Confidence Sets in Nonparametric Calibration of Exponential Lévy Models

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Outline

Exponential Lévy Model

Exponential Lévy Model (Merton 1976)

Let $(e^{-rt}S_t, t \geq 0)$ be a martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t))$, where $r \geq 0$ is the riskless interest rate.

Let $S_t = S_0 e^{rt+X_t}$ with a Lévy process X_t for $t \geq 0$, where $S_0 > 0$ is the present value.

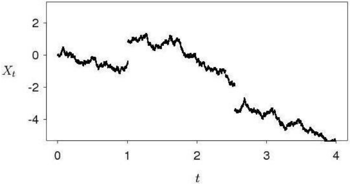
Nonparametric estimation of the Lévy triplet (σ^2, γ, ν) from option data (Cont & Tankov 2004)

Aim: Confidence intervals and confidence sets

Restriction: Let (X_t) have finite intensity and an absolutely continuous jump measure.

Lévy Processes

A Lévy process $(X_t, t \geq 0)$ is a stochastically continuous process with independent and stationary increments, $X_0 = 0$.



Lévy processes are characterized by their Lévy triplets (σ^2, γ, ν), with volatility $\sigma \geq 0$, drift $\gamma \in \mathbb{R}$, jump measure ν and intensity $\lambda = \nu(\mathbb{R})$.

Lévy-Khintchine representation:

$$\varphi_T(u) := \mathbb{E}[e^{iuX_T}] = \exp\left(T\left(-\frac{\sigma^2 u^2}{2} + i\gamma u + \int_{-\infty}^{\infty}(e^{iux} - 1)\nu(x)\mathrm{d}x\right)\right).$$

Observations

$\mathcal{C}(K,T) :=$ Prices of European call options,

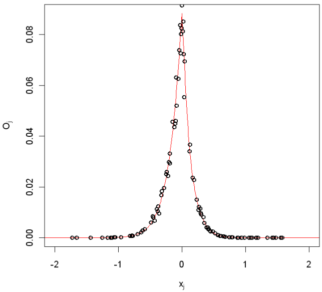
$\mathcal{P}(K,T) :=$ Prices of European put options, K strike, T maturity.

Put-call parity: $\mathcal{C}(K,T) - \mathcal{P}(K,T) = S_0 - e^{-rT}K$.

Substitute K by $x := \log(K/S_0) - rT$.

Define the option function $\mathcal{O}$ by:

$$\mathcal{O}(x) := \begin{cases} S_0^{-1}\mathcal{C}(x,T), & x \geq 0, \\ S_0^{-1}\mathcal{P}(x,T), & x < 0. \end{cases}$$



Observations:

$$O_j = \mathcal{O}(x_j) + \epsilon_j,$$

ϵ_j independent, $\mathbb{E}[\epsilon_j] = 0$ and $\sup_j \mathbb{E}[\epsilon_j^4] < \infty$.

Description of the Method

Putting $\mu(x) := e^x \nu(x)$ and $\mathcal{F}\mathcal{O}(u) := \int_{-\infty}^{\infty} e^{iux}\mathcal{O}(x)\mathrm{d}x$, from an option pricing formula (Carr & Madan 1999) and from the Lévy-Khintchine representation follows:

$$\begin{aligned}\psi(u) &:= \frac{1}{T}\log\Big(1 + iu(1+iu)\mathcal{F}\mathcal{O}(u)\Big)\\ &= -\frac{\sigma^2}{2}u^2 + i(\sigma^2+\gamma)u + (\sigma^2/2+\gamma-\lambda) + \mathcal{F}\mu(u).\end{aligned}$$

Method (Belomestny & Reiß 2006):

1. Interpolate the data (O_j) to obtain a function $\mathcal{O}_\epsilon(x)$.
2. Calculate $\psi_\epsilon(u)$ with $\mathcal{F}\mathcal{O}_\epsilon$ instead of $\mathcal{F}\mathcal{O}$.
3. Determine $\hat{\sigma}^2$, $\hat{\gamma}$, $\hat{\lambda}$ from the coefficients of the quadratic polynomial. $\mathcal{F}\hat{\mu}$ is given by the remainder.

Definition of $\hat{\sigma}^2$

$\psi_\epsilon(u)$ is an empirical version of

$$\psi(u) = -\frac{\sigma^2}{2}u^2 + i(\sigma^2+\gamma)u + \left(\sigma^2/2+\gamma-\lambda\right) + \mathcal{F}\mu(u).$$

Let w_σ^U be a weight function such that

$$\int_{-U}^{U} \frac{-u^2}{2} w_\sigma^U(u)\mathrm{d}u = 1, \quad \int_{-U}^{U} w_\sigma^U(u)\mathrm{d}u = 0, \quad w_\sigma^U(u) = U^{-3}w_\sigma(u/U).$$

