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Explicit characterization of the super-replication strategy in financial markets with partial transaction costs

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Abstract

We consider a continuous time multivariate financial market with proportional transaction costs and study the problem of finding the minimal initial capital needed to hedge, without risk, European-type contingent claims. The model is similar to the one considered in Bouchard and Touzi (2000) except that some of the assets can be exchanged freely, i.e. without paying transaction costs. This is the so-called *non-efficient friction* case. To our knowledge, this is the first time that such a model is considered in a continuous time setting. In this context, we generalize the result of the above paper and prove that the *super-replication price* is given by the cost of the cheapest hedging strategy in which the number of non-freely exchangeable assets is kept constant over time.

Key words: Transaction costs, hedging options, viscosity solutions.
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1 Introduction

Since the 90's, there has been many papers devoted to the proof of the conjecture of Davis and Clark (1994) : in the context of the Black and Scholes model with proportional transaction costs, the cheapest super-hedging strategy for a European call option is just the price (up to initial transaction costs) of the underlying asset. The first proofs of this result were obtained, independently, by Soner, Shreve and Cvitanić (1995) and Levental and Skorohod (1995). In a one-dimensional Markov diffusion model, the result was extended by Cvitanić, Pham and Touzi (1999) for general contingent claims. Their approach relies on the dual formulation of the super-replication cost (see Jouini and Kallal 1995 and Cvitanić and Karatzas 1996).

The multivariate case was then considered by Bouchard and Touzi (2000). In contrast to Cvitanić, Pham and Touzi (1999), they did not use the dual formulation but introduced a family of fictitious markets without transaction costs but with modified price processes evolving in the bid-ask spreads of the original market. Then, they defined the associated super-hedging costs and showed that they provide lower bounds for the original one. By means of a direct dynamic programming principle for stochastic target problems, see e.g. Soner and Touzi (2002), they provided a PDE characterization for the upper bound of these auxiliary super-hedging prices. Using similar arguments as in Cvitanić, Pham and Touzi (1999), they were then able to show that the associated value function is concave in space and non-increasing in time. This was enough to show that it corresponds to the price of the cheapest buy-and-hold strategy in the original market. A different proof relying on the dual formulation for multivariate markets, see Kabanov (1999), was then proposed by Bouchard (2000).

A crucial feature of all the analysis is that transaction costs are *efficient*, i.e. there is no couple of freely exchangeable assets.

In this paper, we propose a first attempt to characterize the super-replication strategy in financial markets with "partial" transaction costs, where some assets can be exchanged freely. To our knowledge, this is the first time that such a model is considered in a continuous time setting.

As a first step, we follow the approach of Bouchard and Touzi (2000). We introduce a family of fictitious markets and provide a PDE characterization similar to the one obtained in this paper. However, in our context, one can only show that the corresponding value function is concave in some directions (the ones where transaction costs are effective - roughly speaking, not equal to zero), and this is not sufficient to provide a precise characterization of the super-hedging strategy.

To overcome this difficulty, we define a new control problem. With the help of a comparison principle for PDE's, we show that it provides a new lower bound for the original super-hedging price. This new approach allows us to characterize the optimal hedging strategy : it consists in keeping constant the number of non-freely exchangeable assets held in the portfolio and hedging the remaining part of the claim by trading dynamically on the freely exchangeable ones.

The paper is organized as follows. After setting some notations in Section 2, we describe the model and the super-replication problem in Section 3. The main result of the paper is stated in Section 4. In Section 5, we introduce an auxiliary super-hedging problem similar to the one considered in Bouchard and Touzi (2000) and derive the PDE associated to the value function. The main idea of this paper, which allows to conclude the proof, is presented in Section 6. Section 7 contains the proof of some intermediary, and more technical, results.

