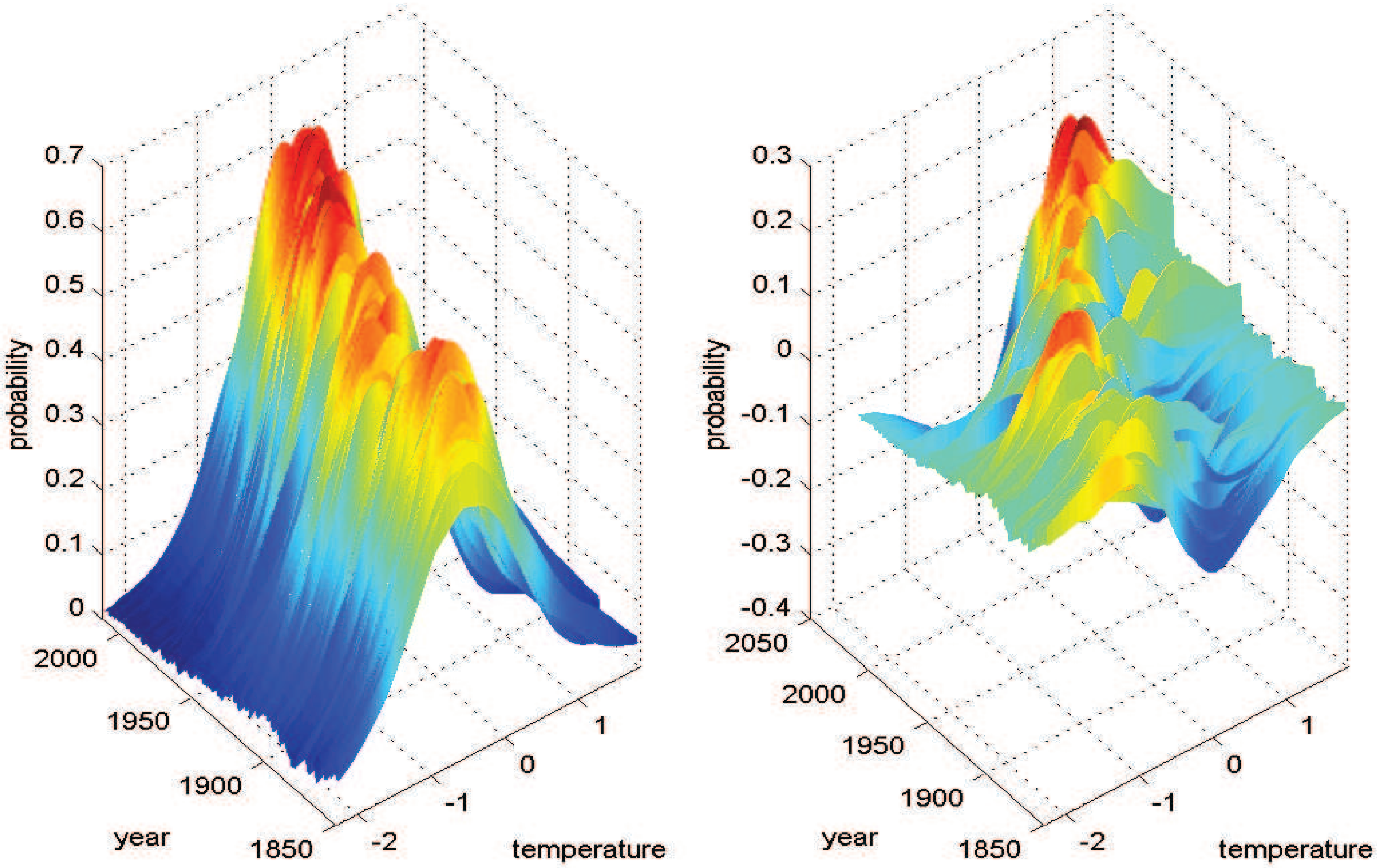


Figure : Annual temperature anomalies distributions for the southern hemisphere (undemeaned and demeaned densities)

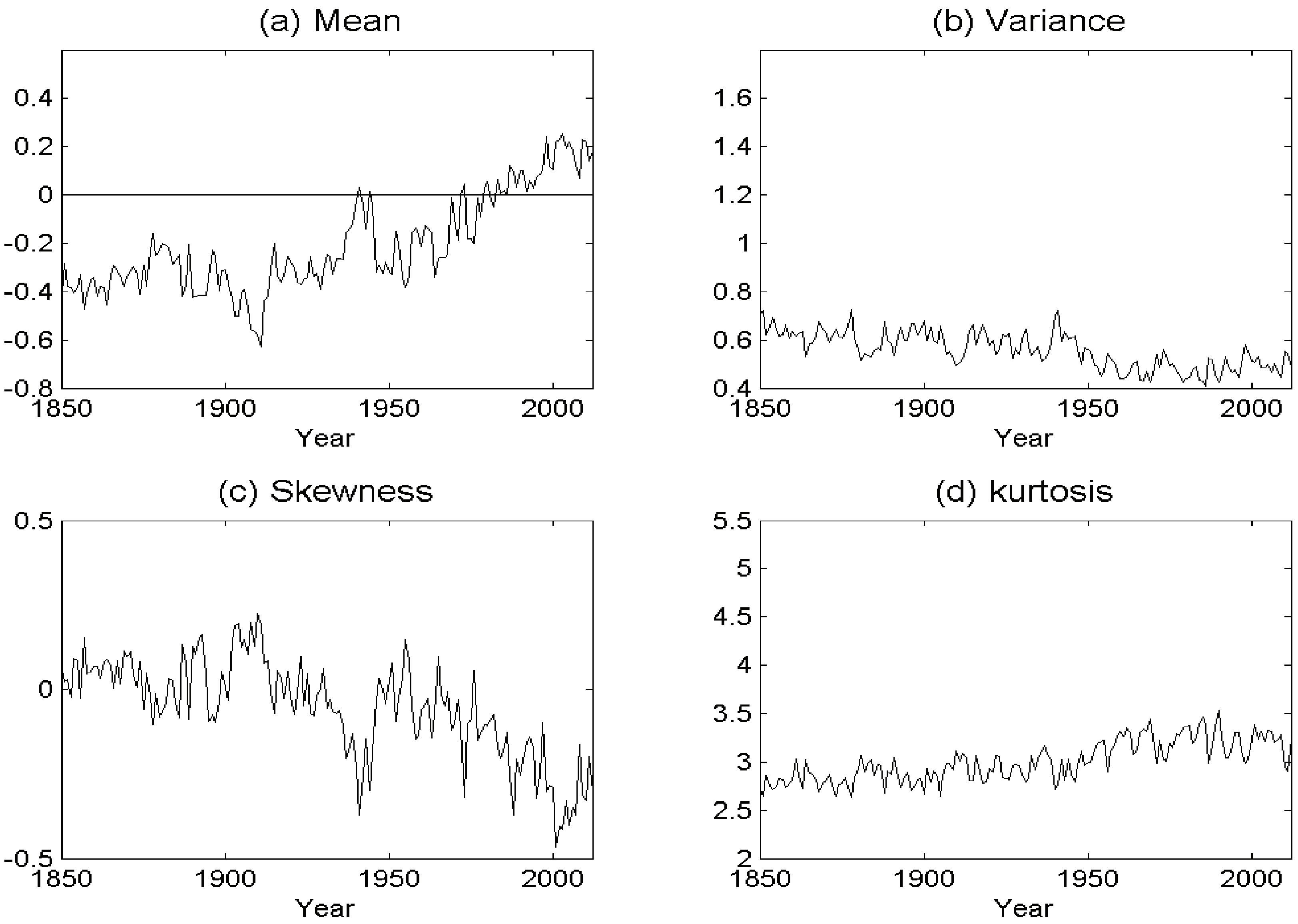


Figure : Mean, Variance, Skewness and Kurtosis for the southern hemisphere (Estimated annual temperature anomalies distributions are used.)

Table of Contents

1 Data

2 Empirical Results

- To analyze the nonstationarity of the time series of state densities, the statistic developed in Chang *et al*(2012) is used.
- We estimate annual temperature anomalies densities for the globe, northern and southern hemisphere, (f_t^g), (f_t^n) and (f_t^s).
- we choose the support that preserves 95% of the total probability mass of the average of (f_t^g), (f_t^n) and (f_t^s).
- For the representation of functions in our Hilbert space as numerical vectors, we use a Daubechies wavelet basis. The Daubechies wavelet is implemented with 1037 basis functions.
- To determine the dimension M of the unit root subspace H_N, we consider the test of the null hypothesis,

  $$H_0 : dim(H_N) = M$$

  against the alternative hypothesis

  $$H_1 : dim(H_N) \leq M - 1$$

Table : Critical Values of the Test Statistics $\hat{\tau}_M^T$

$\hat{\tau}_{M,1}^T$	M=1	M=2	M=3	M=4	M=5
1%	0.0274	0.0175	0.0118	0.0103	0.0085
5%	0.0385	0.0223	0.0154	0.0127	0.0101
10%	0.0478	0.0267	0.0175	0.0139	0.0111
$\hat{\tau}_{M,2}^T$					
99%	0.7487	1.0073	1.2295	1.4078	1.5952
95%	0.4660	0.6787	0.8645	1.0336	1.1892
90%	0.3494	0.5399	0.7066	0.8574	1.0092

Table : Test Results for the Globe

(a) Values of Statistic for Testing M=m						
	M=1	M=2	M=3	M=4		
$\hat{\tau}^T_{M,1}$	0.0531	0.0289	0.0105	0.0097		
$\hat{\tau}^T_{M,2}$	0.0531	0.0536				
(b) Critical Value						
	M=1	M=2	M=3	M=4		
5%	0.0385	0.0223	0.0154	0.0127		
95%	0.4660	0.6787				
(c) Unit Root Proportions in First Seven Moments						
$\hat{\pi}^T_1$	$\hat{\pi}^T_2$	$\hat{\pi}^T_3$	$\hat{\pi}^T_4$	$\hat{\pi}^T_5$	$\hat{\pi}^T_6$	$\hat{\pi}^T_7$
0.516	0.270	0.235	0.188	0.151	0.142	0.117

Table : Test Results for the Northern Hemisphere

(a) Values of Statistic for Testing M=m

	M=1	M=2	M=3	M=4		
$\hat{\tau}^T_{M,1}$	0.0387	0.0379	0.0119	0.0105		
$\hat{\tau}^T_{M,2}$	0.0387	0.0407				

(b) Critical Value

	M=1	M=2	M=3	M=4		
5%	0.0385	0.0223	0.0154	0.0127		
95%	0.4660	0.6787				

(c) Unit Root Proportions in First Seven Moments

$\hat{\pi}^T_1$	$\hat{\pi}^T_2$	$\hat{\pi}^T_3$	$\hat{\pi}^T_4$	$\hat{\pi}^T_5$	$\hat{\pi}^T_6$	$\hat{\pi}^T_7$
0.409	0.205	0.160	0.129	0.101	0.094	0.078

Table : Test Results for the Southern Hemisphere

(a) Values of Statistic for Testing M=m						
	M=1	M=2	M=3	M=4		
$\hat{\tau}^T_{M,1}$	0.0611	0.0219	0.0097	0.0089		
$\hat{\tau}^T_{M,2}$	0.0611					
(b) Critical Value						
	M=1	M=2	M=3	M=4		
5%	0.0385	0.0223	0.0154	0.0127		
95%	0.4660					
(c) Unit Root Proportions in First Seven Moments						
$\hat{\pi}^T_1$	$\hat{\pi}^T_2$	$\hat{\pi}^T_3$	$\hat{\pi}^T_4$	$\hat{\pi}^T_5$	$\hat{\pi}^T_6$	$\hat{\pi}^T_7$
0.633	0.199	0.331	0.168	0.212	0.140	0.157

Evaluating Factor Pricing Models Using High Frequency Panels

Yoosoon Chang

Department of Economics

Indiana University

Joint with Yongok Choi, Hwagyun Kim and Joon Park

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Outline

1. **Background**
2. **New Approach**
3. **Theoretical Derivation**
4. **Regression Formulation**
5. **Statistical Procedure and Theory**
6. **Reexamination of Fama-French Regressions**

Background

Foundation

Modern empirical asset pricing models are exemplified by Fama-French regressions (1992, 1993). At the foundational level, they rely on the fact that

> *Under no arbitrage condition, we have*
>
> $$\mathbb{E}(R_t^i - R_t^f) = \beta_i' \lambda,$$
>
> *where λ is the market price of risk (or market risk premium).*

Note that λ is the same for all assets i and plays the role of pricing factors.

Factor Pricing Model

The actual factor pricing regressions are of the form

$$R_t^i - R_t^f = \alpha_i + \beta_i' X_t + \varepsilon_t^i,$$

where X_t is a set of pricing factors.

- If the CAPM holds, the single market factor (if possible to identify) must be able to explain all of the i's returns by betas (β_i), and $\alpha_i = 0$ holds for all i.
- Fama and French and numerous authors have found that the CAPM based on the single market factor does not work very well, and tried to find other factors. It is important to note that $\alpha_i = 0$ for all i must hold if the model is well specified with correct factors.

Finding Pricing Factors

A typical statistical procedure to find pricing factors is as follows:

1. Come up with a firm characteristic which has a potential to be a pricing factor (size or B/M ratio in case of Fama-French, Momentum for Jegadeesh and Titman (1993), for example.)
2. Group stock returns in terms of this characteristic.
3. Estimate and test the model if those returns are well explained by the exising model.
4. If not, add a factor based on the characteristic used for grouping the stocks.
5. Repeat 3 for the same portfolios and other cross sections of asset returns. If this model with an added factor works well, then you have found a new asset pricing model.

In sum, empirical asset pricing involves constructing panel data sets of returns, and econometric analysis on those data sets.

Econometric Issues

There are many econometric issues that are relevant to the analysis of factor pricing regressions. Of them are the following:

- Despite recent advances in econometric research, most of empirical asset pricing models rely on the simple classical regression framework. Is this desirable?
- Nowadays, a wide variety of high frequency financial data are available at an individual firm level. Is there any way to exploit the high frequency observations and improve the quality of statistical analysis?
- Recently, it is well observed that volatilities in financial returns are endogenous and persistent, as well as time-varying and stochastic. The presence of such endogenous and persistent time-varying and stochastic volatilities may invalidate the standard inference. Are the previous results robust with respect to the introduction of more realistic volatility models?

Our subsequent analysis will address these issues.

New Approach

Main Diagnosis

The Fama-French type regressions have the following characteristics.

- Though the data are available, they cannot be used directly to run regressions at high frequencies. This is because the signal/noise ratio is truly minimal at high frequencies.
- The presence of stochastic volatilities is evident. Moreover, they are strongly endogenous and persistent.

In particular, endogenous and persistent stochastic volatilities may severely distort the OLS estimates and invalidate the standard tests.

Our approach utilizes high frequency data to effectively deal with endogenous and persistent volatilities, and fit the regressions at random low frequencies to boost up the signal/noise ratio.

Our Approach

In our approach, we do the following:

- Develop an error component model in continuous time,
- Use the high frequency observations and apply the method of time change to choose the random sampling intervals, and
- Run the OLS regressions based on the samples collected at random intervals to estimate and test factor pricing models.

Each of the steps will be explained in more detail subsequently.

Theoretical Derivations

Theoretical Background

To derive a continuous-time beta model of asset returns, let the state price density π be given by

$$\frac{d\pi_t}{\pi_t} = \upsilon_t dt + \sum_{j=1}^{J} \tau_{jt} dV_{jt}, \tag{1}$$

and specify the price process (P_i) of security i as

$$\frac{dP_{it}}{P_{it}} = \mu_{it} dt + \sigma_t \left(\sum_{j=1}^{J} \kappa_{ij} dV_{jt} + \sum_{k=1}^{K} \lambda_{ik} dW_k \right) + \omega_{it} dZ_{it}, \tag{2}$$

We regard the instantaneous returns (dP_i/P_i) of a risky asset as the total return from trading gains and the dividends paid between t and $t + dt$. The drift term (μ_i) measures the risk return trade off.

Specification of Error Components

Assumption 2.1 (Z_i), (V_j) and (W_k) are independent Brownian motions such that (ω_i, Z_i) and (σ, V_j, W_k) are independent of each other, and such that (W_k) are Brownian motions independent of (V_j) conditional on σ.

The diffusion term of (dP_i/P_i) includes

- the common volatility σ
- the idiosyncratic volatility (ω_i) specific to asset i.

The common volatility component is then further divided into

- involving (V_j) - results from covariation with π
- consisting only of (W_k), independent of (V_j) - no bearing on π.

In total, we have three terms describing the stochastic evolution of (dP_i/P_i). The first involving (V_j) is related to the pricing factor. Meanwhile, the second and third terms including (W_k) and (Z_i) have no bearing on π, and are not used to pin down the conditional mean. Instead, (W_k) and (Z_i) capture fluctuations of dividends which do not affect investors' discount factors.

