Calibration of self-decomposable Lévy models

— Haindorf Seminar —

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Outline

Exponential Lévy model

Stock price is modeled by

$$S_t = S_0 e^{rt+X_t},$$

where e^{X_t} is a martingale and

- S_0 denotes the present value,
- $r \geq 0$ is the riskless interest rate and
- X_t is a Lévy process with characteristic triplet (σ, γ, ν).

Lévy-Khintchine representation

(for jump parts with finite variation)

$$\mathbb{E}[e^{iuX_t}] = \exp\Big(t\Big(-\frac{\sigma^2}{2}u^2 + i\gamma u + \int_{\mathbb{R}}(e^{iux}-1)\nu(\mathrm{d}x)\Big)\Big)$$

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Self-decomposable distributions

Definition: A random variable X is called self-decomposable, if for any $b \in (0,1)$ there is an independent random variable Y_b such that

$$\varphi_X(u) = \varphi_X(bu)\varphi_{Y_b}(u) \quad \Leftrightarrow \quad X \stackrel{d}{=} bX + Y_b$$

Self-decomposable distributions are relevant as

- limit distributions of generally scaled sums of independent random variables,
- marginal distributions of general Sato processes and
- limit distributions of Lévy driven Ornstein-Uhlenbeck processes.

Self-decomposable Lévy processes

(X_t) is self-decomposable if and only if its jump measure is given by

$$\nu(\mathrm{d}x) = \frac{k(x)}{|x|}\,\mathrm{d}x,$$

with a function $k : x \mapsto k(x)$ that is

- nonnegative,
- increasing on $(-\infty, 0)$ and
- decreasing on $(0, \infty)$.

We suppose $\sigma = 0$ and k to be smooth away from zero.

Selfdecomposable Lévy processes

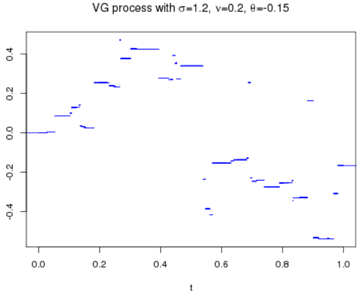
Example:

Variance gamma process (Madan and Seneta, 1990):

$$X_t = \theta G_t + \sigma W_{G_t},$$

where

- W is a standard Brownian motion and
- G is a Gamma process with $G_t \sim \Gamma(t/\nu, 1/\nu)$.



Self-decomposable Lévy processes

Aim: Estimation of

$$\gamma \in \mathbb{R}, \quad \alpha := k(0+) + k(0-) \in [0, \infty) \quad \text{and}$$
$$k_e(x) := \operatorname{sgn}(x) e^x k(x), \quad x \in \mathbb{R}.$$

What is new? (X_t) has infinite activity!

The estimation error is driven by the growth of $|\varphi_T(u - i)|^{-1}$, where $\varphi_T(u) := \mathbb{E}[e^{iuX_T}], u \in \{z \in \mathbb{C} | \operatorname{Im} z \in [-1, 0]\}$.

Lemma

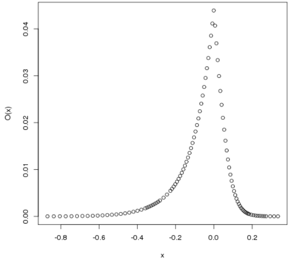
For some constant $C_\varphi > 0$ and all $|u| \geq 1, u \in \mathbb{R}$, we obtain

$$|\varphi_T(u - i)| \geq C_\varphi |u|^{-T\alpha}.$$

Observations

We fix a maturity $T > 0$ and define $x := \log(K/S_0) - rT$.
For call prices $\mathcal{C}(x,T)$ and put prices $\mathcal{P}(x,T)$ the option function $\mathcal{O}$ is given by:

$$\mathcal{O}(x) := \begin{cases} S_0^{-1}\mathcal{C}(x,T), & x \geq 0, \\ S_0^{-1}\mathcal{P}(x,T), & x < 0. \end{cases}$$



We observe

$$O_j = \mathcal{O}(x_j) + \delta_j \varepsilon_j, \quad j = 1, \dots, N,$$

where (ε_j) are centered and independent with $\mathbb{E}[\varepsilon^2] = 1$ and finite fourth moments.