Regularization by spectral cut-off for $|u| > U$:

$$\begin{aligned}\hat{\sigma}^2 &:= \int_{-U}^{U} \mathrm{Re}(\psi_\epsilon(u)) w_\sigma^U(u)\mathrm{d}u,\\ &= \sigma^2 + \underbrace{\int_{-U}^{U} \mathrm{Re}(\mathcal{F}\mu(u)) w_\sigma^U(u)\mathrm{d}u}_{\text{approximation error}} + \underbrace{\int_{-U}^{U} \mathrm{Re}(\psi_\epsilon(u)-\psi(u)) w_\sigma^U(u)\mathrm{d}u}_{\text{stochastic error}}.\end{aligned}$$

Known Results

For these estimators holds (Belomestny & Reiß 2006):

- The Lévy triplet is estimated consistently.
- In general the rates are logarithmic. If $\sigma = 0$ is known, the rates are polynomial.
- The rates depend on the smoothness s of μ.
- The rates are optimal in the minimax sense.

Real Data: DAX options

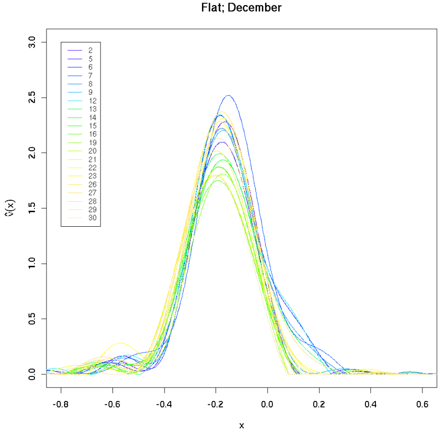


Figure: Estimated Jump Densities by DAX options May 2008

Continuous Observations

It is easier to work in the Gaussian white noise model, where $\mathcal{O}$ is observed continuously:

$$\mathrm{d}\mathcal{O}_\epsilon(x) = \mathcal{O}(x)\mathrm{d}x + \epsilon\,\delta(x)\mathrm{d}W(x),$$

with Brownian motion W, $\delta \in L^2(\mathbb{R})$ and $\epsilon > 0$.

There is an asymptotic equivalence between nonparametric regression and the Gaussian white noise model.

Theorem (Asymptotic Normality for $\sigma = 0$)

If

- $\epsilon U(\epsilon)^{5/2} \to 0$ as $\epsilon \to 0$ (small variance),
- $\epsilon U(\epsilon)^{(2s+5)/2} \to \infty$ as $\epsilon \to 0$ (undersmoothing),

then

$$\frac{1}{\epsilon}\begin{pmatrix} U(\epsilon)^{-1/2}(\hat{\gamma}-\gamma) \\ U(\epsilon)^{-3/2}(\hat{\lambda}-\lambda) \\ U(\epsilon)^{-5/2}(\hat{\mu}(x_1)-\mu(x_1)) \\ \vdots \\ U(\epsilon)^{-5/2}(\hat{\mu}(x_n)-\mu(x_n)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} d(0)\int_0^1 u^2 w_\gamma(u)\mathrm{d}V_0(u) \\ d(0)\int_0^1 u^2 w_\lambda(u)\mathrm{d}W_0(u) \\ d(x_1)\int_0^1 u^2 w_\mu(u)\mathrm{d}W_{x_1}(u)/2\pi \\ \vdots \\ d(x_n)\int_0^1 u^2 w_\mu(u)\mathrm{d}W_{x_n}(u)/2\pi \end{pmatrix}$$

where $V_0, W_0, W_{x_1} \dots, W_{x_n}$ are independent Brownian motions and $d(x) := 2\sqrt{\pi}\,\delta(x+T\gamma)T^{-1}\exp(T(\lambda-\gamma))$.

Theorem (Asymptotic Normality for $\sigma > 0$)

If

- $\epsilon U(\epsilon)^2\sqrt{\log(U(\epsilon))}\exp(T\sigma^2 U(\epsilon)^2/2) \to 0$ as $\epsilon \to 0$ (small variance),
- $\epsilon U(\epsilon)^{s+1}\exp(T\sigma^2 U(\epsilon)^2/2) \to \infty$ as $\epsilon \to 0$ (undersmoothing),

then

$$\frac{1}{\epsilon e^{T\sigma^2 U(\epsilon)^2/2}}\begin{pmatrix} U(\epsilon)^2(\hat{\sigma}^2-\sigma^2) \\ U(\epsilon)(\hat{\gamma}-\gamma) \\ (\hat{\lambda}-\lambda) \\ U(\epsilon)^{-1}(\hat{\mu}(x)-\mu(x)) \end{pmatrix} - \begin{pmatrix} d\, w_\sigma(1) W_\epsilon \\ d\, w_\gamma(1) V_\epsilon \\ d\, w_\lambda(1) W_\epsilon \\ d\, w_\mu(1) Z_\epsilon(x)/2\pi \end{pmatrix} \xrightarrow{\mathbb{P}} 0,$$

where W_ϵ and V_ϵ are normal random variables,

$$\begin{pmatrix} W_\epsilon \\ V_\epsilon \end{pmatrix} \xrightarrow{d} N(0, I_2),$$

$$Z_\epsilon(x) := \cos(U(\epsilon)x)W_\epsilon + \sin(U(\epsilon)x)V_\epsilon,$$

$$d := \sqrt{2}\,\|\delta\|_{L^2(\mathbb{R})}\sigma^{-2}T^{-2}\exp(T(\lambda-\gamma-\sigma^2/2)).$$