Pricing Factors

We specify pricing factors (Q_j) as

$$\frac{dQ_{jt}}{Q_{jt}} = \nu_{jt}dt + \rho_j \sigma_t dV_{jt} \tag{3}$$

for $j = 1, 2, ..., J$.

In our specification, (Q_j) can be understood as the price of a portfolio made out of individual assets so that only the systematic diffusion part relevant for pricing will remain. For example, a portfolio with a long position for small firms and a short position for large firms

Market Factor The first pricing factor Q_1 is set to be the market factor with the unit corresponding coefficient, i.e., $\rho_1 = 1$.

The subsequent derivation of our model depends crucially on the existence of a common volatility movement σ.

No Arbitrage Condition

Under some mild conditions, the total gain process (P_i) admits no arbitrage if and only if the deflated process (πP_i) is a martingale, which yields for all $i = 1, \ldots, I$ and $j = 1, \ldots, J$,

$$\mu_{it} = -\upsilon_t - \sum_{j=1}^{J} \kappa_{ij}\sigma_t\tau_{jt}, \qquad \nu_{jt} = -\upsilon_t - \rho_j\sigma_t\tau_{jt},$$

due to Ito's lemma. Note that the instantaneously riskless rate is given by $r_t^f = -\upsilon_t$, implying

$$\mu_{it} - r_t^f = \sum_{j=1}^{J} \beta_{ij}(\nu_{jt} - r_t^f),$$

where $\beta_{ij} = \kappa_{ij}/\rho_j$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Beta Model of Asset Returns

Consequently, we have the instantaneous regression

$$\frac{dY_{it}}{Y_{it}} = \alpha_i dt + \sum_{j=1}^{J} \beta_{ij} \frac{dX_{jt}}{X_{jt}} + dU_{it} \tag{4}$$

with $\alpha_i = 0$ and the error differential

$$dU_{it} = \sigma_t \sum_{k=1}^{K} \lambda_{ik} dW_{kt} + \omega_{it} dZ_{it}, \tag{5}$$

if we define

$$\frac{dY_{it}}{Y_{it}} = \frac{dP_{it}}{P_{it}} - r_t^f dt, \qquad \frac{dX_{jt}}{X_{jt}} = \frac{dQ_{jt}}{Q_{jt}} - r_t^f dt,$$

for $i = 1, \ldots, I$ and $j = 1, ...J$.

We call $\alpha = (\alpha_1, \ldots, \alpha_I)'$ the pricing error, and testing for $\alpha = 0$ has been a focal point of empirical asset pricing literature.

Regression Formulation

Regressions

Our model (4) is originally formulated as an instantaneous regression, where both the regressand and regressors are measured over an infinitesimal time interval. The actual regressions are estimated using observations that are collected over the intervals defined by

$$0 \equiv T_0 < T_1 < \cdots < T_N \equiv T \tag{6}$$

over the time interval $[0, T]$. The corresponding regression model becomes

$$\int_{T_{n-1}}^{T_n} \frac{dY_{it}}{Y_{it}} = \alpha_i (T_n - T_{n-1}) + \sum_{j=1}^{J} \beta_{ij} \int_{T_{n-1}}^{T_n} \frac{dX_{jt}}{X_{jt}} + (U_{iT_n} - U_{iT_{n-1}}) \tag{7}$$

for $n = 1, \ldots, N$.

Fixed and Random Sampling Schemes

We consider two sampling schemes to obtain estimable regressions, i.e., the fixed and random sampling schemes.

- For the fixed sampling scheme, we set $T_n - T_{n-1}$ to be nonrandom and constant for all $n = 1, \ldots, N$, like a month or a year.
- For the random sampling scheme, we define (T_n) to be a non-decreasing sequence of stopping times or a time change. In particular, we will use the time change given by the volatility process σ in the common volatility factor. As discussed, σ is the volatility process of the market factor introduced below (3).

More on Random Sampling Scheme

Let the instantaneous market excess return be given by

$$dS_t = \frac{dX_{1t}}{X_{1t}} = \frac{dQ_{1t}}{Q_{1t}} - r_t^f dt, \tag{8}$$

and define the time change (T_n) as

$$[S]_{T_n} - [S]_{T_{n-1}} = \int_{T_{n-1}}^{T_n} \sigma_t^2 dt = \Delta \tag{9}$$

for $n = 1, \ldots, N$, where Δ is a fixed constant. This compares with the corresponding fixed sampling scheme (T_n) given by

$$T_n = (n/N)T$$

for $n = 1, \ldots, N$. In particular, if we set

$$T_n = n\Delta = (n/N)[S]_T,$$

then the random sampling scheme yields the same number of observations as the fixed sampling scheme for regression (7).

Motivation for Time Change

It is well known that the market volatility σ is nonstationary with an autoregressive root very close to unity. Also, their leverage effect on the market excess return is quite strongly negative. See Jacquier, Polson and Rossi (1994, 2004) and Kim, Lee and Park (2009). In this situation, the usual LLN and CLT do not hold and hence the usual chi-square tests for inference in regression (7) are invalid. This poses a serious problem in analyzing Fama-French regressions.

Under the random sampling scheme, however, we have

$$\int_{T_{n-1}}^{T_n} \sigma_t dW_{kt} =_d \mathbb{N}(0, \Delta) \tag{10}$$

for all $n = 1, \ldots, N$ and all $k = 1, \ldots, K$, and that they are independent each other. This is due to a theorem by Dambis, Dubins and Schwarz (DDS).

Endogeneity and Persistency of Volatility

The DDS time change allows for very general form of volatilities, which may in particular be endogenous and persistent. It is important that we use samples collected at random intervals. For samples at fixed intervals, we have

$$\int_{T_{n-1}}^{T_n} \sigma_t dW_{kt} \sim \mathbb{MN}\left(0, \int_{T_{n-1}}^{T_n} \sigma_t^2 dt\right),$$

when and only when the volatility σ is independent of W. Moreover, the sequence of conditional variances

$$\int_{T_{n-1}}^{T_n} \sigma_t^2 dt$$

may well be strongly persistent, so that the standard LLN and CLT may not apply.

Idiosyncratic Error Component

Under Assumption 2.1, $(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it})$ is independent across i and has variance

$$\mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it}\right)^2 = \mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it}^2 dt\right), \tag{11}$$

for each $i = 1, \ldots, I$.

Assumption 2.2 For all $i = 1, \ldots, I$,

$$\frac{1}{N} \sum_{n=1}^{N} \int_{T_{n-1}}^{T_n} \omega_{it}^2 dt \to_p \varpi_i^2,$$

as $N \to \infty$, for some $\varpi_i^2 > 0$.

Assumption 2.2 requires that a LLN hold for the idiosyncratic error, which is satisfied widely if (ω_i) is stationary.

Error Variance Estimation

Under the random sampling scheme, we have

$$\mathbb{E}(U_{iT_n} - U_{iT_{n-1}})^2 = \Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it}^2 dt\right)$$

$$\mathbb{E}(U_{iT_n} - U_{iT_{n-1}})(U_{jT_n} - U_{jT_{n-1}}) = \Delta \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk}$$

for all $0 \leq i \leq I$ and $0 \leq i \neq j \leq I$. Therefore, if we define $U_{T_n} - U_{T_{n-1}} = (U_{1T_n} - U_{1T_{n-1}}, \ldots, U_{IT_n} - U_{IT_{n-1}})'$, then we would expect to have

$$\mathbb{E}(U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})' \approx \Sigma$$

asymptotically, where Σ is a matrix with the i-th diagonal entry $\Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \varpi_i^2$ and (i,j)-th off-diagonal entry $\Delta \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk}$. Subsequently, we call Σ the asymptotic error covariance matrix for our regression (7).

Conventional Estimator

As in the conventional approach, we may estimate Σ by

$$\hat{\Sigma} = \frac{1}{N}\sum_{n=1}^{N}(U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})'. \tag{12}$$

Clearly, we have $\hat{\Sigma} \to_p \Sigma$ as $N \to \infty$, if we assume some extra regularity conditions to ensure that

$$\frac{1}{N}\sum_{n=1}^{N}\left(\int_{T_{n-1}}^{T_n}\omega_{it}dZ_{it}\right)^2 \to_p \varpi_i^2, \; \frac{1}{N}\sum_{n=1}^{N}\left(\int_{T_{n-1}}^{T_n}\omega_{it}dZ_{it}\right)\left(\int_{T_{n-1}}^{T_n}\omega_{jt}dZ_{jt}\right) \to_p 0 \tag{13}$$

for all i and for all $i \neq j$. It is easy to see that (13) holds under appropriate assumptions, due in particular to (11) and Assumption 2.2, and the independence of $(\int_{T_{n-1}}^{T_n}\omega_{it}dZ_{it})$ across $i = 1, \ldots, I$.

New Estimator

We may estimate the asymptotic error covariance matrix Σ using

$$\tilde{\Sigma} = \frac{1}{N} \int_0^T [U, U']_t \, dt, \tag{14}$$

where $[U, U']$ is the matrix of quadratic variations and covariations of $U = (U_1, \ldots, U_I)'$. Note that

$$[U_i]_{T_n} - [U_i]_{T_{n-1}} = \Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \int_{T_{n-1}}^{T_n} \omega_{it}^2 dt$$

$$[U_i, U_j]_{T_n} - [U_i, U_j]_{T_{n-1}} = \Delta \sum_{k=1}^{K} \lambda_{ik} \lambda_{jk}$$

for all $1 \leq i \leq I$ and $1 \leq i \neq j \leq I$. Therefore, we have $\tilde{\Sigma} \to_p \Sigma$ as $N \to \infty$, which holds without any extra regularity conditions.

Validity of OLS

Due to Assumption 2.1, the usual exogeneity condition for the regressors in (7) holds and the OLS procedure is valid for regression (7) for both random and fixed sampling schemes. To see this, let

$$\mathcal{F}_n = \sigma\Big((U_{it}, i = 1, \ldots, I, \; t \leq T_n), (X_{jt}, j = 1, \ldots, J, \; t \leq T_{n+1}) \Big),$$

$n = 1, \ldots, N$, for our fixed or random sampling scheme (T_n). Then we may easily see that the regressors $(\int_{T_{n-1}}^{T_n} dX_{jt}/X_{jt})$, $j = 1, \ldots, J$, are all $\mathcal{F}_{n-1}$-measurable, and the regression errors $(U_{iT_n} - U_{iT_{n-1}})$ satisfy the orthogonality condition

$$\mathbb{E}\left[(U_{iT_n} - U_{iT_{n-1}}) \Big| \mathcal{F}_{n-1} \right] = 0$$

for $i = 1, \ldots, I$, as required for the validity of the OLS regression in (7).

Statistical Procedure and Theory

Regression Model

To develop the actual statistical procedure, rewrite our model (7) in continuous time as a more conventional regression as

$$y_{ni} = \alpha_i c_n + \sum_{j=1}^{J} \beta_{ij} x_{nj} + u_{ni}, \tag{15}$$

where

$$\begin{aligned} y_{ni} &= \int_{T_{n-1}}^{T_n} \frac{dY_{it}}{Y_{it}}, \qquad & c_n &= T_n - T_{n-1}, \\ x_{nj} &= \int_{T_{n-1}}^{T_n} \frac{dX_{jt}}{X_{jt}}, \qquad & u_{ni} &= U_{iT_n} - U_{iT_{n-1}} \end{aligned} \tag{16}$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, I$.

Regression Model

$$y_{ni} = \alpha_i c_n + \sum_{j=1}^{J} \beta_{ij} x_{nj} + u_{ni},$$

Under Assumptions 2.1 and 2.2, our choice of random sampling time (T_n) yields a regression model with errors, which are devoid of endogenous nonstationarity in volatility and have asymptotically stationary volatilities. Note in particular that the regression errors (u_n), $u_n = (u_{n1}, \ldots, u_{nI})'$, are approximately multivariate normal with mild heterogeneity. Let $y_n = (y_{n1}, \ldots, y_{nI})'$ and $x_n = (x_{n1}, \ldots, x_{nJ})'$.

Assumption 3.1 $N^{-1} \sum_{n=1}^{N} x_n x_n' \to_p \Lambda > 0$ and $N^{-1/2} \sum_{n=1}^{N} x_n u_n' \to_d \mathbb{N}(0, \Lambda \otimes \Sigma)$, as $N \to \infty$.

Assumption 3.1 is necessary for all regression asymptotics, and holds under very general conditions. Of course, (y_n) and (x_n) are not directly observable, and have to be estimated from discrete observations.

Observations

We assume that a sample providing observations for

$$(Y_{i,m\delta}, X_{j,m\delta}) \tag{17}$$

is available for $m = 0, \ldots, M$ with δ-interval in time, for each of $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Moreover, from $(X_{1,m\delta})$ we obtain the observations $(S_{m\delta})$ for the excess market return process S introduced in (8) as

$$S_{m\delta} = \frac{X_{1,m\delta} - X_{1,(m-1)\delta}}{X_{1,(m-1)\delta}} = \frac{Q_{1,m\delta} - Q_{1,(m-1)\delta}}{Q_{1,(m-1)\delta}} - r^f_{(m-1)\delta}\,\delta$$

for $m = 1, \ldots, M$. We let $M\delta = T$, so that T is the horizon of the sample with size M collected at δ-interval in time. Our subsequent procedure is based on the asymptotic theory requiring $\delta \to 0$. Therefore, δ should be small.

Estimation of Time Change

To implement our random time approach based on regression (15), we need to estimate the time change (T_n). If δ is small, we may estimate the quadratic variation $[S]$ of the excess market return process S using $(S_{m\delta})$. Indeed, if we set

$$[S]_t^\delta = \sum_{m\delta \leq t} (S_{m\delta} - S_{(m-1)\delta})^2,$$

then we may expect $[S]^\delta \approx [S]$ for δ small. Once, we obtain an estimate $[S]^\delta$ of $[S]$, the corresponding estimate of the time change (T_n) may easily be obtained, accordingly as in (9), for a prescribed value of Δ. We propose the estimate (T_n^δ) of (T_n), which is given by

$$T_n^\delta = \delta \operatorname*{argmin}_{1 \leq \ell \leq M} \left| \sum_{m=1}^{\ell} (S_{m\delta} - S_{(m-1)\delta})^2 - n\Delta \right| \tag{18}$$

for each $n = 1, \ldots, N$.

Regression Constructed from Discrete Samples

Let $M_n = \delta^{-1}T_n^\delta$ for random time and $M_n = \delta^{-1}T_n = \delta^{-1}(n/N)T$ for fixed time sampling scheme, and define

$$y_{ni}^\delta = \sum_{m=M_{n-1}+1}^{M_n} \frac{Y_{i,m\delta} - Y_{i,(m-1)\delta}}{Y_{i,(m-1)\delta}}, \qquad c_n^\delta = T_n^\delta - T_{n-1}^\delta,$$

$$x_{nj}^\delta = \sum_{m=M_{n-1}+1}^{M_n} \frac{X_{j,m\delta} - X_{j,(m-1)\delta}}{X_{j,(m-1)\delta}}, \tag{19}$$

correspondingly as (16), and consider

$$y_{ni}^\delta = \alpha_i c_n^\delta + \sum_{j=1}^{J} \beta_{ij} x_{nj}^\delta + u_{ni}^\delta \tag{20}$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, I$. Note that we have a sample of size N to fit regression (20), which is formulated using a sample of size M in (17) with $M > N$. We call the latter *the original sample*, and the former *the regression sample*.

Asymptotic Theory

We need to introduce some technical conditions to ensure that the regression (20) constructed from discrete samples is asymptotically equivalent to our original regression (15) in continuous time.

Assumption 3.2 We let (a) $a_T(t-s) \leq \int_s^t \sigma_u^2 du \leq b_T(t-s)$ for all $0 \leq s \leq t \leq T$ with (a_T) and (b_T) depending only upon T, (b) $\sup_{t\geq 0} |\nu_{jt} - r_t^f| = O_p(1)$ for all $j = 1, \ldots, J$, (c) $\inf_t X_{jt} > 0$ and $\sup_{0\leq t\leq T} X_{jt} = O_p(c_T)$ for all $j = 1, \ldots, J$ with (c_T) depending only upon T, and (d) $\sup_{t\geq 0} \omega_{it} = O_p(1)$ for all $i = 1, \ldots, I$. Furthermore, we set (e) $\delta = O(T^{-4-\varepsilon}(a_T^2/b_T^7 c_T^4))$ for some $\varepsilon > 0$.