2 Notations

For the reader's convenience, we first introduce the main notations of this paper. All elements of $I\!R^n$ are identified with column vectors and the scalar product is denoted by $\cdot$. We denote by $I\!M^{n,p}$ the set of all real-valued matrices with n rows and p columns, and by $I\!M_+^{n,p}$ the subset of $I\!M^{n,p}$ whose elements have non-negative entries. If $n = p$, we write $I\!M^n$ and $I\!M_+^n$ for $I\!M^{n,n}$ and $I\!M_+^{n,n}$. The Euclidean norm is simply denoted by $|\cdot|$, transposition is denoted by $'$. For $M \in I\!M^n$, we set $\mathrm{Tr}[M] := \sum_{i=1}^n M^{ii}$ the associated trace. For $x \in I\!R^n$, $\mathrm{diag}[x]$ denotes the diagonal matrix of $I\!M^n$ whose i-th diagonal element is x^i. We denote by $\mathbf{1}_i$ the vector of $I\!R^n$ defined by $\mathbf{1}_i^j = 1$ if $j = i$ and 0 otherwise. Given a smooth function φ mapping $I\!R^n$ into $I\!R^p$, we denote by $D_z\varphi$ the (partial) Jacobian matrix of φ with respect to its z variable. In the case $p = 1$, we denote by $D^2_{zs}\varphi$ the matrix defined as $(D^2_{zs}\varphi)^{ij} = \partial^2\varphi/\partial z^i \partial s^j$. If φ depends only on z, we simply write $D\varphi$ and $D^2\varphi$ in place of $D_z\varphi$ and $D^2_{zz}\varphi$. All inequalities involving random variables have to be understood in the $\mathbb{P}-$ a.s. sense.

3 The model

3.1 The financial market

We first explain in details the financial market we shall consider in this paper and outline the difference with the literature.

3.1.1 The risky assets and the structure of transaction costs

We consider a financial market which consists of one bank account, with constant price process normalized to unity, and two different types of risky assets. The non-risky asset is taken as the "numéraire", sometimes called "cash" or thereafter.

The first m risky assets, $P = (P^1, \ldots, P^m)$, can be exchanged freely with the numéraire, while any exchange involving the d other assets, $Q = (Q^1, \ldots, Q^d)$, is subject to proportional transaction costs.

Transaction costs are described by a matrix $\lambda = (\lambda^{ij})_{i,j=0}^d \in \mathbb{M}_+^{d+1}$ satisfying

$$(\mathbf{H}_\lambda) \quad : \quad \lambda^{ij} + \lambda^{ji} > 0 \ \text{ for all } \ i, j = 0, \ldots, d, \ i \neq j \ .$$

The buying price (resp. selling price) in numéraire at time t of one unit of Q^i is given by $\pi_t^{0i+} := (1 + \lambda^{0i})Q^i(t)$ (resp. $\pi_t^{0i-} := Q^i(t)/(1 + \lambda^{i0})$). Thus, $[\pi_t^{0i-} \ , \ \pi_t^{0i+}]$ is the bid-ask spread price of Q^i in terms of cash.

As in Bouchard and Touzi (2000) and Kabanov (1999), we also allow for direct exchanges between the assets $(Q^i)_i$. Let $\tau_t^{ij} := Q^j(t)/Q^i(t)$ be the exchange rate between Q^i and Q^j before transaction costs at time t. To obtain one unit of Q^j one has to pay $\pi_t^{ij+} := \tau_t^{ij}(1 + \lambda^{ij})$ units of Q^i. When selling one unit of Q^j, one receives $\pi_t^{ij-} := \tau_t^{ij}/(1 + \lambda^{ji})$ units of Q^i. Here again, $[\pi_t^{ij-} \ , \ \pi_t^{ij+}]$ is the bid-ask spread price of Q^j in terms of Q^i.