Strategy of Proof I

The stochastic errors involve the difference:

$$\psi_\epsilon(u) - \psi(u) \;=\; \frac{1}{T}\log\left(1 + \frac{\epsilon\, iu(1+iu)}{\varphi_T(u-i)}\int_{-\infty}^{\infty} e^{iux}\delta(x)\mathrm{d}W(x)\right),$$

Linearization:

$$\mathcal{L}_\epsilon(u) := \frac{\epsilon\, iu(1+iu)}{T\varphi_T(u-i)}\int_{-\infty}^{\infty} e^{iux}\delta(x)\mathrm{d}W(x), \quad \text{Gaussian process,}$$

$$\mathcal{R}_\epsilon(u) := \psi_\epsilon(u) - \psi(u) - \mathcal{L}_\epsilon(u), \quad \text{remainder term.}$$

Proposition

$$\mathbb{E}\left[\sup_{u\in[-U,U]}|\mathcal{L}_\epsilon(u)|\right] \lesssim \epsilon U^2\sqrt{\log(U)}\exp(T\sigma^2U^2/2) \quad \text{as } U\to\infty.$$

Strategy of Proof II

Stochastic error:

$$\int_{-1}^{1} \mathrm{Re}(\psi_\epsilon(Uu) - \psi(Uu)) w_\sigma(u) \mathrm{d}u$$
$$= \underbrace{\int_{-1}^{1} \mathrm{Re}(\mathcal{L}_\epsilon(Uu)) w_\sigma(u) \mathrm{d}u}_{\text{Normal random variable}} + \int_{-1}^{1} \mathrm{Re}(\mathcal{R}_\epsilon(Uu)) w_\sigma(u) \mathrm{d}u.$$

- Derive the asymptotic distribution of the first term. Behaves differently for $\sigma = 0$ and $\sigma > 0$.
- Second term negligible by Taylor expansion and the bound on the supremum of the Gaussian process $\mathcal{L}_\epsilon$.

Confidence Sets

For $\sigma = 0$ and for $\sigma > 0$ known:

Confidence intervals:

$$\lim_{\epsilon \to 0} \inf_{\mathcal{T}} \mathbb{P}(\rho \in I_{\rho,\epsilon}) = 1 - \alpha$$

for all $\rho \in \{\gamma, \lambda, \nu(x) | x \in \mathbb{R}\}$ from quantiles of the normal distribution.

Confidence sets:

$$\lim_{\epsilon \to 0} \inf_{\mathcal{T}} \mathbb{P}((\gamma, \lambda)^\top \in A_\epsilon) = 1 - \alpha$$

from quantiles of the chi-square distribution.

Confidence Intervals

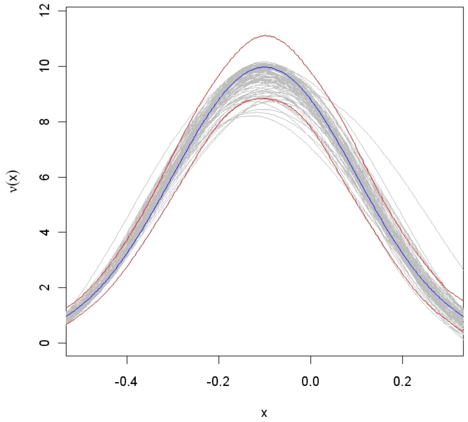


Figure: True Lévy density with pointwise 95% confidence intervals and 100 estimated Lévy densities from a Monte Carlo simulation

Conclusion

- Derived joint asymptotic distribution in a nonlinear ill-posed inverse problem.
- For $\sigma = 0$:
  - variance depends on the noise level δ locally
  - variance depends on the whole weight function $w_\cdot$
  - asymptotically independent
- For $\sigma > 0$:
  - variance depends on δ globally
  - variance depends on $w_\cdot$ only through $w_\cdot(1)$
  - covariances do not converge
- Construction of confidence intervals and confidence sets.

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Thank you for your attention!

Fitted Option Functions

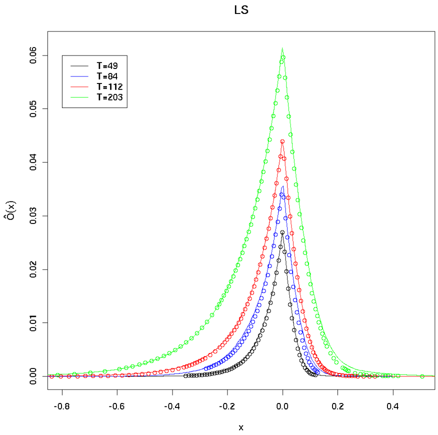

Figure: Option data and fitted option functions, May 29, 2008