Theorem 3.1 Under Assumption 3.2, we have

$$\max_{1\leq n\leq N} |c_n^\delta - c_n|, \max_{1\leq n\leq N} |x_{nj}^\delta - x_{nj}|, \max_{1\leq n\leq N} |u_{ni}^\delta - u_{ni}|, \max_{1\leq n\leq N} |y_{ni}^\delta - y_{ni}| = o_p(N^{-1/2})$$

as $N \to \infty$, for all $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Error Variance Estimate

We need to estimate the variance of the error term for the test of a joint hypothesis involving multiple regression coefficients across $i = 1, \ldots, I$. For the estimation of error variance in regression (20), we may follow the usual two step procedure: In the first step, we estimate (α_i) and (β_{ij}) for each i by the single equation method. Then in the second step, use the fitted residuals to estimate the error variance Σ by

$$\hat{\Sigma}^\delta = \frac{1}{N} \sum_{n=1}^{N} \hat{u}_n^\delta \hat{u}_n^{\delta\prime}, \tag{21}$$

where $\hat{u}_n^\delta = (\hat{u}_{n1}^\delta, \ldots, \hat{u}_{nI}^\delta)'$ with $(\hat{u}_{ni}^\delta)$ being the fitted residual from regression (20).

The error variance estimate $\hat{\Sigma}^\delta$ is expected to behave well only when $N >> I$, i.e., the size of regression sample is substantially bigger than the number of cross sectional units.

New Variance Estimator Using Original Sample

This estimator is robust and useful especially when N is small relative to I. It is indeed well defined even if $N < I$, as long as the size M of the original sample is large enough. To introduce the estimator explicitly, let $(\hat{\alpha}_i)$ and $(\hat{\beta}_{ij})$ be the OLS estimators of (α_i) and (β_{ij}) obtained from (20) and define

$$\hat{U}_{i,m\delta} - \hat{U}_{i,(m-1)\delta} = \frac{Y_{i,m\delta} - Y_{i,(m-1)\delta}}{Y_{i,(m-1)\delta}} - \hat{\alpha}_i - \sum_{j=1}^{J} \hat{\beta}_{ij} \frac{X_{j,m\delta} - X_{j,(m-1)\delta}}{X_{j,(m-1)\delta}}$$

and

$$\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta} = \left(\hat{U}_{1,m\delta} - \hat{U}_{1,(m-1)\delta}, \ldots, \hat{U}_{I,m\delta} - \hat{U}_{I,(m-1)\delta}\right)'.$$

Then the asymptotic error variance of regression (20) can be estimated by

$$\tilde{\Sigma}^{\delta} = \frac{1}{N} \sum_{m=1}^{M} \left(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta}\right)\left(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta}\right)'. \qquad (22)$$

Asymptotics for Error Variance Estimators

From Theorem 3.1, it is well expected that

Corollary 3.2 We have

$$\hat{\Sigma}^{\delta} = \hat{\Sigma} + O_p(N^{-1/2}), \qquad \tilde{\Sigma}^{\delta} = \tilde{\Sigma} + O_p(N^{-1/2})$$

for all large N, where

$$\hat{\Sigma} = \frac{1}{N}\sum_{n=1}^{N}(U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})' \to_p \Sigma$$

$$\tilde{\Sigma} = \frac{1}{N}\int_0^T [U, U']_t \, dt \to_p \Sigma.$$

Wald Test

In CAPM and Fama-French regressions, it is one of the main interests to test for the hypothesis

$$\mathbb{H}_0 : \alpha_1 = \cdots = \alpha_I = 0.$$

The rejection of the hypothesis implies that the proposed model is not a true model and presumably requires a new factor.

The Wald test $\tau(\alpha)$ for the hypothesis can be easily formulated in our model (20), which is defined by

$$\tau(\alpha) = \left(c'c - c'X(X'X)^{-1}X'c\right)\hat{\alpha}'\bar{\Sigma}^{-1}\hat{\alpha}, \tag{23}$$

where c is an N-dimensional vector with c_n^δ as its n-th component and X is an $N \times J$ matrix with x_{nj}^δ as its (n, j)-th element, and $\bar{\Sigma} = \hat{\Sigma}$ or $\tilde{\Sigma}$. The test statistic $\tau(\alpha)$ has chi-square limit distribution with I-degrees of freedom.

Jumps

Our theoretical development thus far assumes that the error process is given by a process with a continuous sample path a.s. However, we may easily accommodate the presence of discontinuities in sample paths due to jumps, as we implement our methodology. We may first use a test by, e.g., Lee and Mykland (2008), to find the locations of jumps. Once we find their locations, we may readily identify the sampling intervals $[T_{n-1}^{\delta}, T_n^{\delta}]$ to which they belong, and simply discard the corresponding regression samples.

It is also possible that we test for the presence of jumps in each of the time intervals $[T_{n-1}^{\delta}, T_n^{\delta}]$, $n = 1, \ldots, N$, using the test developed by, e.g., Barndorff-Nielson and Shepard (2004), and delete the regression samples from any of the time intervals which are tested positive. This procedure, however, makes sense only when enough number of the original samples exist in all of the time intervals.

Data and Preliminary Analysis

Data

- We make use of decile portfolios stratified by sizes, book-to-market ratios (B/M), and past performances. We also use 25 portfolios sorted by sizes and B/M and 30 industry portfolios.
- All the data sets are obtained from Kenneth French's web page ¡http://mba.tuck.darthmouth.edu/pages/faculty/ken.french¿
- For pricing factors, we adopt the market (MKT), the size (SMB), and the B/M (HML), often referred to as the Fama-French factors, and the momentum factor (MMT).
- The data sets cover the period of July, 1963 and December, 2008, and all of the returns in the data sets are of the daily frequency and annualized.

Preliminary Analysis

Our factor pricing model specified in (4) and (5) imposes some special error structure in the Fama-French regressions, which motivated us to invent a new methodology. Before we reexamine the Fama-French regressions using our methodology, it is therefore necessary that we investigate whether various specifications of our model are empirically justifiable.

To obtain empirical evidence, we consider the conventional 3-factor Fama-French regression which uses 25 portfolio returns sorted by size and book-to-market ratio as regressands and Fama-French factors (MKT, SMB, HML) as regressors.

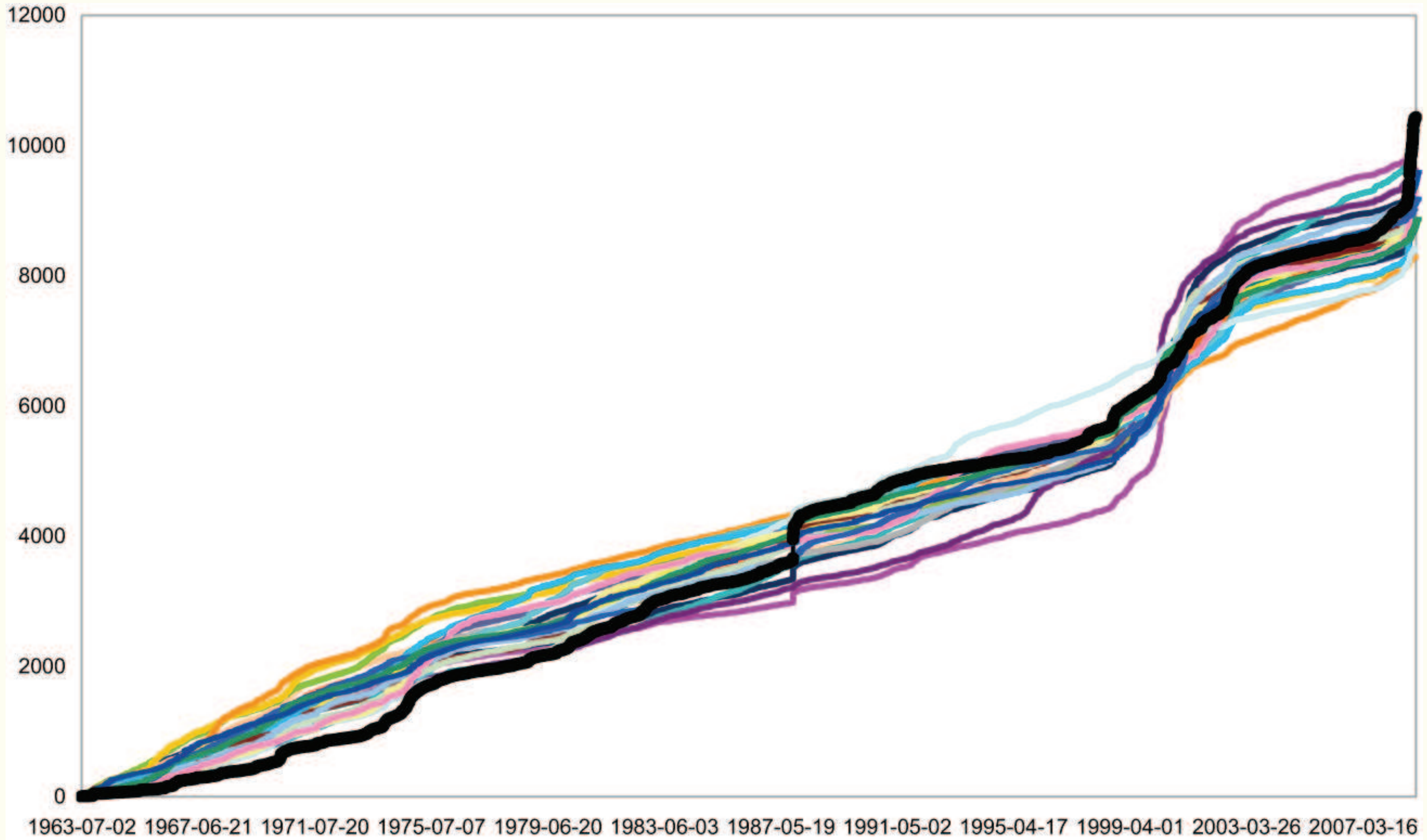
Cross-Correlations of Errors

Table : Test of Diagonality of Variance-Covariance Matrix

Model	LM Test	p-Value
Fixed Time Regression	15185.9766	0.0000
Random Time Regression	6974.5703	0.0000

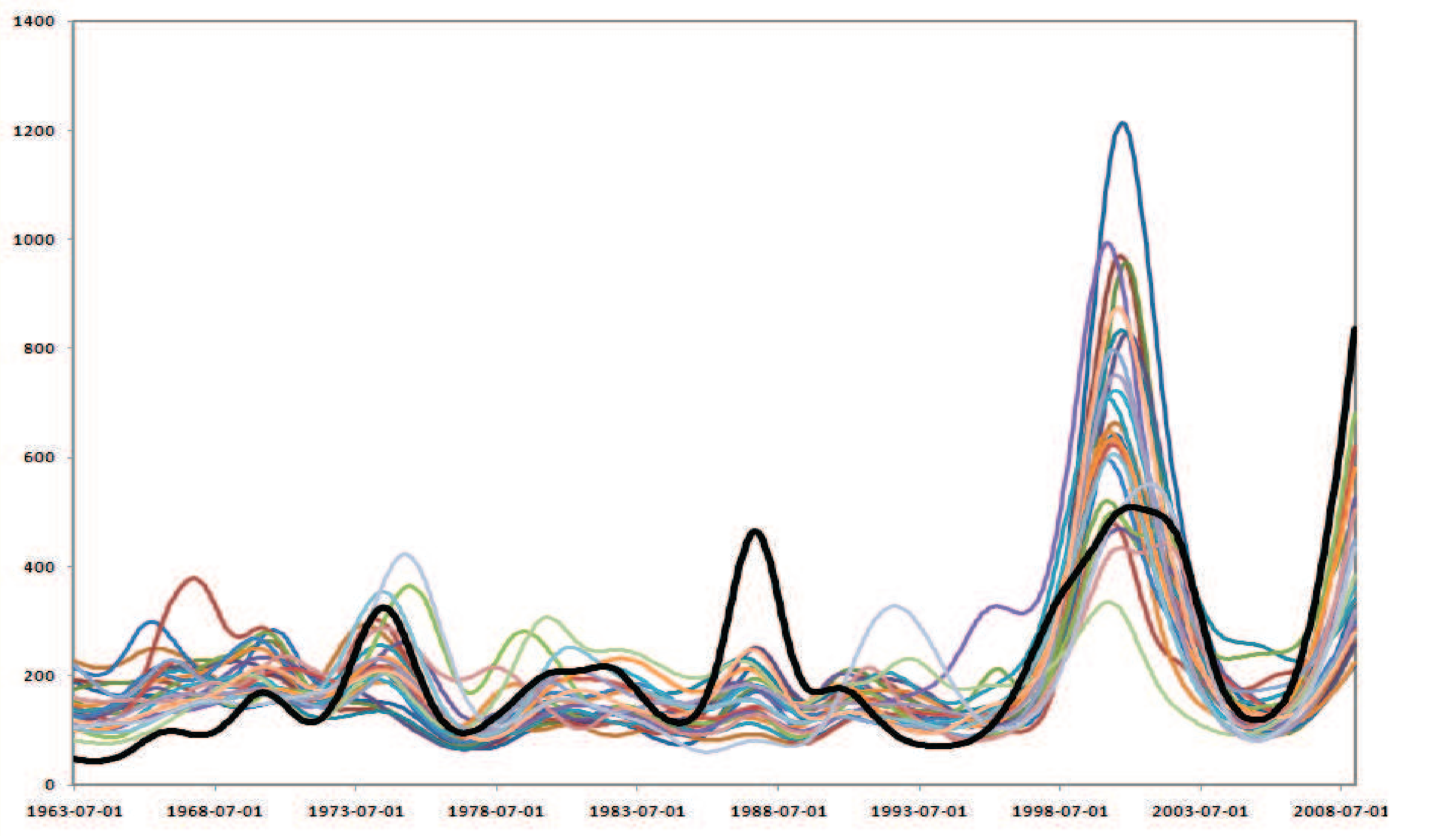
Common Volatility Component

Figure : Quadratic Variations of Market Return and Fitted Residuals



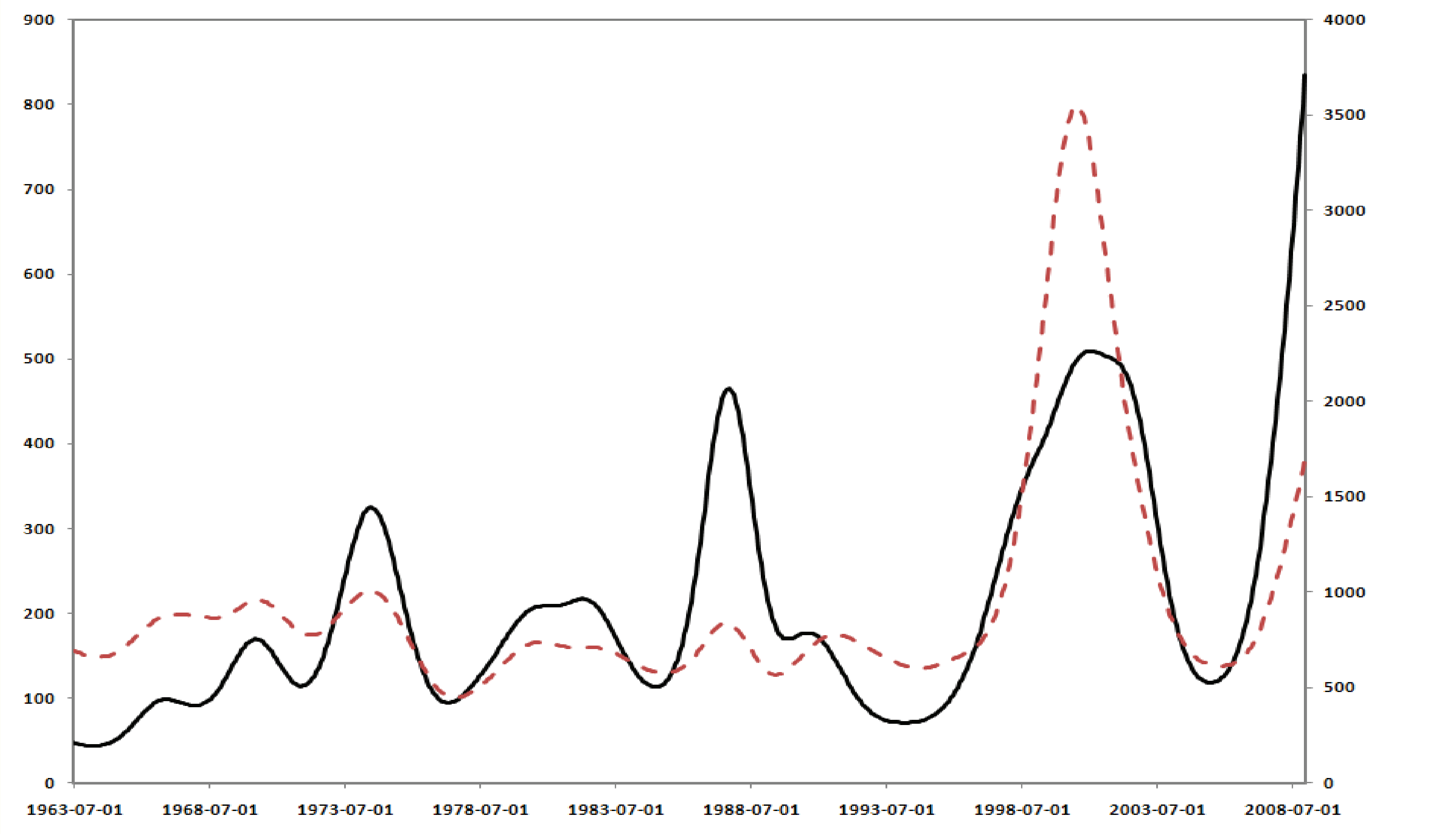
Common Volatility Component

Figure : Instantaneous Variances of Market Return and Fitted Residuals



Common Volatility Component

Figure : Instantaneous Variances of Market Return and Leading Factor of Fitted Residuals



Stochastic Volatility Model

To empirically measure the degree of persistency in the volatilities of the regression errors in (15) we fit a stochastic volatility model

$$u_{ni} = \sqrt{f_i(v_{ni})}\,\varepsilon_{ni}$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, I$ with $\mathbb{E}\varepsilon_{ni}^2 = 1$, where (v_{ni}) is the latent volatility factor generated as

$$v_{ni} = \rho_i v_{n-1,i} + \eta_{ni},$$

with $(\eta_{ni}) \sim$ iid $\mathbb{N}(0,1)$, and (f_i) is the volatility function. We use the logistic function for the volatility function f_i, and allow for nonzero correlation between (ε_{ni}) and (η_{ni}), which represents the leverage effect. The stochastic volatility model is fitted for each i using the fitted residuals from regression (20) based on the fixed sampling scheme, and extract the latent volatility factor using the conventional density-based Kalman filter method.