Remark 3.1 If we want to avoid direct exchanges between Q^i and Q^j, it suffices to choose λ^{ij} and λ^{ji} such that $(1+\lambda^{i0})(1+\lambda^{0j}) = 1+\lambda^{ij}$ and $(1+\lambda^{j0})(1+\lambda^{0i}) = 1+\lambda^{ji}$. With this choice of λ, making a direct exchange between Q^i and Q^j or passing through the cash account to make the corresponding exchange are two equivalent strategies. Thus everything works as if direct exchanges where prohibited.

Remark 3.2 The assumption $(\mathbf{H}_\lambda)$ is usually called *efficient friction* case. It means that the assets $(Q^i)_i$ can actually not be exchanged freely with the cash or between themselves, or equivalently that the bid-ask spreads $([\pi_t^{ij-} \ , \ \pi_t^{ij+}])_{1 \leq j \neq i \leq d}$ and $[\pi_t^{0i-} \ , \ \pi_t^{0i+}]$ are not reduced to a singleton.

Remark 3.3 Observe that we can always assume that

$$(1+\lambda^{ij}) \;\le\; (1+\lambda^{ik})(1+\lambda^{kj}) \;\;,\; i,\; j,\; k=0,\ldots,d\;, \tag{3.1}$$

since otherwise it would be cheaper to transfer money from the account i to j by passing through k rather than directly. Then, for any "optimal" strategy the effective cost between i and j would be $\tilde{\lambda}^{ij} := (1+\lambda^{ik})(1+\lambda^{kj})-1$. Thus, after possibly modifying λ, we can obtain a new market, equivalent to the previous one, such that (3.1) holds.

In this paper, the price process of all the risky assets $S := (P(t), Q(t))_{t\le T}$ is assumed to be a $\mathbb{R}_+^{m+d}$-valued stochastic process defined by the following stochastic differential system

$$dS(t) \;=\; \mathrm{diag}[S(t)]\;\sigma(t,S(t))\;dW(t)\;, \qquad t\;\le\;T\;, \tag{3.2}$$

where T is a finite time horizon and $\{W(t),\ 0\le t\le T\}$ is a $m+d$-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the usual assumptions. In the following, we shall denote by $\mathbb{F}=\{\mathcal{F}_t,\ 0\le t\le T\}$ the $\mathbb{P}$-augmentation of the filtration generated by W.

Here $\sigma(.,.)$ is an $\mathbb{M}^{m+d}$-valued function. We shall assume all over the paper that the function $\mathrm{diag}[s]\sigma(t,s)$ satisfies the usual Lipschitz and linear growth conditions in order for the process S to be well-defined and that $\sigma(t,s)$ is invertible with $\sigma(t,s)^{-1}$ locally bounded, for all $(t,s)\in[0,T]\times\mathbb{R}_+^{m+d}$.

Remark 3.4 As usual, there is no loss of generality in defining S as a martingale since we can always reduce the model to this context by an appropriate change of measure (under mild conditions on the initial coefficients).

3.1.2 The wealth process

The initial holdings is described by a vector $x=(x^0,\ldots,x^d)\in\mathbb{R}^{d+1}$: x^0 is the initial dotation in cash and x^i the initial amount invested in Q^i, $i\le d$. Since P^1 up to P^m can be freely exchanged with the numéraire, we do not need to isolate the initial dotations in theses assets.

A trading strategy is described by a pair $\nu=(\phi,L)$ where ϕ is a $\mathbb{R}^m$-valued predictable process satisfying $\int_0^T|\phi(t)|^2dt<\infty$ and $L=[L^{ij}]_{i,j=0}^d$ is an $\mathbb{M}_+^{1+d}$-valued process with initial value $L(0-) = 0$, such that L^{ij} is $\mathbb{F}-$adapted, right-continuous, and nondecreasing for all $i, j = 0, \ldots, d$.