Representation of the characteristic exponent

From an option pricing formula (Carr and Madan, 1999) and from the Lévy-Khintchine representation follows:

$$\psi(u) := \frac{1}{T}\log(\varphi_T(u-i)) = \frac{1}{T}\log\Big(1 + iu(1+iu)\mathcal{F}\mathcal{O}(u)\Big)$$
$$= i\gamma u + \gamma + \int_0^1 i(u-i)\mathcal{F}(\operatorname{sgn}(x)k(x))((u-i)t)\,\mathrm{d}t.$$

Let $k \in C^s(\mathbb{R}\setminus\{0\})$. We consider the compensated function:

$$g(x) := \operatorname{sgn}(x)k(x) - \sum_{j=0}^{s-2}\alpha_j x^j e^{-x}\mathbf{1}_{[0,\infty)}(x), \qquad x \in \mathbb{R}.$$

Choosing α_j properly gives $g \in C^{s-2}(\mathbb{R}) \cap C^s(\mathbb{R}\setminus\{0\})$, especially $\alpha_0 = \alpha$.

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Representation of the characteristic exponent

Proposition

For $s \geq 2$ and k properly integrable there exist functions $D : \{-1, 1\} \to \mathbb{C}$ and $\rho : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ such that for $u \neq 0$

$$\psi(u) = D(\operatorname{sgn}(u)) + i\gamma u - \alpha \log(|u|) + \sum_{j=1}^{s-2} \frac{i^j (j-1)! \alpha_j}{u^j} + \rho(u)$$

and $|u^{s-1}\rho(u)|$ is bounded in u.

Estimation of α

Analogously to Belomestny and Reiß (2006) we proceed as follows:

1. Fit a function $\tilde{\mathcal{O}}$ to the data (O_j).
2. Calculate $\tilde{\psi}(u)$ with $\mathcal{F}\tilde{\mathcal{O}}$ instead of $\mathcal{F}\mathcal{O}$ and a trimmed log.
3. Estimate α using a spectral cut-off value $U > 0$:

$$\hat{\alpha} := \int_{-U}^{U} \mathrm{Re}(\tilde{\psi}(u)) w_\alpha^U(u)\,\mathrm{d}u.$$

The weight function w_α^U satisfies $|w_\alpha^U(u)| \lesssim U^{-s}|u|^{s-1}$ and

$$\int_{-U}^{U} \log(|u|) w_\alpha^U(u)\,\mathrm{d}u = -1, \qquad \int_0^U w_\alpha^U(\pm u)\,\mathrm{d}u = 0,$$

$$\int_{-U}^{U} u^{-j} w_\alpha^U(u)\,\mathrm{d}u = 0, \qquad \text{for } 2 \le j \le s-2, \text{even}.$$

Estimation of k_e

Differentiation yields

$$\psi'(u) = \frac{(i-2u)\mathcal{F}\mathcal{O}(u) - (u+iu^2)\mathcal{F}\big(x\mathcal{O}(x)\big)(u)}{T(1+(iu-u^2)\mathcal{F}\mathcal{O}(u))} = i\gamma + i\mathcal{F}k_e(u).$$

Hence, we define the estimator

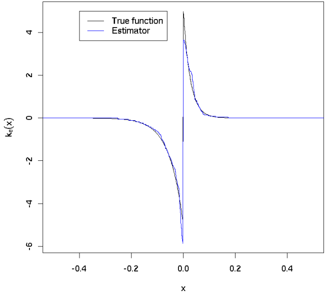
$$\hat{k}_e(x) := \begin{cases} \mathcal{F}^{-1}\big[(-\hat{\gamma} - i\tilde{\psi}'(u))\mathcal{F}W_k(u/U)\big](x), & x > 0, \\ \mathcal{F}^{-1}\big[(-\hat{\gamma} - i\tilde{\psi}'(u))\mathcal{F}W_k(-u/U)\big](x), & x < 0 \end{cases}$$