Figure : Extracted Volatility Factors for Fitted Residuals

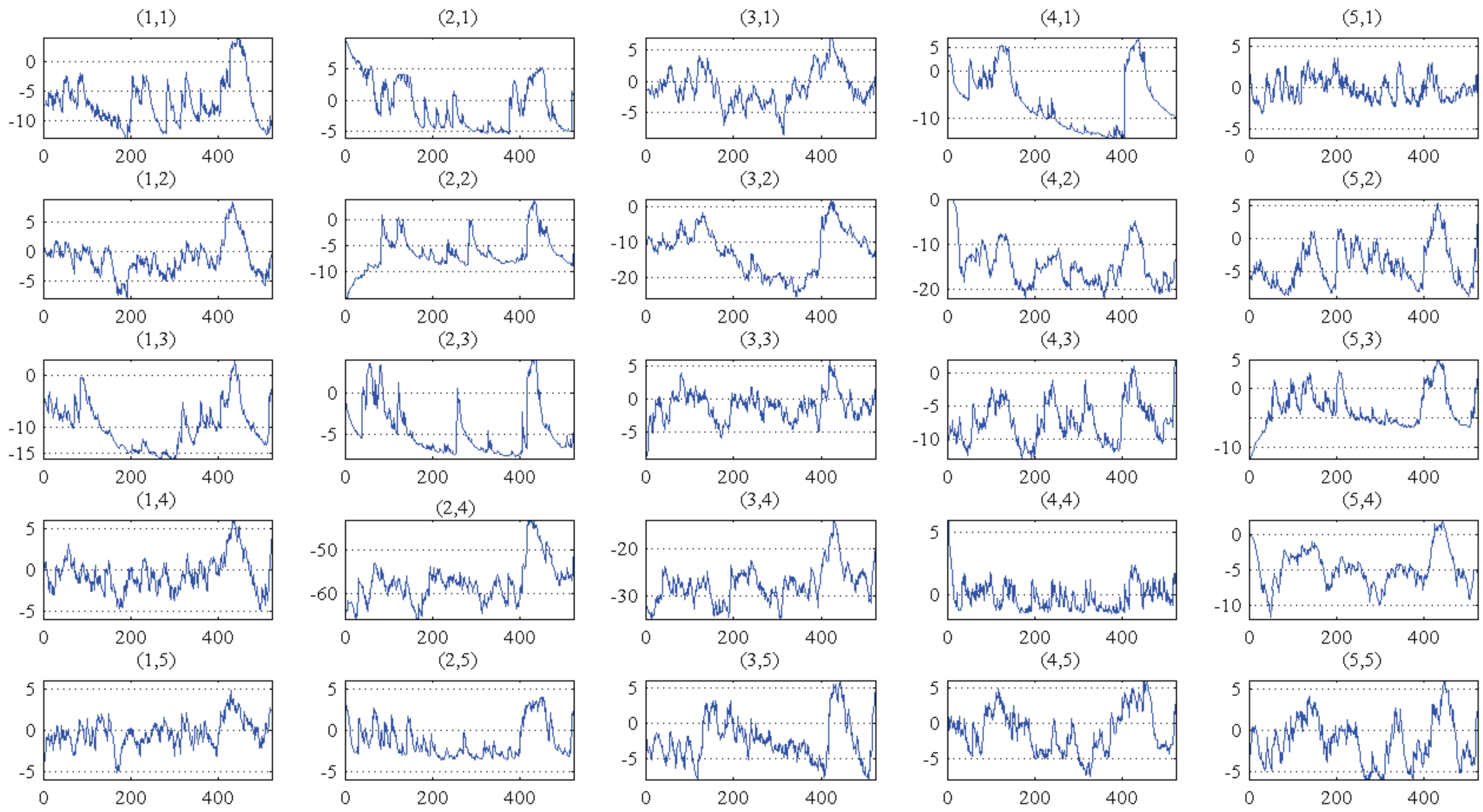


Table : AR Coefficients of Extracted Volatility Factors

(Size,B/M)	AR Coefficients
(1,1)	0.980 (0.020)
(1,2)	0.970 (0.024)
(1,3)	0.961 (0.029)
(1,4)	0.994 (0.007)
(1,5)	0.928 (0.054)
(2,1)	0.954 (0.028)
(2,2)	0.980 (0.012)
(2,3)	0.992 (0.009)
(2,4)	0.972 (0.039)
(2,5)	0.970 (0.001)
(3,1)	0.969 (0.024)
(3,2)	0.989 (0.010)
(3,3)	0.949 (0.035)
(3,4)	0.981 (0.012)
(3,5)	0.969 (0.017)
(4,1)	0.989 (0.015)
(4,2)	0.992 (0.023)
(4,3)	0.974 (0.013)
(4,4)	0.986 (0.016)
(4,5)	0.970 (0.020)
(5,1)	0.944 (0.034)
(5,2)	0.956 (0.029)
(5,3)	0.980 (0.012)
(5,4)	0.962 (0.027)
(5,5)	0.964 (0.021)
Average	0.971 (0.021)

Figure : Extracted Volatility Factor from Excess Market Return

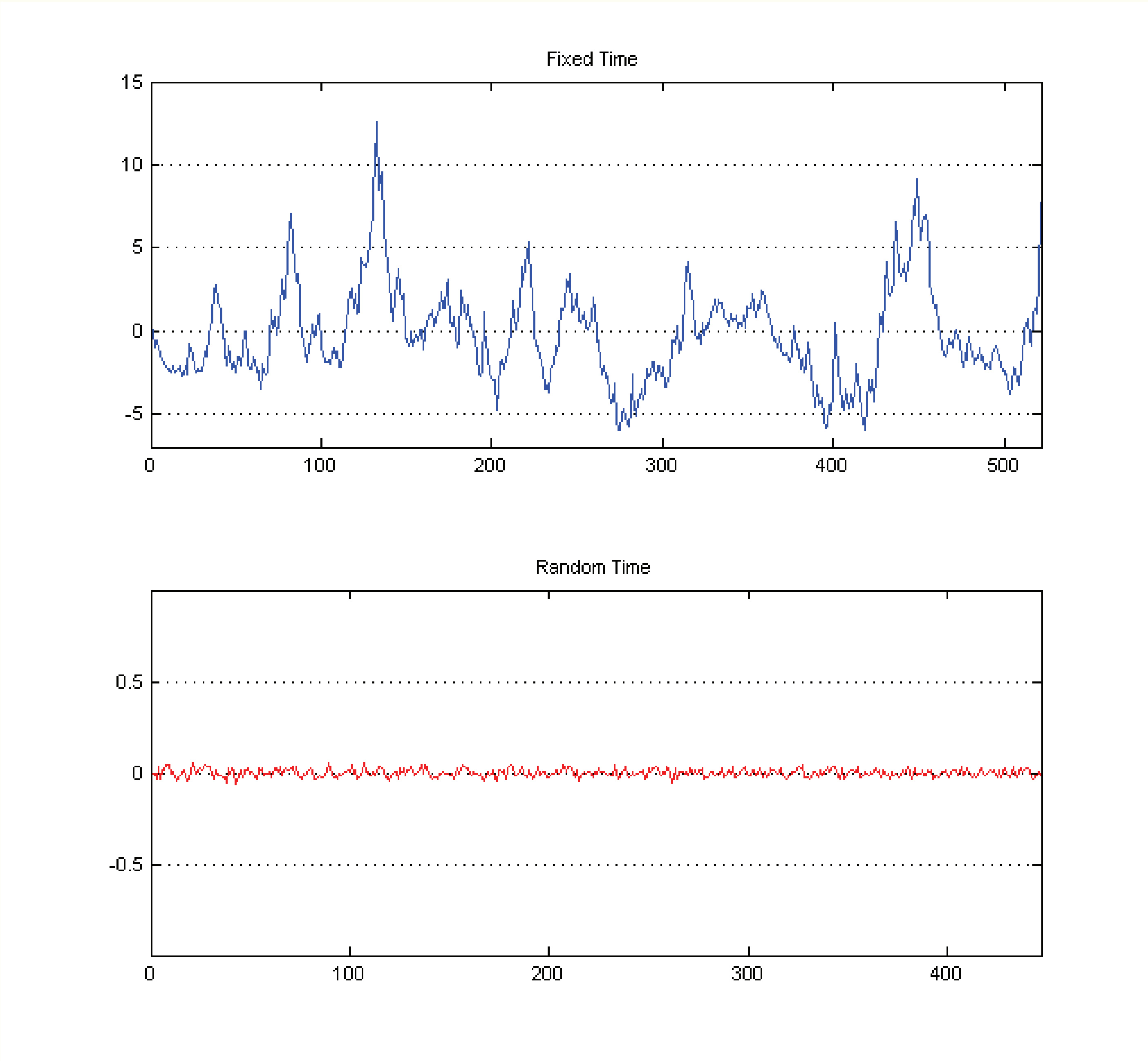


Table : Correlations between Random Time Residuals and Regressors

	Correlation Coefficients				|Correlation Coefficients|			
(Size,B/M)	Alpha	MKT	SMB	HML	Alpha	MKT	SMB	HML
(1,1)	0.0686	-0.0713	0.1501	-0.0028	0.0686	0.0713	0.1501	0.0028
(1,2)	-0.0151	-0.0377	0.0717	0.0100	0.0151	0.0377	0.0717	0.0100
(1,3)	-0.1221	-0.0473	0.0295	-0.0566	0.1221	0.0473	0.0295	0.0566
(1,4)	0.0006	0.0343	0.1107	-0.0912	0.0006	0.0343	0.1107	0.0912
(1,5)	0.1674	0.0352	0.1616	-0.0580	0.1674	0.0352	0.1616	0.0580
(2,1)	-0.0567	-0.0301	0.0746	0.0378	0.0567	0.0301	0.0746	0.0378
(2,2)	-0.0793	0.0464	0.0524	-0.0836	0.0793	0.0464	0.0524	0.0836
(2,3)	0.0469	0.0660	-0.0220	-0.0643	0.0469	0.0660	0.0220	0.0643
(2,4)	0.0075	-0.0200	-0.0240	-0.0415	0.0075	0.0200	0.0240	0.0415
(2,5)	-0.1031	0.0467	-0.1434	0.0123	0.1031	0.0467	0.1434	0.0123
(3,1)	-0.0250	0.0851	0.0136	-0.0036	0.0250	0.0851	0.0136	0.0036
(3,2)	0.0425	0.0548	0.1423	-0.0853	0.0425	0.0548	0.1423	0.0853
(3,3)	0.0787	0.0205	0.1066	-0.0300	0.0787	0.0205	0.1066	0.0300
(3,4)	0.0553	0.0813	0.1083	-0.1135	0.0553	0.0813	0.1083	0.1135
(3,5)	-0.0834	0.0793	0.0645	-0.0490	0.0834	0.0793	0.0645	0.0490
(4,1)	-0.0949	0.0360	-0.0039	-0.0520	0.0949	0.0360	0.0039	0.0520
(4,2)	-0.0856	-0.0431	0.0324	-0.0800	0.0856	0.0431	0.0324	0.0800
(4,3)	0.0560	-0.0409	0.1215	-0.0588	0.0560	0.0409	0.1215	0.0588
(4,4)	0.0103	-0.0409	0.0511	-0.0135	0.0103	0.0409	0.0511	0.0135
(4,5)	0.0498	0.0495	0.1390	0.0107	0.0498	0.0495	0.1390	0.0107
(5,1)	0.0834	0.0949	0.0626	-0.1246	0.0834	0.0949	0.0626	0.1246
(5,2)	-0.1344	-0.0034	-0.0817	0.0398	0.1344	0.0034	0.0817	0.0398
(5,3)	0.0575	0.0040	0.0059	0.0939	0.0575	0.0040	0.0059	0.0939
(5,4)	-0.0225	-0.0675	0.0763	-0.0332	0.0225	0.0675	0.0763	0.0332
(5,5)	0.0547	0.0909	0.0543	-0.0616	0.0547	0.0909	0.0543	0.0616
Mean	-0.0017	0.0169	0.0542	-0.0359	0.0640	0.0491	0.0762	0.0523
Stdev	0.0771	0.0534	0.0731	0.0518	0.0409	0.0252	0.0486	0.0343

Reexamination of Fama-French Regressions

CAPM Anomaly

- Estimate and test the CAPM on five data sets of daily equity returns
- Use both the conventional fixed sampling and the newly proposed random sampling schemes.
- Five data sets are used: (i) three data sets consisting of 10 portfolio returns sorted by size, book-to-market ratio, and momentum respectively, (ii) 30 industry portfolios, and (iii) 25 portfolios sorted by (size,book-to-market ratio).

Table : Summary Statistics of Factors and Portfolio Returns

Panel A: Factors

			Correlations			
	Mean	Stdev	MKT	SMB	HML	MMT
MKT	0.0446	0.1515	1.0000	-0.2200	-0.4302	-0.0644
SMB	0.0152	0.0798	-0.2200	1.0000	-0.0623	0.0667
HML	0.0511	0.0746	-0.4302	-0.0623	1.0000	-0.1215
MMT	0.0985	0.1025	-0.0644	0.0667	-0.1215	1.0000

Panel B: Decile Portfolio Returns

Size	Mean	Stdev	B/M	Mean	Stdev	Momentum	Mean	Stdev
1 Small	0.0562	0.1267	1 Growth	0.0316	0.1779	1 Losers	-0.0584	0.2321
2	0.0559	0.1544	2	0.0449	0.1619	2	0.0068	0.1913
3	0.0646	0.1558	3	0.0499	0.1545	3	0.0382	0.1669
4	0.0619	0.1551	4	0.0499	0.1542	4	0.0389	0.1594
5	0.0656	0.1546	5	0.0477	0.1521	5	0.0295	0.1529
6	0.0581	0.1478	6	0.0572	0.1447	6	0.0448	0.1493
7	0.0595	0.1493	7	0.0665	0.1436	7	0.0450	0.1493
8	0.0561	0.1528	8	0.0724	0.1486	8	0.0757	0.1529
9	0.0507	0.1510	9	0.0831	0.1512	9	0.0685	0.1631
10 Big	0.0406	0.1608	10 Value	0.0878	0.1631	10 Winners	0.1240	0.2044
1-10	0.0156	0.1230	10-1	0.0562	0.1220	10-1	0.1824	0.1906