Here, $\phi^i(t)$ denotes the number of units of P^i held in this portfolio at time t. If $L = 0$, the part of the portfolio invested in cash and in freely exchangeable assets, X^0_x, evolves as usual according to $X^0_x(t) = x^0 + \int_0^t \phi(r) \cdot dP(r)$, while the account invested in the asset Q^i has the dynamics

$$X^i_x(t) \;=\; x^i + \int_0^t \frac{X^i_x(r)}{Q^i(r)}\, dQ^i(r)\,.$$

For $i, j \geq 0$, $L^{ij}(t)$ denotes the cumulated amount of money transferred to the account X^j_x by selling units of Q^i (of cash if $i = 0$) up to time t. In view of the structure of transaction costs described in Section 3.1.1, the wealth process X^ν_x induced by $\nu = (\phi, L)$ solves

$$\begin{aligned} X^0(t) &= x^0 + \int_0^t \phi(r) \cdot dP(r) + \sum_{j=1}^{d} \left[L^{j0}(t) - (1 + \lambda^{0j}) L^{0j}(t) \right] \qquad (3.3) \\ X^i(t) &= x^i + \int_0^t \frac{X^i(r)}{Q^i(r)}\, dQ^i(r) + \sum_{j=0}^{d} \left[L^{ji}(t) - (1 + \lambda^{ij}) L^{ij}(t) \right] \quad \text{for } 1 \leq i \leq d\,. \end{aligned}$$

We conclude this section by insisting on the difference between this model and the existing literature. In Bouchard and Touzi (2000) and Kabanov (1999), see also the references quoted in the introduction, there is no couple of freely exchangeable assets. In our model, this corresponds to the case $m = 0$ or $P \equiv 0$. This assumption is crucial in the proofs of the above papers. To our knowledge, this is the first time that such a model is considered in a continuous time setting (see Schachermayer 2004, Kabanov et al. 2003 and the references therein for discrete time models).

3.2 The super-replication problem

Following Kabanov (1999), we define the *solvency region* :

$$K := \left\{ x \in \mathbb{R}^{1+d} \;:\; \exists\, a \in \mathbb{M}^{1+d}_+,\; x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda^{ij}) a^{ij}) \;\geq\; 0 \;\; \forall\, i = 0, \ldots, d \right\}.$$

The elements of K can be interpreted as the vectors of portfolio holdings such that the no-bankruptcy condition is satisfied, i.e. the liquidation value of the portfolio holdings x, through some convenient transfers $(a^{ij})_{ij}$, is nonnegative.

Clearly, the set K is a closed convex cone containing the origin. It induces the partial ordering on $I\!R^d$:

$$x_1 \succeq x_2 \quad \text{if and only if} \quad x_1 - x_2 \in K\ .$$

A trading strategy $\nu = (\phi, L)$ is said to be *admissible* if there is some some $c, \delta \in I\!R$ and γ in $I\!R^m$ such that

$$X_0^\nu(t) \ \succeq \ -(c + \gamma \cdot P(t), \delta\, Q(t)) \quad \text{for all } t \le T\ . \tag{3.4}$$

We denote by $\mathcal{A}$ the set of such trading strategies. Observe that, if X_0^ν satisfies the above condition, then, after possibly changing (c, δ, γ), it holds for X_x^ν too, $x \in I\!R^{1+d}$.

Remark 3.5 In Bouchard and Touzi (2000), the admissibility condition corresponds to

$$X_0^\nu(t) \ \succeq \ 0 \quad \text{for all } t \le T\ .$$

We relax this condition by allowing the wealth process to be bounded from below, in terms of K, by a portfolio made of a constant number of units of cash and of the different risky assets. This assumption is sufficient in our context to obtain a kind of super-martingale property for the portfolio process, see e.g. Touzi (1999) and Bouchard (2000). This will allow us to consider a more general class of contingent claims than the one used in Bouchard and Touzi (2000), see below.