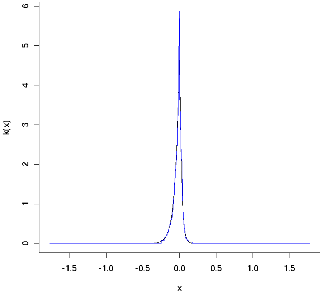
where

- $\tilde{\psi}'$ is a trimmed empirical version of ψ' and
- $W_k \in H^{T\bar{\alpha}+2}(\mathbb{R})$ is a one-sided kernel function of the order $s-1$ satisfying $x^{2s-1}W_k(x) \in L^1(\mathbb{R})$.

Simulations in the variance gamme model

	N	50	100	100
	Noise level	0.01	0.01	0.05
RMSE	$\hat{\gamma}$	0.0014	0.0004	0.0074
	$\hat{\alpha}$	0.4196	0.6669	2.0231
RMISE	$\hat{k}_e$	0.5930	0.4980	0.4754





Asymptotics

Consider asymptotics of a growing number of observations with

$$\Delta := \max_{j=2,\dots,N}(x_j - x_{j-1}) \to 0 \quad \text{and} \quad A := \min(x_N, -x_1) \to \infty.$$

We analyze the risk of the estimators in terms of the noise level

$$\varepsilon := \Delta^{3/2} + \Delta^{1/2}\|\delta\|_{l^\infty}.$$

Uniform convegence rates are derived in the classes $\mathcal{G}_s(R,\bar{\alpha})$ and $\mathcal{H}_s(R,\bar{\alpha})$ of pairs $\mathcal{P} = (\gamma, k)$ that satisfy

$$\alpha \in [0,\bar{\alpha}] \quad \text{and} \quad k \in C^s(\mathbb{R}\setminus\{0\})$$

as well as L^1-type or L^2-type smoothness conditions, respectively.

Convergence Rates for the parameters

Theorem

Let $s \geq 2, R, \bar{\alpha} > 0$ *and assume* $e^{-A} \lesssim \Delta^2$ *and* $\Delta\|\delta\|_{l^2}^2 \lesssim \|\delta\|_{l^\infty}^2$. *We choose the cut-off value* $U_{\bar{\alpha}} := \varepsilon^{-2/(2s+2T\bar{\alpha}+1)}$ *to obtain*

$$\sup_{\mathcal{P}\in\mathcal{G}_s(R,\bar{\alpha})} \mathbb{E}_{\mathcal{P}}[|\hat{\gamma} - \gamma|^2]^{\frac{1}{2}} \lesssim \varepsilon^{\frac{2s}{2s+2T\bar{\alpha}+1}},$$

$$\sup_{\mathcal{P}\in\mathcal{G}_s(R,\bar{\alpha})} \mathbb{E}_{\mathcal{P}}[|\hat{\alpha}_0 - \alpha|^2]^{\frac{1}{2}} \lesssim \varepsilon^{\frac{2(s-1)}{2s+2T\bar{\alpha}+1}}.$$