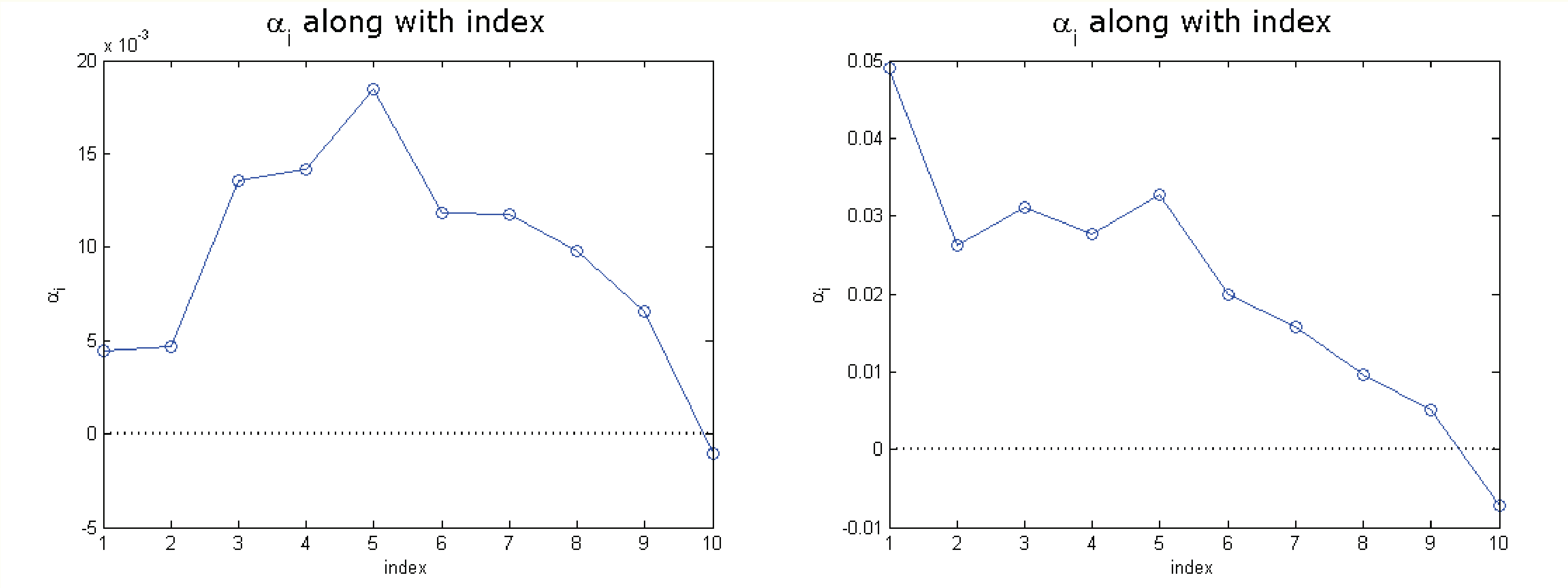
Table : Test of CAPM on Size Portfolios

	Fixed Time		Random Time	
Size	Alpha	Beta	Alpha	Beta
1 Small	0.0045	1.0772	0.0489	1.0215
	(0.0205)	(0.0394)	(0.0173)	(0.0419)
2	0.0047	1.1564	0.0262	1.1400
	(0.0170)	(0.0326)	(0.0142)	(0.0344)
3	0.0135	1.1536	0.0311	1.1364
	(0.0145)	(0.0279)	(0.0120)	(0.0292)
4	0.0141	1.1231	0.0277	1.1221
	(0.0135)	(0.0259)	(0.0108)	(0.0262)
5	0.0185	1.1048	0.0327	1.1047
	(0.0113)	(0.0218)	(0.0095)	(0.0232)
6	0.0118	1.0789	0.0198	1.0793
	(0.0097)	(0.0185)	(0.0082)	(0.0198)
7	0.0118	1.0798	0.0158	1.0619
	(0.0081)	(0.0156)	(0.0066)	(0.0161)
8	0.0098	1.0695	0.0096	1.0516
	(0.0070)	(0.0134)	(0.0059)	(0.0142)
9	0.0065	0.993	0.0051	0.9851
	(0.0057)	(0.0109)	(0.0046)	(0.0112)
10 Big	-0.0010	0.9217	-0.0073	0.9352
	(0.0054)	(0.0104)	(0.0045)	(0.0108)
1-10 Size Strategy	0.0055	0.1555	0.0562	0.0863
	(0.0250)	(0.0480)	(0.0211)	(0.0512)
Wald	12.2747 (0.2671)		54.6970 (0.0000)	

- Table reports results for the decile size portfolios, and the size strategy (1st$-$10th decile) portfolio.
- Beta estimates in both sampling schemes are close to each other.
- MKT mildly captures exposures to taking risks for small firms (i.e., beta is higher for small firms).
- However, comparing the alpha estimates, one can clearly see that there is a huge difference between the fixed sampling and the random sampling schemes.
- There exists a significant risk component not captured by the market factor according to small firms' alpha estimates in the random sampling case, whereas the fixed sampling result is much weaker.
- Take a look at the estimated alphas in the next slide.

Figure : Alphas of Size Portfolios

Fixed Time **Random Time**



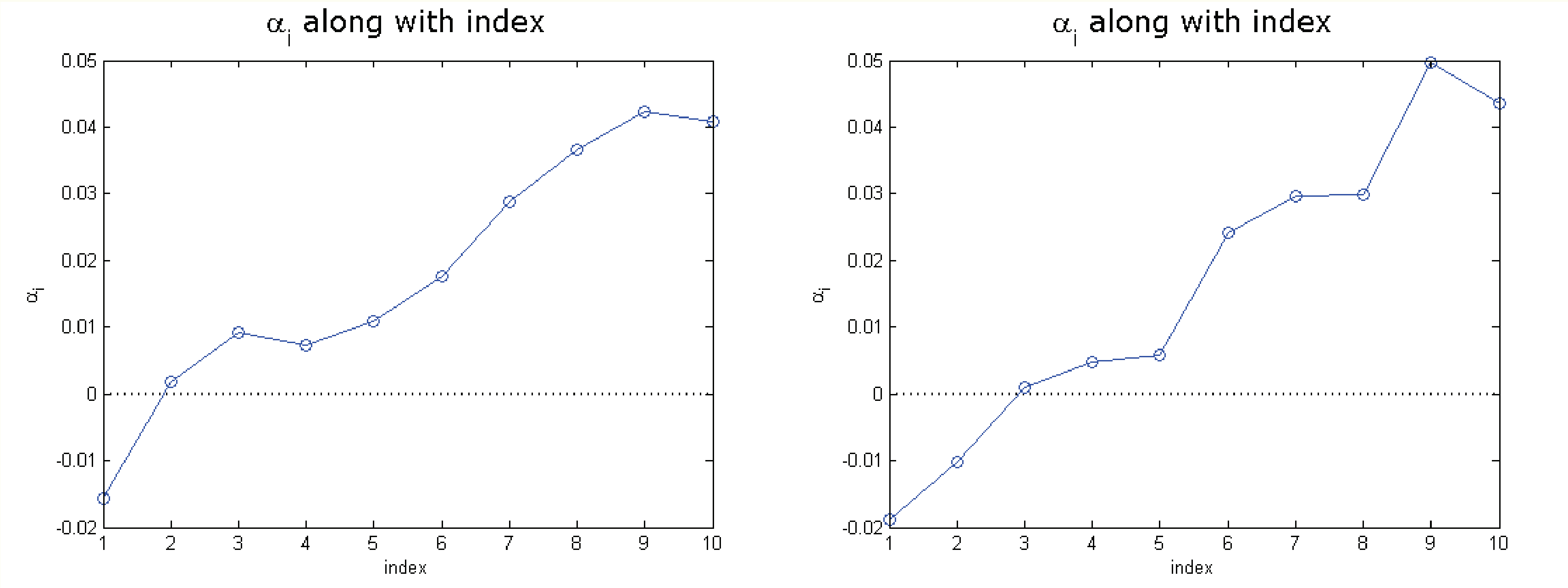
- Quantitatively, the alpha estimates from the fixed sampling are much smaller than those from the random sampling.
- Not so surprisingly, according to Wald test statistics, fixed time regression does not reject the model, while the result from the random time regression says that the model is wrong!
- Which one is true? Take a look at the decile Book-to-Market portfolios.

Table : Test of CAPM on Book-to-Market Portfolios

	Fixed Time		Random Time	
Book-to-Market	Alpha	Beta	Alpha	Beta
1 Growth	-0.0157	1.0872	-0.0188	1.1208
	(0.0101)	(0.0194)	(0.0089)	(0.0217)
2	0.0019	1.0101	-0.0101	1.0188
	(0.0075)	(0.0143)	(0.0061)	(0.0147)
3	0.0091	0.9677	0.0010	0.9916
	(0.0079)	(0.0152)	(0.0065)	(0.0159)
4	0.0074	0.9606	0.0048	0.9544
	(0.0092)	(0.0177)	(0.0077)	(0.0187)
5	0.0108	0.8679	0.0057	0.8817
	(0.0097)	(0.0186)	(0.0081)	(0.0197)
6	0.0176	0.8873	0.0242	0.8779
	(0.0094)	(0.0181)	(0.0079)	(0.0191)
7	0.0289	0.8379	0.0296	0.8086
	(0.0112)	(0.0214)	(0.0089)	(0.0215)
8	0.0367	0.8368	0.0298	0.8136
	(0.0116)	(0.0224)	(0.0094)	(0.0228)
9	0.0423	0.8850	0.0496	0.8638
	(0.0122)	(0.0235)	(0.0105)	(0.0256)
10 Value	0.0409	0.9898	0.0436	0.9914
	(0.0165)	(0.0316)	(0.0138)	(0.0336)
10-1 Book-to-Market Strategy	0.0566	-0.0974	0.0625	-0.1294
	(0.0232)	(0.0446)	(0.0203)	(0.0491)
Wald	15.7134 (0.1081)		46.2893 (0.0000)	

Figure : Alphas of Book-to-Market Portfolios

Fixed Time **Random Time**



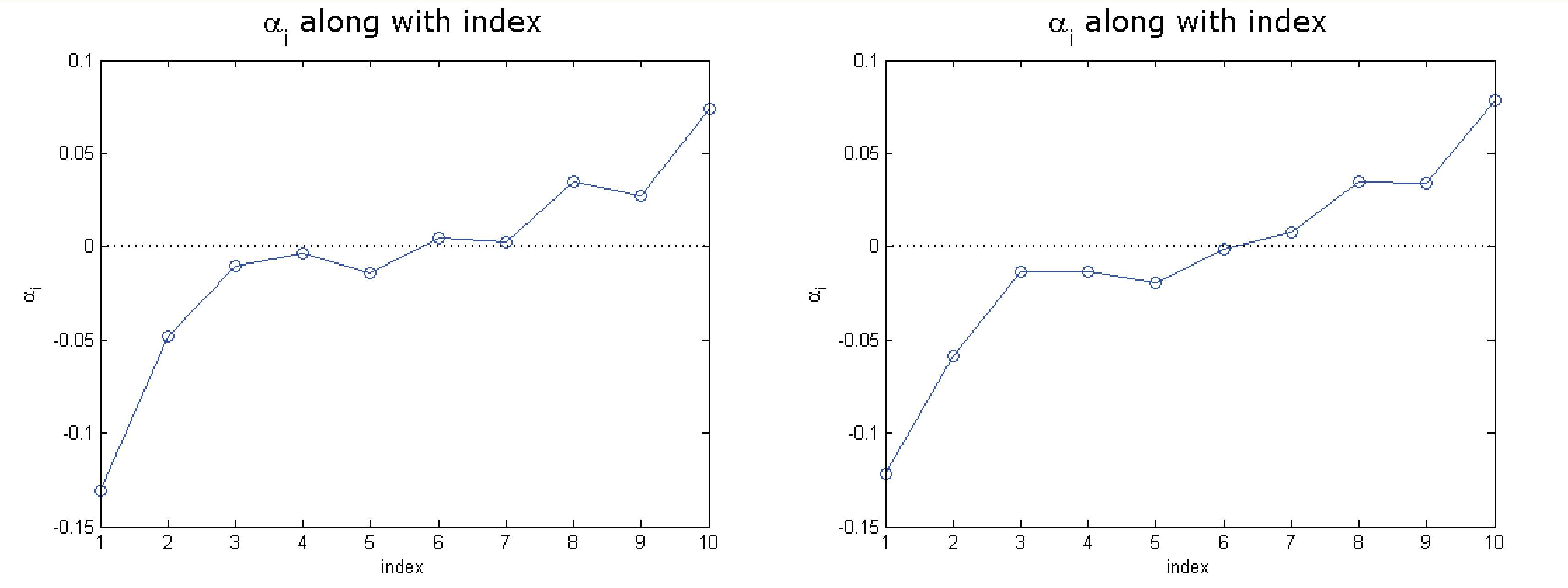
- In this case, the estimated betas show a clear symptom for the difficulty of the CAPM to explain cross sectional variation of portfolio returns.
- Betas decrease as the decile increases (That is, the more distressed the firms are, the less riskier are their equities.)
- That is, the market factor fails to capture this risk source. We expect high and significant alphas to fit the model which means that the model is wrong.
- Despite a significant alpha value for 10-1 B/M strategy, fixed time regression does not reject the model even at 10%, while the result from the random time regression says that the model is again wrong!
- Graphical illustration for the estimated alphas show not much difference. Fixed sampling seems to have trouble rejecting the model.
- Which one is true? Take a look at the Momentum portfolios.

Table : Test of CAPM on Momentum Portfolios

	Fixed Time		Time Change	
Momentum	Alpha	Beta	Alpha	Beta
1 Losers	-0.1307	1.4675	-0.1220	1.4962
	(0.0205)	(0.0394)	(0.0198)	(0.0481)
2	-0.0483	1.1983	-0.0586	1.2041
	(0.0170)	(0.0326)	(0.0142)	(0.0344)
3	-0.0105	1.0247	-0.0134	1.0299
	(0.0145)	(0.0279)	(0.0113)	(0.0274)
4	-0.0037	0.9665	-0.0136	1.0014
	(0.0135)	(0.0259)	(0.0095)	(0.0231)
5	-0.0143	0.9158	-0.0194	0.9236
	(0.0113)	(0.0218)	(0.0083)	(0.0202)
6	0.0045	0.8758	-0.0010	0.9008
	(0.0097)	(0.0185)	(0.0074)	(0.0179)
7	0.0026	0.9129	0.0075	0.9111
	(0.0081)	(0.0156)	(0.0077)	(0.0186)
8	0.0346	0.8969	0.0349	0.8997
	(0.0070)	(0.0134)	(0.0082)	(0.0198)
9	0.0275	0.9855	0.0343	0.9675
	(0.0057)	(0.0109)	(0.0088)	(0.0212)
10 Winners	0.0739	1.1602	0.0785	1.1750
	(0.0054)	(0.0104)	(0.0139)	(0.0338)
10-1 Momentum Strategy	0.2046	-0.3073	0.2005	-0.3212
	(0.0348)	(0.0669)	(0.0291)	(0.0706)
Wald	69.1309 (0.0000)		104.1411 (0.0000)	

Figure : Alphas of Momentum Portfolios

Fixed Time **Random Time**



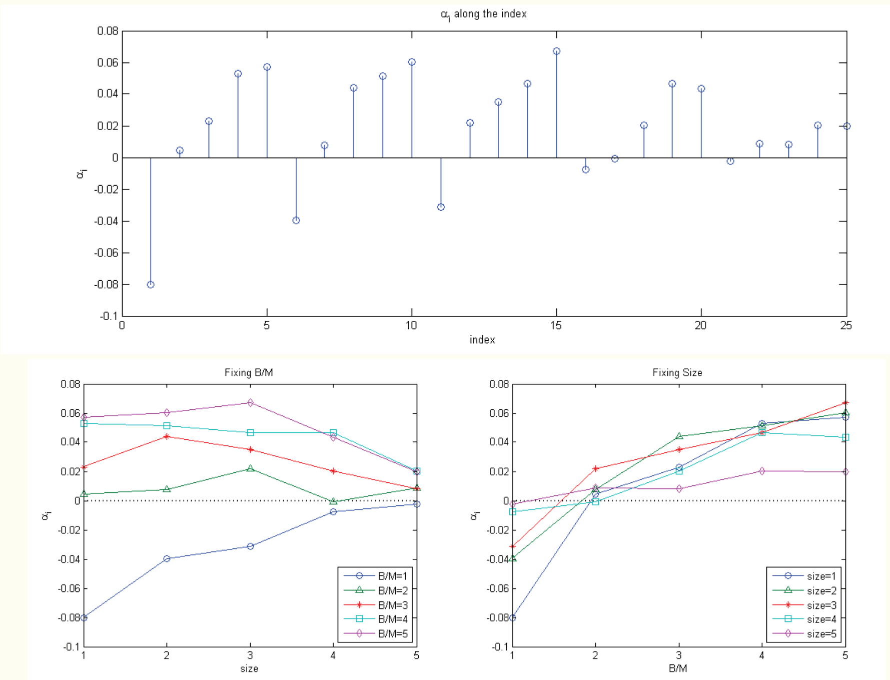
- For the momentum, the effect is really obvious. 20% abnormal return per year!
- Now, the fixed sampling scheme seems to work fine. That is, we conjecture that the fixed sampling scheme has trouble rejecting a model correctly, unless it is obvious. Weak power of a test?
- We keep going on with testing the CAPM on the 25 portfolios with both size and B/M.
- Recall that fixed sampling scheme did not reject none of these when separately sorted.