A *contingent claim* is a $(1+d)$-dimensional $\mathcal{F}_T$-measurable random variable $g(S(T)) = (g^0(S(T)), \ldots, g^d(S(T)))$. Here, g maps $I\!R_+^{m+d}$ into $I\!R^{1+d}$ and satisfies

$$g(p,q) \succeq -(c + \gamma \cdot p, \delta\, q) \qquad \text{for all } (p,q) \in I\!R_+^m \times I\!R_+^d \tag{3.5}$$

for some $c,\ \delta \in I\!R$ and γ in $I\!R^m$.

In the rest of the paper, we shall identify a contingent claim with its pay-off function g. For $i = 1, \ldots, d$, the random variable $g^i(S(T))$ represents a target position in the asset Q^i, while $g^0(S(T))$ represents a target position in numéraire.

The *super-replication problem* of the contingent claim g is then defined by

$$p(0, S(0)) \ := \ \inf\left\{ w \in I\!R \ :\ \exists\, \nu \in \mathcal{A},\ X_{w\mathbf{1}_0}^\nu(T) \succeq g(S(T)) \right\}\ ,$$

where $w\mathbf{1}_0 = (w, 0, \ldots, 0) \in I\!R^{1+d}$. The quantity $p(0, S(0))$ is the minimal initial capital which allows to hedge the contingent claim g by means of some admissible trading strategy.

4 The explicit characterization

Before to state our main result, we need to define some additional notations. We first introduce the positive polar of K

$$\begin{aligned} K^* &:= \{\xi \in I\!R^{1+d} \ : \ \xi \cdot x \geq 0, \ \ \forall x \in K\} \qquad (4.1) \\ &= \{(\xi^0, \ldots, \xi^d) \in I\!R^{1+d}_+ \ : \ \xi^j \leq \xi^i (1+\lambda^{ij}) \ \forall \, i \neq j \in \{0, \ldots, d\}\} \ , \end{aligned}$$

together with its (compact) section

$$\Lambda := \{(\xi^0, \ldots, \xi^d) \in K^* \ : \ \xi^0 = 1\} \ \subset \ (0, \infty)^{1+d} \ . \qquad (4.2)$$

One easily checks that Λ is not empty since it contains the vector of $I\!R^{1+d}$ with all components equal to one. Moreover, it is a standard result in convex analysis that the partial ordering $\succeq$ can be characterized in terms of Λ by

$$x_1 \ \succeq \ x_2 \ \ \text{if and only if} \ \ \xi \cdot (x_1 - x_2) \geq 0 \quad \text{for all} \ \ \xi \in \Lambda \ , \qquad (4.3)$$

see e.g. Rockafellar (1970).

For $\xi = (\xi^0, \ldots, \xi^d) \in I\!R^{1+d}$, we define $\underline{\xi} = (\xi^1, \ldots, \xi^d)$ the vector of $I\!R^d$ obtained by removing the first component. With these notations, we set

$$G(p, q) \ := \ \sup_{\xi \in \Lambda} \ \xi \cdot g \left(p, \mathrm{diag}[\underline{\xi}]^{-1} q\right) \quad \text{for} \ \ (p, q) \ \text{in} \ I\!R^m_+ \times I\!R^d_+ \ ,$$

and denote by G^{conc} the concave envelope of G with respect to q.

4.1 Main result

The main result of this paper requires the additional conditions :
$(\mathbf{H}_\sigma)$: For all $i \leq m$ and $t \leq T$, $[\sigma(t, s)^{ij}]_{j \leq m+d}$ depends only on the m first components of s.
$(\mathbf{H_g})$: g is lower-semicontinuous, G^{conc} is continuous and has linear growth.

The assumption $(\mathbf{H}_\sigma)$ means that the volatility matrix $[\sigma(t, S(t))^{ij}]_{i \leq m, \, j \leq m+d}$ of the freely exchangeable assets P depends only on t and $P(t)$ but not on $Q(t)$. The more technical assumption $(\mathbf{H_g})$ is necessary for our PDE based approach.