Convergence Rate for the k-function

Theorem

Let $s \geq 1, R, \bar{\alpha} > 0$ *and assume* $Ae^{-A} \lesssim \Delta^2$ *and* $\Delta(\|(x_j\delta_j)_j\|_{l^2}^2 + \|\delta_j\|_{l^2}^2) \lesssim \|\delta\|_{l^\infty}^2$. *Furthermore, let* $\hat{\gamma}$ *satisfy* $\sup_{\mathcal{P}} \mathbb{E}_{\mathcal{P}}[|\hat{\gamma} - \gamma|^2]^{1/2} \lesssim \varepsilon^{(2s+1)/(2s+2T\bar{\alpha}+5)}$. *The choice of the cut-off value* $U_{\bar{\alpha}}$ *yields*

$$\sup_{\mathcal{P} \in \mathcal{H}_s(R,\bar{\alpha})} \mathbb{E}_{\mathcal{P}}[\|\hat{k}_e - k_e\|_{L^2}^2]^{\frac{1}{2}} \lesssim \varepsilon^{\frac{2s}{2s+2T\bar{\alpha}+5}}.$$

Remark: For $\bar{\alpha} \leq (5s-1)/(2T)$ the $\hat{\gamma}$ from above satisfies the assumption and thus the theorem, restricted to $\mathcal{P} \in \mathcal{G}_s(R,\bar{\alpha}) \cap \mathcal{H}_s(R,\bar{\alpha})$, is applicable to the proposed procedure.

Asymptotic equivalence

- Nonparametric regression model:

$$O_j = \mathcal{O}(x_j) + \delta_j \varepsilon_j \quad j = 1, \ldots, N,$$

with $x_j = H_N^{-1}(j/(N+1))$ and $\delta_j = \delta(x_j)$ for a c.d.f. H_N with $h_N := H_N' > 0$ and $\delta \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

- Continuous white noise model:

$$\mathrm{d}Z_{\mathcal{P}}(x) = \mathcal{O}_{\mathcal{P}}(x)\,\mathrm{d}x + N^{-1/2}\lambda_N(x)\,\mathrm{d}W(x), \quad x \in [-A_N, A_N],$$

with a Brownian motion W, an option function $\mathcal{O}_{\mathcal{P}}$ and $A_N \to \infty$.

- Brown and Low (1996): asymptotic equivalence holds for

$$\lambda_N(x) = \delta^2(x)/h_N(x), \quad x \in [-A_N, A_N].$$

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Lower bounds

Theorem

In the white noise model we obtain for some $\beta > 1$ and any $s \geq 2, R, \bar{\alpha} > 0$:

$$\inf_{\hat{\gamma}} \sup_{\mathcal{P} \in \mathcal{G}_s(R,\bar{\alpha})} \mathbb{E}_{\mathcal{P}}[|\hat{\gamma} - \gamma|^2]^{\frac{1}{2}} \gtrsim \left(\varepsilon(\log \varepsilon^{-1})^{-\frac{\beta}{2}}\right)^{\frac{2s}{(2s+2T\bar{\alpha}+1)}},$$

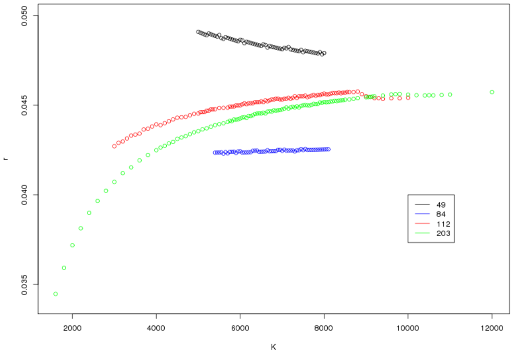
$$\inf_{\hat{\alpha}_j} \sup_{\mathcal{P} \in \mathcal{G}_s(R,\bar{\alpha})} \mathbb{E}_{\mathcal{P}}[|\hat{\alpha} - \alpha|^2]^{\frac{1}{2}} \gtrsim \left(\varepsilon(\log \varepsilon^{-1})^{-\frac{\beta}{2}}\right)^{\frac{2(s-1)}{2s+2T\bar{\alpha}+1}} \quad \textit{and}$$

$$\inf_{\hat{k}_e} \sup_{\mathcal{P} \in \mathcal{H}_s(R,\bar{\alpha})} \mathbb{E}_{\mathcal{P}}[\|\hat{k}_e - k_e\|_{L^2}^2]^{\frac{1}{2}} \gtrsim \left(\varepsilon(\log \varepsilon^{-1})^{-\frac{\beta}{2}}\right)^{\frac{2s}{2s+2T\bar{\alpha}+5}}.$$

Remark: For the estimation of γ and α we can even consider $\beta = 0$ in the case $s \geq 3$.