Table : Test of CAPM on (Size,B/M) Portfolios

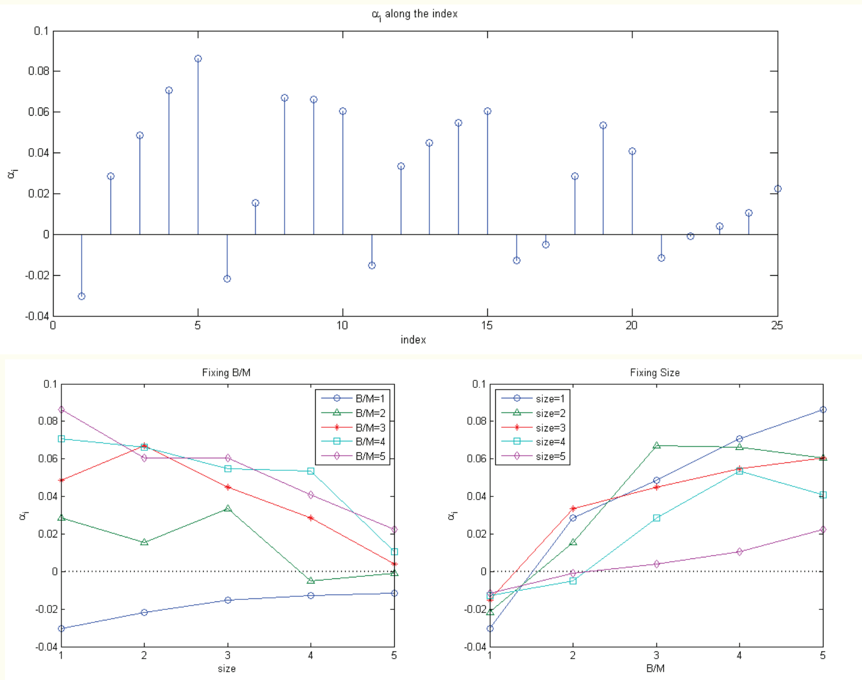
	Fixed Time		Random Time	
(Size,B/M)	Alpha	Beta	Alpha	Beta
(1,1)	-0.0801 (0.0246)	1.4533 (0.0472)	-0.0304 (0.0212)	1.4014 (0.0515)
(1,2)	0.0042 (0.0211)	1.2044 (0.0406)	0.0285 (0.0167)	1.1609 (0.0406)
(1,3)	0.0231 (0.0178)	1.0469 (0.0341)	0.0487 (0.0151)	1.0133 (0.0366)
(1,4)	0.0529 (0.0172)	0.9567 (0.0331)	0.0708 (0.0143)	0.9444 (0.0347)
(1,5)	0.0572 (0.0191)	1.0020 (0.0367)	0.0862 (0.0159)	0.9675 (0.0385)
(2,1)	-0.0395 (0.0187)	1.4118 (0.0359)	-0.0216 (0.0153)	1.3927 (0.0372)
(2,2)	0.0075 (0.0151)	1.1287 (0.0290)	0.0153 (0.0124)	1.1304 (0.0301)
(2,3)	0.0442 (0.0143)	1.0017 (0.0274)	0.0672 (0.0115)	1.0066 (0.0280)
(2,4)	0.0512 (0.0142)	0.9533 (0.0273)	0.0662 (0.0113)	0.9362 (0.0275)
(2,5)	0.0603 (0.0172)	1.0087 (0.0330)	0.0605 (0.0138)	1.0255 (0.0335)
(3,1)	-0.0314 (0.0156)	1.3526 (0.0300)	-0.0152 (0.0129)	1.3385 (0.0314)
(3,2)	0.0219 (0.0117)	1.0735 (0.0225)	0.0336 (0.0105)	1.0875 (0.0254)
(3,3)	0.0349 (0.0121)	0.9439 (0.0232)	0.0449 (0.0099)	0.9337 (0.0240)
(3,4)	0.0466 (0.0127)	0.8833 (0.0244)	0.0549 (0.0103)	0.8936 (0.0249)
(3,5)	0.0673 (0.0156)	0.9410 (0.0300)	0.0604 (0.0128)	0.9576 (0.0312)
(4,1)	-0.0078 (0.0115)	1.2331 (0.0221)	-0.0129 (0.0094)	1.2470 (0.0228)
(4,2)	-0.0010 (0.0100)	1.0550 (0.0192)	-0.0051 (0.0082)	1.0388 (0.0199)
(4,3)	0.0204 (0.0114)	0.9844 (0.0220)	0.0287 (0.0089)	0.9296 (0.0216)
(4,4)	0.0464 (0.0119)	0.9110 (0.0229)	0.0537 (0.0095)	0.8874 (0.0231)
(4,5)	0.0432 (0.0150)	0.9658 (0.0288)	0.0407 (0.0128)	0.9658 (0.0311)
(5,1)	-0.0026 (0.0088)	0.9998 (0.0169)	-0.0117 (0.0077)	1.0317 (0.0186)
(5,2)	0.0087 (0.0087)	0.9177 (0.0168)	-0.0010 (0.0072)	0.9329 (0.0174)
(5,3)	0.0083 (0.0106)	0.8352 (0.0204)	0.0040 (0.0089)	0.8506 (0.0216)
(5,4)	0.0203 (0.0126)	0.7907 (0.0241)	0.0107 (0.0101)	0.7602 (0.0246)
(5,5)	0.0200 (0.0158)	0.8337 (0.0303)	0.0223 (0.0129)	0.8432 (0.0313)
Wald	121.7332 (0.0000)		317.2629 (0.0000)	

Figure : Alphas of (Size,B/M) Portfolios

Fixed Time



Random Time



- Interestingly, now both fixed sampling and random sampling schemes reject the model decisively, although the Wald test statistics is still much smaller for the fixed sampling case.
- Therefore, test results based on the fixed sampling lack logical consistency.
- This means that statistical inference based on fixed sampling can be troublesome, when applied to high-frequency data.
- OK. But what if we just run OLS regressions using low frequency data? Well, maybe or maybe not.

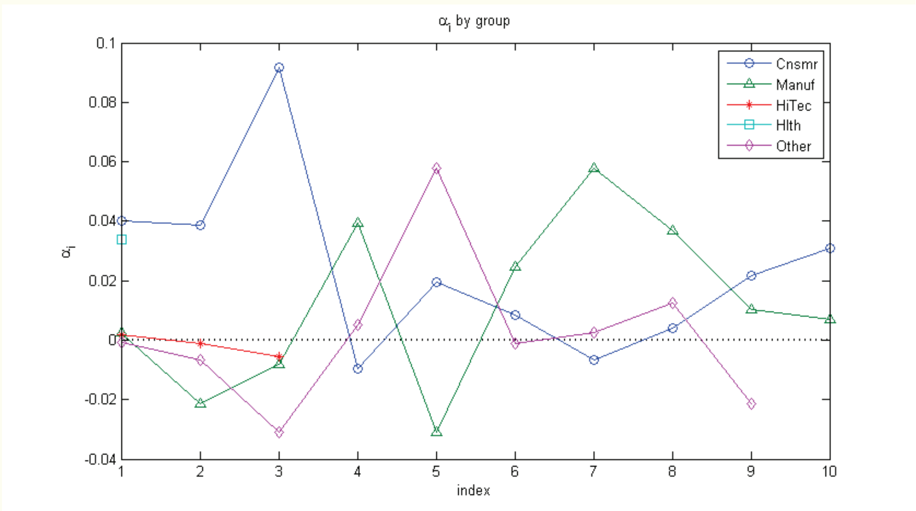
- We did this with monthly frequency data (using only two points of time for each month to construct return data). In this case, a test by Gibbons, Ross, and Shanken (1989) is a popular choice.
- We have p-values of 0.031 and 0.038 for the size portfolios and the value portfolios. At 3%, we can't reject the model for the period. Similar things happen with 25 portfolios. The CAPM is rejected with p-values of 0.000. Although weaker, the issue we raised is still there.
- In addition, when we vary the time periods covered, we have p-values varying between 0.002 and 0.208. This one, of course, may remind you of time-varying beta model (i.e., conditional beta model), but even for the conditional beta models, you need to test whether or not the long-run average of alphas is zero.
- More slippage and move on to the industry portfolios.

Table: Test of CAPM on Industry Portfolios

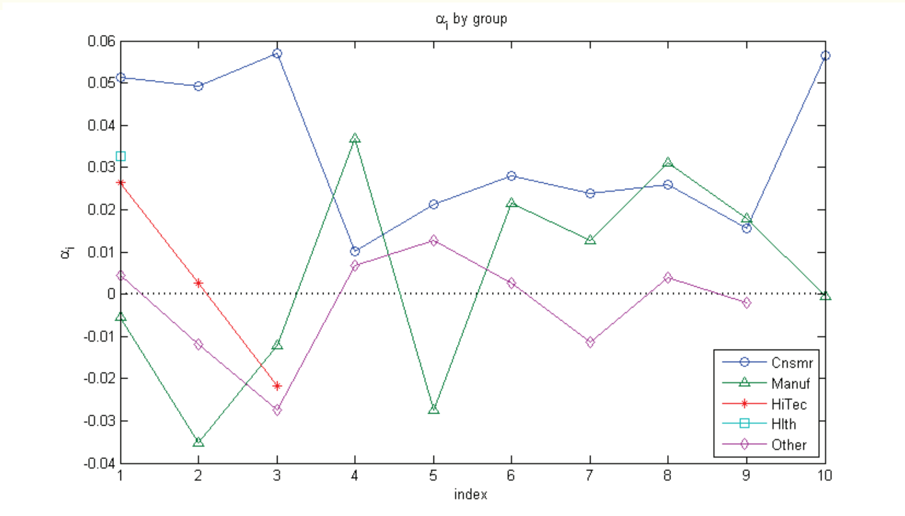
Industry	Fixed Time Alpha	Fixed Time Beta	Random Time Alpha	Random Time Beta
Food	0.0402 (0.0161)	0.7140 (0.0309)	0.0514 (0.0132)	0.6860 (0.0321)
Beer	0.0388 (0.0219)	0.8004 (0.0420)	0.0491 (0.0180)	0.7811 (0.0438)
Smoke	0.0919 (0.0294)	0.6287 (0.0566)	0.0571 (0.0243)	0.6583 (0.0590)
Games	-0.0007 (0.0223)	1.3058 (0.0428)	0.0044 (0.0174)	1.3323 (0.0421)
Books	-0.0095 (0.0166)	1.0430 (0.0319)	0.0100 (0.0139)	1.0281 (0.0337)
Hshld	0.0196 (0.0162)	0.8189 (0.0312)	0.0212 (0.0141)	0.8387 (0.0341)
Clths	0.0085 (0.0221)	1.0917 (0.0425)	0.0280 (0.0180)	1.1390 (0.0437)
Hlth	0.0339 (0.0168)	0.8485 (0.0322)	0.0326 (0.0144)	0.8941 (0.0351)
Chems	0.0024 (0.0160)	0.9740 (0.0308)	-0.0054 (0.0142)	0.9846 (0.0346)
Txtls	-0.0068 (0.0242)	0.9961 (0.0465)	0.0240 (0.0194)	1.0537 (0.0471)
Cnstr	-0.0065 (0.0154)	1.1475 (0.0295)	-0.0120 (0.0134)	1.1410 (0.0325)
Steel	-0.0214 (0.0234)	1.2390 (0.0449)	-0.0352 (0.0192)	1.2135 (0.0466)
FabPr	-0.0080 (0.0155)	1.1848 (0.0298)	-0.0120 (0.0127)	1.1839 (0.0307)
ElcEq	0.0394 (0.0168)	1.1628 (0.0323)	0.0368 (0.0140)	1.1811 (0.0339)
Autos	-0.0308 (0.0232)	1.0665 (0.0446)	-0.0274 (0.0181)	1.0683 (0.0439)
Carry	0.0245 (0.0216)	1.0758 (0.0415)	0.0215 (0.0175)	1.1023 (0.0424)
Mines	0.0050 (0.0329)	0.9203 (0.0631)	0.0067 (0.0272)	0.9220 (0.0659)
Coal	0.0578 (0.0431)	1.1051 (0.0827)	0.0127 (0.0340)	1.1016 (0.0826)
Oil	0.0368 (0.0216)	0.7588 (0.0414)	0.0311 (0.0191)	0.7483 (0.0463)
Util	0.0104 (0.0174)	0.5469 (0.0334)	0.0180 (0.0142)	0.4965 (0.0344)
Telcm	0.0017 (0.0170)	0.7654 (0.0327)	0.0264 (0.0137)	0.7500 (0.0332)
Servs	-0.0010 (0.0167)	1.3796 (0.0320)	0.0026 (0.0149)	1.3322 (0.0361)
BusEq	-0.0057 (0.0208)	1.3192 (0.0399)	-0.0219 (0.0169)	1.3429 (0.0411)
Paper	0.0071 (0.0157)	0.9096 (0.0301)	-0.0006 (0.0126)	0.9144 (0.0307)
Trans	0.0026 (0.0185)	1.0520 (0.0355)	-0.0114 (0.0150)	1.0343 (0.0363)
Whlsl	0.0041 (0.0162)	1.1093 (0.0312)	0.0259 (0.0137)	1.0832 (0.0332)
Rtail	0.0218 (0.0169)	0.9991 (0.0325)	0.0155 (0.0143)	1.0223 (0.0348)
Meals	0.0310 (0.0227)	1.1117 (0.0436)	0.0566 (0.0205)	1.1370 (0.0496)
Fin	0.0125 (0.0144)	1.0293 (0.0276)	0.0038 (0.0125)	1.0528 (0.0303)
Other	-0.0215 (0.0171)	1.0719 (0.0328)	-0.0021 (0.0130)	1.0960 (0.0316)
Wald	44.2420 (0.0453)		80.3369 (0.0000)	

Figure : Alphas of Industry Portfolios

Fixed Time



Random Time



- Even with industry portfolios, we basically have the same pattern.
- Fixed sampling scheme fails to reject the model at 4.53%.
- One interesting finding emerges when the alphas connecting each industry that belongs to one of more broadly defined five groups of industries are plotted. The random sampling result shows that the industries in the consumer sector (blue line) has significantly positive alphas, while the fixed time regression produces a mixed bag of results.
- This suggests that consumption growth may be a valid pricing factor together with the financial market factor, which is reminiscent of consumption-based pricing models. Including a consumption industry factor seems to be a sensible strategy.
- Summing up, the random sampling method works reliably in a high-frequency environment, contrary to its fixed sampling counterpart. More importantly, all the test results for the CAPM based on the random sampling provide a strong case for multi-factor models. From now on, we add more factors into the CAPM.