Theorem 4.1 *Assume that* $(\mathbf{H}_\lambda)$-$(\mathbf{H}_\sigma)$-$(\mathbf{H_g})$ *hold. Then,*

$$p(0,S(0)) \;=\; \min\left\{w \in I\!R \;:\; \exists\, \nu \in \mathcal{A}^{BH},\; X^{\nu}_{w\mathbf{1_0}}(T) \succeq g(S(T))\right\}\;,$$

where

$$\mathcal{A}^{BH} \;:=\; \{\nu = (\phi, L) \in \mathcal{A} \;:\; L(t) = L(0) \text{ for all } 0 \le t \le T\}\;.$$

Moreover, there is some $\hat{\Delta} \in I\!R^d$ *such that*

$$p(0,S(0)) \;=\; \mathbb{E}\left[C(P(T);\hat{\Delta})\right] + \sup_{\xi\in\Lambda} \underline{\xi} \cdot \operatorname{diag}[\hat{\Delta}]Q(0)$$

where, for $\Delta \in I\!R^d$,

$$C(P(T);\Delta) := \sup_{q\in(0,\infty)^d} \; G^{conc}\left(P(T), q\right) - \Delta \cdot q\;,$$

and there is an optimal hedging strategy $(\phi, L) \in \mathcal{A}^{BH}$ *satisfying* $L = \hat{\Delta}$ *on* $[0,T]$.

The proof of this result will be provided in the subsequent sections.

As in the papers quoted in the introduction, we obtain that the cheapest hedging strategy consists in keeping the number of non-freely exchangeable assets, Q, constant in the portfolio (equal to $\hat{\Delta}$). The cost in numéraire of such a portfolio is equal to $\sup_{\xi\in\Lambda} \underline{\xi} \cdot \operatorname{diag}[\hat{\Delta}]Q(0)$, this follows from (4.3).

But here there is a remaining part, namely $g(S(T)) - (0, \operatorname{diag}[\hat{\Delta}]Q(T))$, which has to be hedged dynamically by investing in the freely exchangeable assets, P. It is done by hedging $C(P(T);\hat{\Delta})$. Under the assumption $(\mathbf{H}_\sigma)$, the law of P is unchanged under any equivalent probability measure which preserves its martingale feature. It follows that the hedging-price of $C(P(T);\hat{\Delta})$ is simply given by its expectation (recall that P is already a martingale under the original probability measure and that the interest rate is equal to 0).

Remark 4.1 In Bouchard and Touzi (2000), the authors make the assumption:

$$\mathbb{P}\left[Q(u) \in A|\mathcal{F}_t\right] \;>\; 0 \quad \mathbb{P}-\text{a.s.} \quad 0 \le t < u \le T$$

for all Borel subset A of $(0,\infty)^d$. It turns out to be not necessary, the important property being that $\sigma(t,s)$ is invertible with locally bounded inverse.

From Theorem 4.1, we can deduce an explicit formulation for $p(0, S(0))$.

Corollary 4.1 *Let the conditions of Theorem 4.1 hold. Then,*

$$p(0, S(0)) \;\; = \;\; \min_{\Delta \in \mathbb{R}^d} \left\{ \mathbb{E}\left[C(P(T); \Delta)\right] + \sup_{\xi \in \Lambda} \underline{\xi} \cdot \mathrm{diag}[\Delta] Q(0) \right\} \; .$$

Moreover, if $\hat{\Delta}$ solves the above optimization problem, then there is an optimal hedging strategy $(\phi, L) \in \mathcal{A}^{BH}$ which satisfies $L = \hat{\Delta}$ on $[0, T]$.