Real data: Put-call parity violated?

ODAX prices from May 29, 2008:



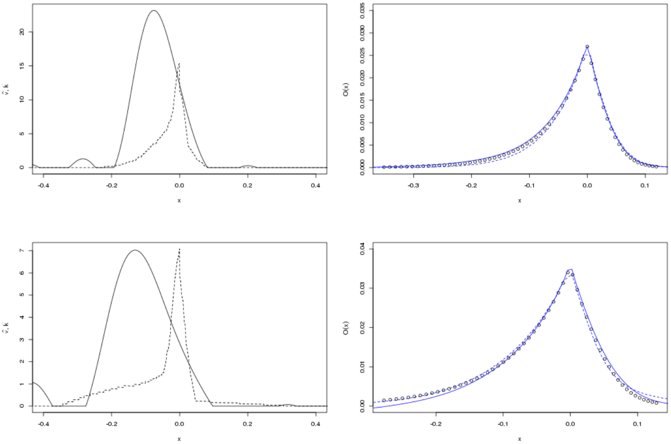
Interest rate r given by $C(K,T) - P(K,T) = S_0 - Ke^{-rT}$ for each pair of put and call options with the same maturity and strike.

Real data

We calibrate ODAX prices from May 29, 2008, with two and three months to maturity to the finite activity Lévy model (FA) and the self-decomposable Lévy model (SD):

	N	61	55
	T	0.136	0.233
FA	$\hat{\sigma}$	0.108	0.117
	$\hat{\gamma}$	0.224	0.151
	$\hat{\lambda}$	3.462	1.493
	Residuals	0.004	0.008
SD	$\hat{\gamma}$	0.413	0.240
	$\hat{\alpha}$	17.500	3.973
	Residuals	0.007	0.007

We use a least squares choice of the tuning parameter U.



Left: Estimated k-Functions (dashed) and jump densities (solid).
Right: Calibrated option functions in both models.

Conclusions

- The nonparametric calibration of exponential Lévy models could be extended to self-decomposable processes that is to an infinite activity case.
- This nonlinear inverse problem is mildly ill-posed.
- The proposed estimation method achieves minimax rates (up to a logarithmic factor).
- The model and the calibration method are suitable to fit real data.

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Thank you very much for your attention!

Concentration of $\hat{\alpha}_0$

Proposition

Let $\alpha \in [0, \bar{\alpha}]$. We assume there are $C_1, C_2 \in (0, \infty)$ such that (ε_j) fulfill for all $\kappa, N > 0$ and $a_j \in \mathbb{R}, j = 1, \dots, N$:

$$\mathbb{P}\Big(\Big|\sum_{j=1}^N a_j \varepsilon_j\Big| \geq \kappa\Big) \leq C_1 \exp\Big(-C_2 \frac{\kappa^2}{\sum_{j=1}^N a_j^2}\Big).$$

Then there is a constant $c > 0$ and for all $\kappa > 0$ there is an $\varepsilon_0 \sim \kappa^{(2s+2T\bar{\alpha}+1)/(2s-2)}$, such that $\hat{\alpha}$ satisfies for all $\varepsilon < \varepsilon_0 \wedge 1$:

$$\mathbb{P}(|\hat{\alpha} - \alpha| \geq \kappa) \leq ((7N+1)C_1 + 2) \exp\Big(-c(\kappa^2 \wedge \kappa^{\frac{1}{2}})\varepsilon^{-\frac{s-1}{2s+2T\bar{\alpha}+1}}\Big).$$

This exponential concentration yield an α-adaptive estimation procedure by sample splitting.