Table : Test of Two-Factor Model on Size Portfolios

	Fixed Time			Random Time		
Size	Alpha	Beta_MKT	Beta_SMB	Alpha	Beta_MKT	Beta_SMB
1 Small	-0.0038	0.8466	1.1876	0.0109	0.8010	1.2398
	(0.0095)	(0.0189)	(0.0272)	(0.0073)	(0.0184)	(0.0276)
2	-0.0026	0.9551	1.0362	-0.006	0.9534	1.049
	(0.0060)	(0.0121)	(0.0174)	(0.0051)	(0.0127)	(0.0191)
3	0.0074	0.9832	0.8771	0.0039	0.9790	0.8847
	(0.0056)	(0.0112)	(0.0161)	(0.0045)	(0.0111)	(0.0167)
4	0.0086	0.9682	0.7975	0.0036	0.9828	0.7831
	(0.0057)	(0.0114)	(0.0164)	(0.0043)	(0.0109)	(0.0163)
5	0.0139	0.9799	0.6434	0.0123	0.9863	0.6659
	(0.0056)	(0.0112)	(0.0162)	(0.0045)	(0.0113)	(0.0170)
6	0.0085	0.9872	0.4718	0.0046	0.9908	0.4978
	(0.0064)	(0.0128)	(0.0184)	(0.0052)	(0.0131)	(0.0197)
7	0.0093	1.0119	0.3494	0.0047	0.9976	0.3617
	(0.0061)	(0.0122)	(0.0175)	(0.0048)	(0.0121)	(0.0181)
8	0.0082	1.0249	0.2296	0.0026	1.0112	0.2271
	(0.0060)	(0.0120)	(0.0173)	(0.0052)	(0.0129)	(0.0194)
9	0.0063	0.9852	0.0402	0.0043	0.9808	0.0240
	(0.0057)	(0.0113)	(0.0163)	(0.0046)	(0.0116)	(0.0174)
10 Big	0.0010	0.9788	-0.2942	0.0016	0.9867	-0.2891
	(0.0030)	(0.0060)	(0.0087)	(0.0026)	(0.0064)	(0.0096)
1-10 Size Strategy	-0.0049	-0.1323	-1.4818	0.0092	-0.1857	1.5290
	(0.0250)	(0.0500)	(0.0719)	(0.0212)	(0.0531)	(0.0797)
Wald	12.0613 (0.2810)			31.3234 (0.0005)		

Table : Test of Two-Factor Model on Book-to-Market Portfolios

	Fixed Time			Random Time		
Book-to-Market	Alpha	Beta_MKT	Beta_HML	Alpha	Beta_MKT	Beta_HML
1 Growth	0.0157	0.9528	-0.5052	0.0136	0.9626	-0.5646
	(0.0075)	(0.0156)	(0.0239)	(0.0062)	(0.0161)	(0.0245)
2	0.0069	0.9885	-0.0811	-0.0045	0.9914	-0.0976
	(0.0075)	(0.0155)	(0.0239)	(0.0061)	(0.0160)	(0.0244)
3	0.0052	0.9845	0.0631	-0.0017	1.0050	0.0479
	(0.0080)	(0.0165)	(0.0254)	(0.0067)	(0.0175)	(0.0266)
4	-0.0084	1.0281	0.2538	-0.0114	1.0336	0.2824
	(0.0087)	(0.0180)	(0.0276)	(0.0072)	(0.0187)	(0.0285)
5	-0.0092	0.9537	0.3224	-0.014	0.9782	0.344
	(0.0088)	(0.0181)	(0.0279)	(0.0072)	(0.0189)	(0.0288)
6	-0.0063	0.9894	0.3837	0.0026	0.9832	0.3759
	(0.0080)	(0.0165)	(0.0254)	(0.0067)	(0.0175)	(0.0266)
7	-0.0049	0.9827	0.5443	0.0009	0.9486	0.4996
	(0.0085)	(0.0176)	(0.0271)	(0.0068)	(0.0179)	(0.0272)
8	-0.0061	1.0201	0.6890	-0.0075	0.9957	0.6497
	(0.0070)	(0.0145)	(0.0223)	(0.0057)	(0.0148)	(0.0225)
9	-0.0001	1.0665	0.6825	0.0093	1.0605	0.7017
	(0.0081)	(0.0168)	(0.0258)	(0.0068)	(0.0178)	(0.0271)
10 Value	-0.0126	1.2189	0.8612	-0.0050	1.2283	0.8454
	(0.0118)	(0.0243)	(0.0374)	(0.0100)	(0.0260)	(0.0396)
10-1 Book-to-Market Strategy	-0.0284	0.2661	1.3663	-0.0186	0.2658	1.4099
	(0.0237)	(0.0489)	(0.0751)	(0.0208)	(0.0543)	(0.0827)
Wald	9.8776 (0.4513)			16.7380 (0.0804)		

Table : Test of Fama-French Three-Factor Model

	Fixed Time			
(Size,B/M)	Alpha	Beta_MKT	Beta_SMB	Beta_HML
(1,1)	-0.0714 (0.0115)	1.1092 (0.0242)	1.3694 (0.0327)	-0.2936 (0.0367)
(1,2)	-0.0107 (0.0084)	0.9804 (0.0176)	1.2834 (0.0238)	0.0949 (0.0267)
(1,3)	-0.0043 (0.0070)	0.9231 (0.0147)	1.0747 (0.0199)	0.3192 (0.0224)
(1,4)	0.0178 (0.0070)	0.8809 (0.0148)	1.0080 (0.0200)	0.4508 (0.0225)
(1,5)	0.0074 (0.0073)	0.9737 (0.0155)	1.0771 (0.0209)	0.6798 (0.0235)
(2,1)	-0.0234 (0.0087)	1.1208 (0.0184)	0.9896 (0.0249)	-0.3714 (0.0280)
(2,2)	-0.0089 (0.0076)	1.0048 (0.0161)	0.8668 (0.0217)	0.1671 (0.0244)
(2,3)	0.0137 (0.0073)	0.9580 (0.0154)	0.7772 (0.0208)	0.4028 (0.0234)
(2,4)	0.0100 (0.0069)	0.9708 (0.0145)	0.7102 (0.0196)	0.5842 (0.0221)
(2,5)	0.0065 (0.0074)	1.0467 (0.0155)	0.8581 (0.0210)	0.7691 (0.0236)
(3,1)	-0.0095 (0.0082)	1.0944 (0.0172)	0.7330 (0.0232)	-0.4356 (0.0261)
(3,2)	0.0058 (0.0087)	1.0242 (0.0182)	0.5264 (0.0246)	0.1990 (0.0277)
(3,3)	0.0017 (0.0085)	0.9869 (0.0179)	0.4431 (0.0242)	0.4850 (0.0272)
(3,4)	0.0037 (0.0082)	0.9800 (0.0172)	0.3885 (0.0232)	0.6469 (0.0261)
(3,5)	0.0143 (0.0095)	1.0495 (0.0200)	0.5287 (0.0270)	0.7936 (0.0304)
(4,1)	0.0160 (0.0079)	1.0492 (0.0166)	0.3656 (0.0224)	-0.4244 (0.0252)
(4,2)	-0.0180 (0.0092)	1.0811 (0.0194)	0.2064 (0.0262)	0.2487 (0.0295)
(4,3)	-0.0114 (0.0094)	1.0844 (0.0198)	0.1621 (0.0267)	0.4940 (0.0301)
(4,4)	0.0069 (0.0086)	1.0330 (0.0181)	0.2090 (0.0245)	0.6110 (0.0275)
(4,5)	-0.0085 (0.0104)	1.1331 (0.0219)	0.2414 (0.0296)	0.8050 (0.0333)
(5,1)	0.0228 (0.0065)	0.9505 (0.0137)	-0.2645 (0.0185)	-0.3786 (0.0208)
(5,2)	0.0029 (0.0079)	0.9935 (0.0167)	-0.2270 (0.0226)	0.1191 (0.0254)
(5,3)	-0.0083 (0.0091)	0.9611 (0.0191)	-0.2432 (0.0258)	0.2957 (0.0290)
(5,4)	-0.0181 (0.0080)	1.0051 (0.0168)	-0.2246 (0.0228)	0.6419 (0.0256)
(5,5)	-0.0285 (0.0115)	1.0631 (0.0243)	-0.0982 (0.0328)	0.7906 (0.0369)
Wald	95.5585 (0.0000)			

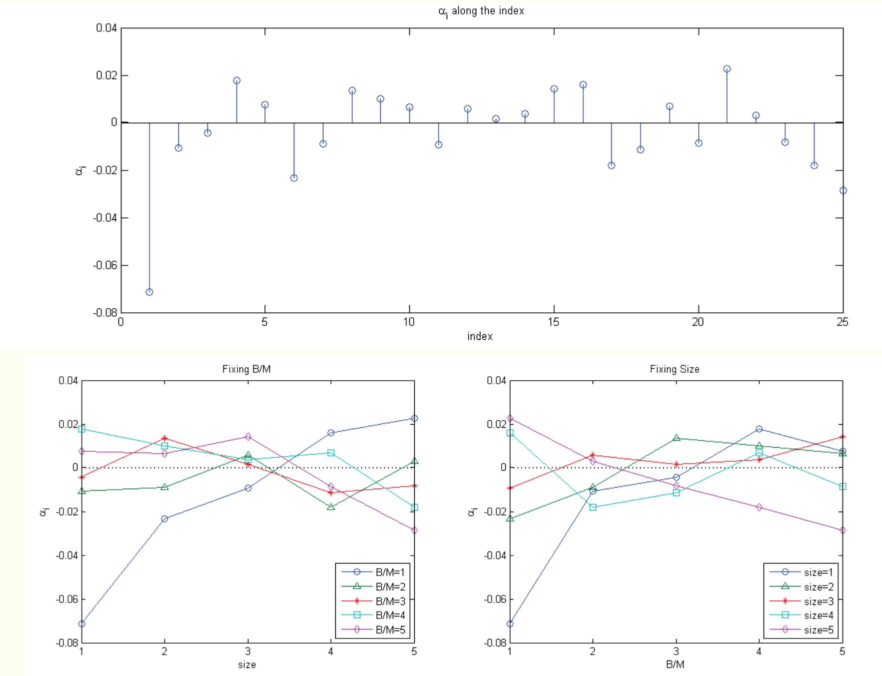
Table : Test of Fama-French Three-Factor Model

	Random Time			
(Size,B/M)	Alpha	Beta_MKT	Beta_SMB	beta_HML
(1,1)	-0.0591 (0.0096)	1.0659 (0.0253)	1.4512 (0.0355)	-0.2758 (0.0384)
(1,2)	-0.0159 (0.0064)	0.9706 (0.0168)	1.2413 (0.0236)	0.1091 (0.0255)
(1,3)	-0.0034 (0.0058)	0.9019 (0.0153)	1.1160 (0.0214)	0.3107 (0.0231)
(1,4)	0.0152 (0.0055)	0.8736 (0.0145)	1.0433 (0.0203)	0.4096 (0.0219)
(1,5)	0.0143 (0.0057)	0.9556 (0.0149)	1.1072 (0.0209)	0.6604 (0.0226)
(2,1)	-0.0321 (0.0070)	1.1188 (0.0184)	0.9924 (0.0258)	-0.3473 (0.0279)
(2,2)	-0.0205 (0.0059)	1.0148 (0.0154)	0.8854 (0.0216)	0.1498 (0.0233)
(2,3)	0.0217 (0.0059)	0.9762 (0.0156)	0.7690 (0.0219)	0.3799 (0.0237)
(2,4)	0.0131 (0.0052)	0.9636 (0.0137)	0.7073 (0.0192)	0.5468 (0.0208)
(2,5)	-0.0074 (0.0065)	1.0955 (0.0171)	0.7967 (0.0239)	0.7553 (0.0259)
(3,1)	-0.0124 (0.0070)	1.0894 (0.0183)	0.7190 (0.0256)	-0.4325 (0.0277)
(3,2)	0.0061 (0.0071)	1.0144 (0.0187)	0.6323 (0.0262)	0.1404 (0.0283)
(3,3)	0.0041 (0.0068)	0.9756 (0.0179)	0.4790 (0.0251)	0.4538 (0.0271)
(3,4)	0.0086 (0.0068)	0.9768 (0.0177)	0.4348 (0.0249)	0.5728 (0.0269)
(3,5)	-0.0021 (0.0074)	1.0847 (0.0195)	0.5426 (0.0274)	0.7979 (0.0296)
(4,1)	0.0012 (0.0065)	1.0669 (0.0170)	0.3400 (0.0238)	-0.4269 (0.0257)
(4,2)	-0.0242 (0.0077)	1.0533 (0.0202)	0.2397 (0.0284)	0.2040 (0.0307)
(4,3)	-0.0019 (0.0075)	1.0095 (0.0197)	0.2106 (0.0276)	0.4189 (0.0299)
(4,4)	0.0136 (0.0069)	1.0032 (0.0181)	0.2430 (0.0254)	0.5673 (0.0274)
(4,5)	-0.0155 (0.0086)	1.1524 (0.0226)	0.2662 (0.0317)	0.8349 (0.0342)
(5,1)	0.0208 (0.0055)	0.9614 (0.0145)	-0.2688 (0.0203)	-0.4216 (0.0219)
(5,2)	-0.0013 (0.0065)	1.0091 (0.0172)	-0.2286 (0.0241)	0.1270 (0.0260)
(5,3)	-0.0086 (0.0076)	0.9843 (0.0199)	-0.2211 (0.0279)	0.3369 (0.0301)
(5,4)	-0.0163 (0.0067)	0.9642 (0.0177)	-0.2197 (0.0248)	0.5882 (0.0268)
(5,5)	-0.0213 (0.0092)	1.0758 (0.0242)	-0.0612 (0.0340)	0.7911 (0.0367)
Wald	166.0789 (0.0000)			

- In this case, both sampling schemes reject the model.
- To see where the rejections come from, we plot a few graphs.
- As Fama and French reported, it is the small stocks in the smallest B/M group.
- Unlike other groups, this one is not at all controlled by the size and B/M factors.

Figure : Fama-French Alphas of (Size,B/M) Portfolios

Fixed Time



Random Time

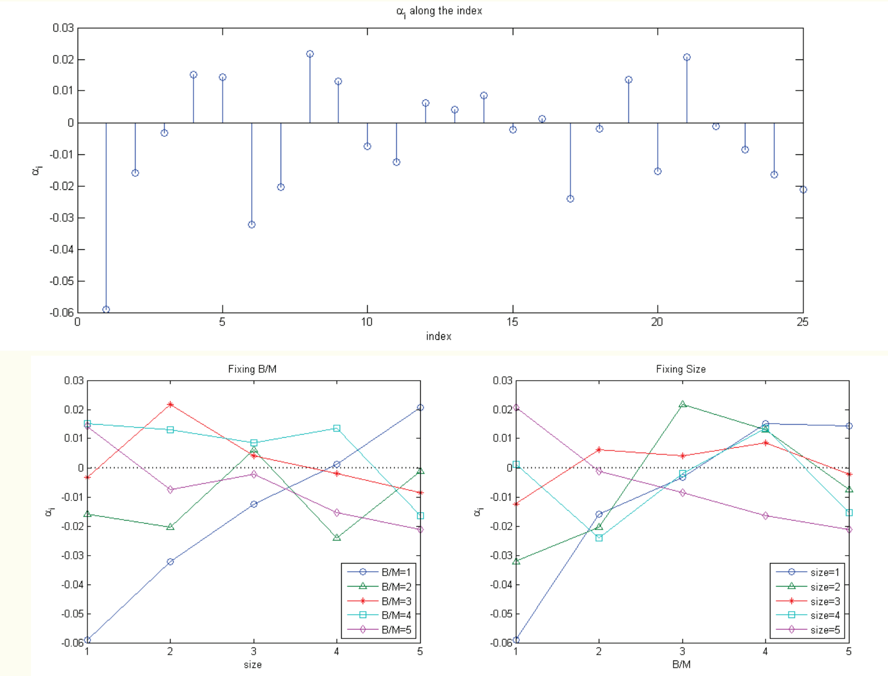
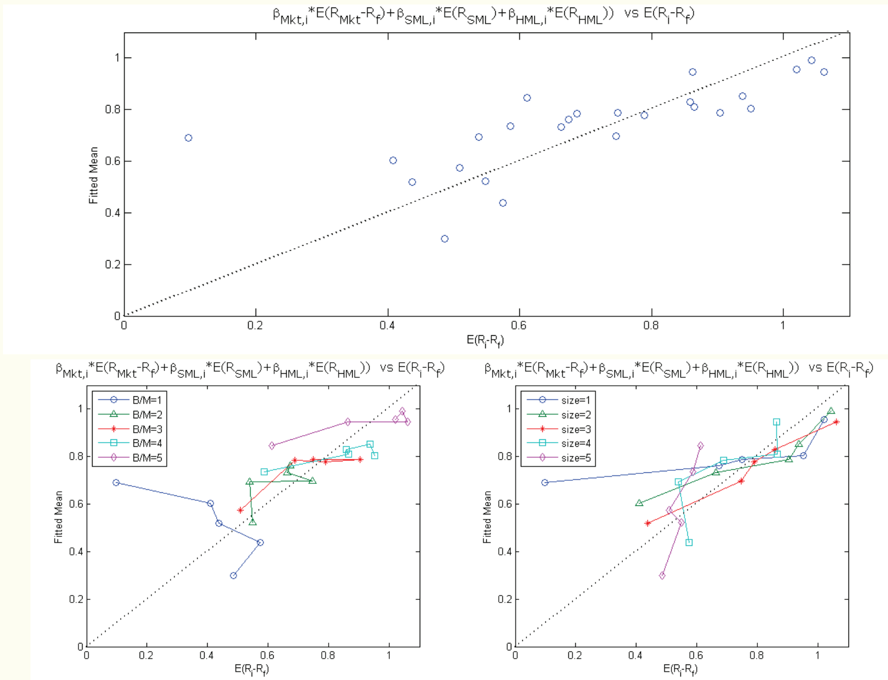
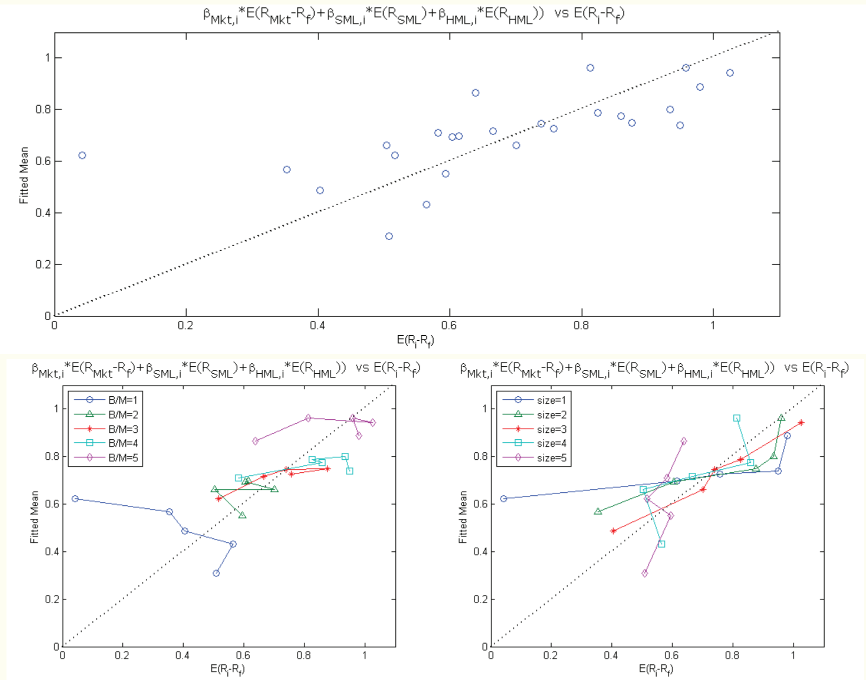


Figure : Fama-French Betas of (Size,B/M) Portfolios

Fixed Time



Random Time



- From the graphs, the deviations of alphas from zero are much smaller than those of the CAPM, although the large deviations from the small firm in the low B/M phenomenon still persists.
- We conjecture that sizes of market equity are explained by more than one risk component.
- In sum, Fama-French model is a significant improvement over CAPM, but it is still not sufficient. Even the four-factor version of the Fama-French model with Momentum factor is decisively rejected for 25 portfolios.
- Therefore, a new factor is still desired to capture this dimension. We believe that our econometric methodology can shed light on this issue by allowing a reliable testing procedure for potential pricing factors.

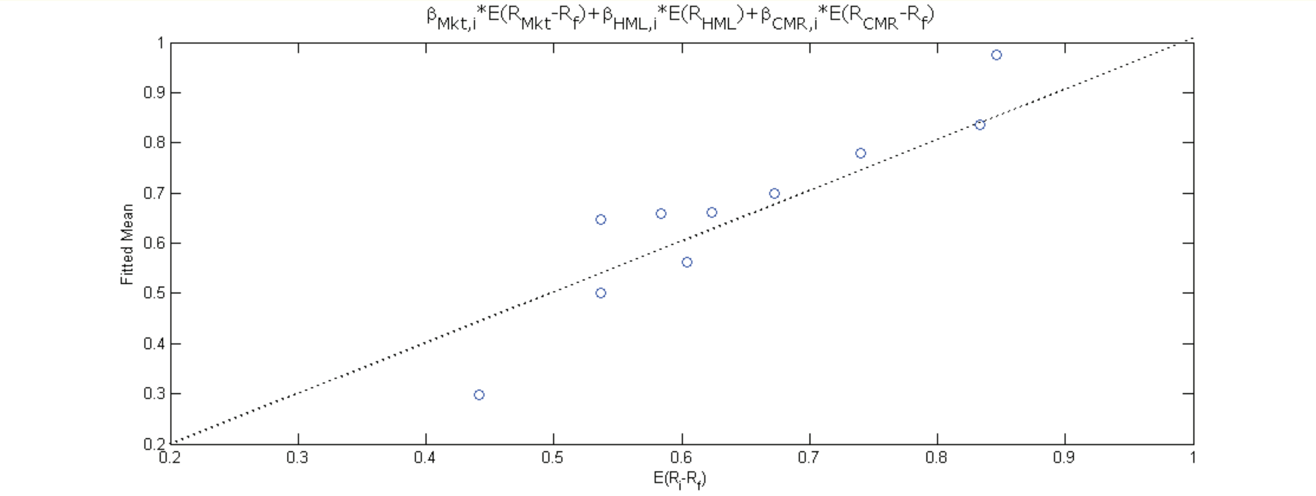
Table : Tests of Three-Factor Models with Consumer Industry Factor

Panel A (Random Time)	Alpha	Beta_MKT	Beta_HML	Beta_CMR
1 Growth	0.0122 (-0.0058)	0.7432 (0.0328)	-0.5940 (0.0234)	0.2276 (0.0301)
2	-0.0064 (-0.0055)	0.6999 (0.0308)	-0.1367 (0.0221)	0.3026 (0.0284)
3	-0.0034 (0.0063)	0.7548 (0.0353)	0.0143 (0.0252)	0.2597 (0.0325)
4	-0.0120 (0.0071)	0.9383 (0.0402)	0.2696 (0.0287)	0.0989 (0.0369)
5	-0.0140 (0.0073)	0.9807 (0.0409)	0.3444 (0.0292)	-0.0027 (0.0376)
6	0.0027 (0.0067)	0.9919 (0.0378)	0.3770 (0.0270)	-0.0091 (0.0347)
7	0.0002 (0.0068)	0.8518 (0.0383)	0.4867 (0.0274)	0.1004 (0.0352)
8	-0.0080 (0.0056)	0.9151 (0.0317)	0.6389 (0.0227)	0.0837 (0.0292)
9	0.0087 (0.0068)	0.9646 (0.0381)	0.6889 (0.0272)	0.0995 (0.0350)
10 Value	-0.0052 (0.0100)	1.1857 (0.0562)	0.8397 (0.0402)	0.0442 (0.0517)
10-1 Book-to-Market Strategy	-0.0174 (0.0208)	0.4425 (0.1173)	1.4337 (0.0839)	-0.1834 (0.1079)
Wald		15.8061 (0.1053)		

Panel B (Random Time)	Alpha	Beta_MKT	Beta_SMB	Beta_CMR
1 Small	0.0103 (0.0074)	0.7624 (0.0401)	1.2405 (0.0276)	0.0415 (0.0383)
2	-0.0072 (0.0051)	0.8768 (0.0275)	1.0502 (0.0189)	0.0824 (0.0263)
3	0.0024 (0.0044)	0.8754 (0.0237)	0.8864 (0.0163)	0.1114 (0.0227)
4	0.0016 (0.0042)	0.8509 (0.0227)	0.7853 (0.0156)	0.1418 (0.0217)
5	0.0108 (0.0044)	0.8892 (0.0242)	0.6675 (0.0167)	0.1044 (0.0231)
6	0.0026 (0.0051)	0.8625 (0.0278)	0.5000 (0.0191)	0.1379 (0.0266)
7	0.0032 (0.0047)	0.8978 (0.0258)	0.3633 (0.0178)	0.1073 (0.0247)
8	0.0016 (0.0051)	0.9471 (0.0280)	0.2282 (0.0192)	0.0690 (0.0267)
9	0.0025 (0.0045)	0.8611 (0.0245)	0.0260 (0.0168)	0.1287 (0.0234)
10 Big	0.0013 (0.0026)	0.9685 (0.0140)	-0.2888 (0.0096)	0.0195 (0.0134)
1-10 Size Strategy	0.0089 (0.0213)	-0.2061 (0.1160)	1.5293 (0.0798)	0.0219 (0.1108)
Wald			27.2450 (0.0024)	

Figure : Betas of Three Factor Models with Consumer Factor

On B/M Portfolios



On Size Portfolios

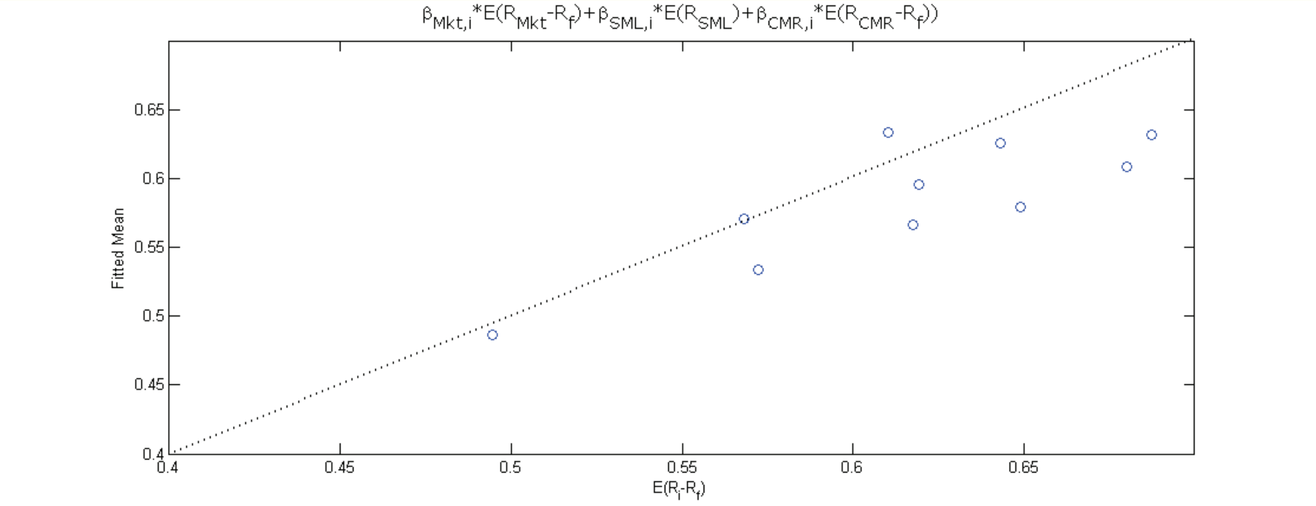


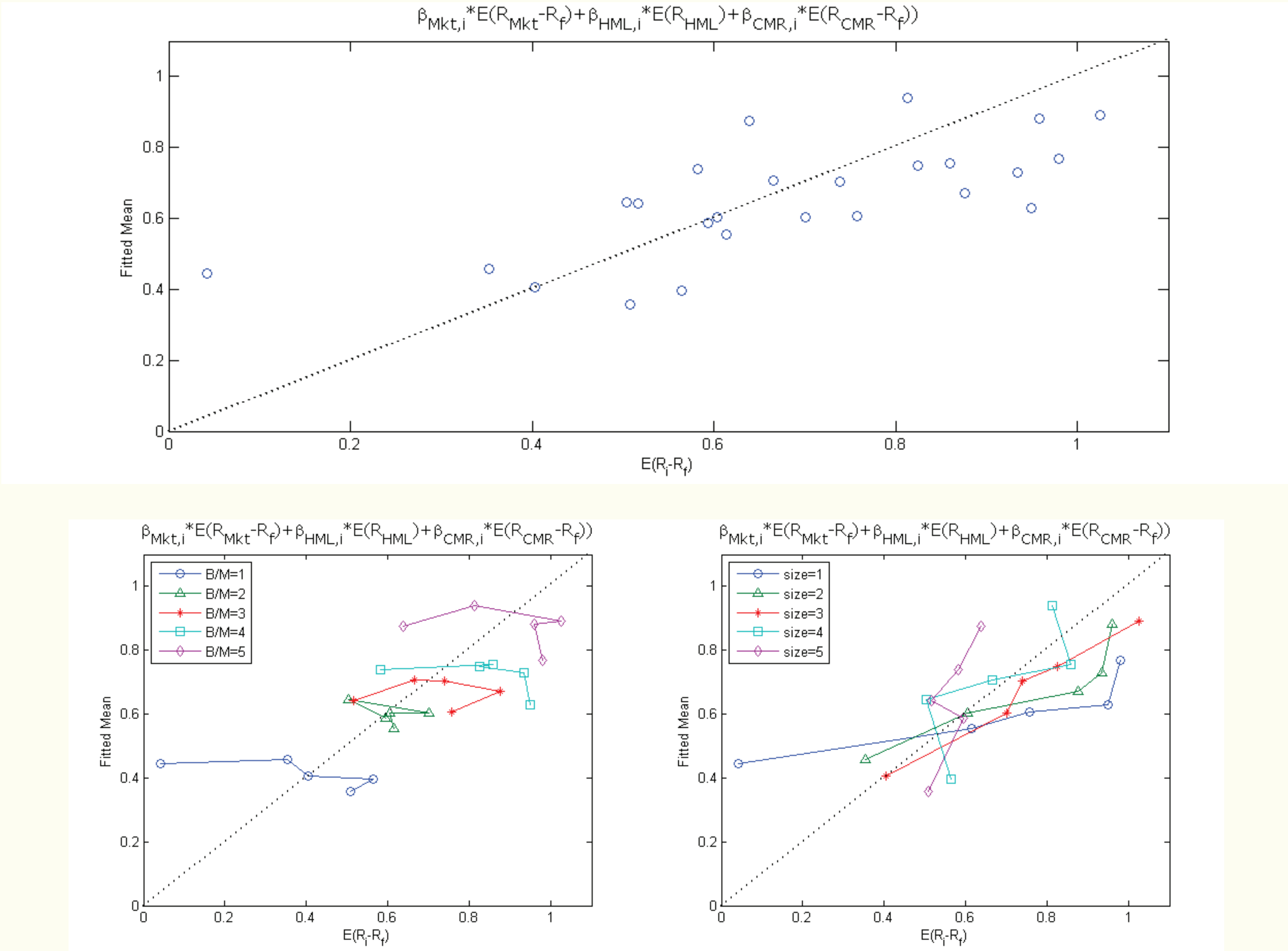
Table : Tests of Three-Factor Models with Consumer Industry Factor

Mkt, B/M, CMR (Random Time)

(Size, B/M)	Alpha	Beta_MKT	Beta_HML	Beta_CMR
(1,1)	0.0005 (0.0209)	1.4048 (0.1178)	-0.5015 (0.0843)	-0.1552 (0.1084)
(1,2)	0.0343 (0.0172)	1.1327 (0.0968)	-0.1011 (0.0692)	-0.0001 (0.0890)
(1,3)	0.0408 (0.0154)	0.9009 (0.0867)	0.1021 (0.0620)	0.1522 (0.0798)
(1,4)	0.0565 (0.0144)	0.8747 (0.0813)	0.2149 (0.0581)	0.1402 (0.0747)
(1,5)	0.0582 (0.0153)	0.9690 (0.0859)	0.4554 (0.0615)	0.1360 (0.0790)
(2,1)	0.0076 (0.0145)	1.1832 (0.0818)	-0.5241 (0.0585)	0.0675 (0.0752)
(2,2)	0.0140 (0.0126)	0.9340 (0.0712)	-0.0265 (0.0509)	0.2038 (0.0655)
(2,3)	0.0518 (0.0114)	0.9256 (0.0643)	0.2295 (0.0460)	0.1566 (0.0592)
(2,4)	0.0405 (0.0103)	0.8850 (0.0582)	0.4041 (0.0416)	0.1773 (0.0535)
(2,5)	0.0238 (0.0120)	1.0446 (0.0676)	0.5996 (0.0484)	0.1607 (0.0622)
(3,1)	0.0165 (0.0115)	1.1652 (0.0649)	-0.5566 (0.0464)	0.0187 (0.0597)
(3,2)	0.0300 (0.0105)	0.8426 (0.0590)	-0.0008 (0.0422)	0.2639 (0.0543)
(3,3)	0.0221 (0.0089)	0.8329 (0.0503)	0.3451 (0.0360)	0.2130 (0.0462)
(3,4)	0.0249 (0.0085)	0.8401 (0.0480)	0.4732 (0.0343)‘	0.2007 (0.0441)
(3,5)	0.0186 (0.0100)	0.9682 (0.0562)	0.6809 (0.0402)	0.1943 (0.0517)
(4,1)	0.0149 (0.0077)	1.0997 (0.0436)	-0.4860 (0.0312)	0.0119 (0.0401)
(4,2)	-0.0160 (0.0079)	0.8547 (0.0446)	0.1326 (0.0319)	0.2386 (0.0410)
(4,3)	0.0052 (0.0076)	0.8214 (0.0430)	0.3543 (0.0307)	0.2237 (0.0395)
(4,4)	0.0226 (0.0074)	0.8996 (0.0416)	0.5080 (0.0297)	0.1404 (0.0383)
(4,5)	-0.0057 (0.0091)	1.0395 (0.0510)	0.7700 (0.0365)	0.1533 (0.0469)
(5,1)	0.0081 (0.0059)	0.6528 (0.0332)	-0.4128 (0.0238)	0.2839 (0.0305)
(5,2)	-0.0116 (0.0069)	0.8168 (0.0391)	0.1439 (0.0279)	0.1687 (0.0359)
(5,3)	-0.0170 (0.0080)	1.0384 (0.0449)	0.3854 (0.0321)	-0.0860 (0.0413)
(5,4)	-0.0259 (0.0072)	0.8291 (0.0404)	0.6111 (0.0289)	0.1104 (0.0371)
(5,5)	-0.0242 (0.0091)	1.0052 (0.0515)	0.7930 (0.0368)	0.0650 (0.0473)
Wald		181.2525 (0.0000)		

Figure : Betas of Three Factor Models with Consumer Factor

On (Size,B/M) Portfolios



Summary

- Using our new framework, we reexamined CAPM anomalies and Fama-French regressions.
- Our empirical results strongly suggest that the conventional OLS method based on fixed sampling scheme is not appropriate to use on financial data sampled at high frequencies.
- This is due to the inability of the conventional method in handling persistent stochastic volatilities chronic in asset return data. This issue becomes more serious as data have higher frequencies.
- Meanwhile, when our random sampling scheme is applied instead, this problem virtually disappears. The results are consistent and robust to all data sets we constructed.
- Our new empirical findings suggest: (i) size premium is important but size factor alone cannot fully explain size-based portfolios and (ii) some potential role to be played by consumer goods industry factor.