The proof will be provided in Section 6. We conclude this section with a remark which provides a characterization of the set of initial wealth which allow to hedge g:

$$\Gamma(g) \;\; := \;\; \left\{ x \in \mathbb{R}^{1+d} \; : \; \exists \, \nu \in \mathcal{A}, \; X_x^\nu(T) \succeq g(S(T)) \right\} \; .$$

Remark 4.2 Let the conditions of Theorem 4.1 hold.
1. In Touzi (1999), the result of Bouchard and Touzi (2000) is generalized to the case where the initial wealth, before to be increased by the super-replication price, is non-zero, i.e. the following problem is considered :

$$p(0, S(0); x) \;\; := \;\; \inf \left\{ w \in \mathbb{R} \; : \; \exists \, \nu \in \mathcal{A}, \; X_{x + w\mathbf{1}_0}^\nu(T) \succeq g(S(T)) \right\} \;\; ,$$

$x \in \mathbb{R}^{1+d}$. Our result can be easily extended to this case. Indeed, it suffices to observe from the wealth dynamics (3.3) that

$$X_{x+w\mathbf{1}_0}^\nu(T) \succeq g(S(T)) \;\; \Longleftrightarrow \;\; X_{w\mathbf{1}_0}^\nu(T) \succeq g(S(T)) - \left(x^0, \mathrm{diag}[Q(0)]^{-1} \mathrm{diag}[Q(T)] \underline{x}\right)$$

where $\underline{x}$ is obtained from x by dropping the first component. Hence, to characterize $p(0, S(0); x)$, it suffices to replace g by

$$g(s; x) \;\; := \;\; g(s) - (x^0, \mathrm{diag}[Q(0)]^{-1} \mathrm{diag}[q] \underline{x}) \;\; , \;\; s = (p, q) \in (0, \infty)^m \times (0, \infty)^d \; .$$

We then deduce from Theorem 4.1 and Corollary 4.1 that, for some $\hat{\Delta}(x) \in \mathbb{R}^d$,

$$\begin{aligned} p(0, S(0); x) \;\; &= \;\; \min \left\{ w \in \mathbb{R} \; : \; \exists \, \nu \in \mathcal{A}^{BH}, \; X_{x+w\mathbf{1}_0}^\nu(T) \succeq g(S(T)) \right\} \\ &= \;\; \mathbb{E}\left[C(P(T); \hat{\Delta}(x), x) \right] + \sup_{\xi \in \Lambda} \underline{\xi} \cdot \mathrm{diag}[\hat{\Delta}(x)] Q(0) \\ &= \;\; \min_{\Delta \in \mathbb{R}^d} \left\{ \mathbb{E}\left[C(P(T); \Delta, x)\right] + \sup_{\xi \in \Lambda} \underline{\xi} \cdot \mathrm{diag}[\Delta] Q(0) \right\} \end{aligned} \tag{4.4}$$

where

$$C(P(T);\Delta,x) := \sup_{q\in(0,\infty)^d}\left\{\ G^{conc}\left(P(T),q\right) - \left(x^0, \mathrm{diag}[Q(0)]^{-1}\mathrm{diag}[q]\underline{x}\right) - \Delta\cdot q\ \right\}\ .$$

2. The set of initial wealth which allow to hedge g can then be written

$$\Gamma(g) \;=\; \left\{x\in I\!R^{1+d}\ :\ p(0,S(0);x)\le 0\right\}\ .$$

3. In the limit case where $m=0$, we recover the result of Bouchard and Touzi (2000) and Touzi (1999).

4.2 Example

We conclude this section with a simple example. We consider a two dimensional Black and Scholes model, i.e. $m=d=1$, $\sigma(t,s)=\sigma\in I\!M^2$ with σ invertible. In this case, we have

$$\Lambda \;=\; \left\{(1,y)\in I\!R^2\ :\ \frac{1}{1+\lambda^{10}}\le y\le 1+\lambda^{01}\right\}\ ,\quad \lambda^{10}+\lambda^{01}\;>\;0\ .$$

We take g of the form

$$g(p,q) \;=\; \left([p-K]^+\mathbf{1}_{\{q>\bar K\}}\right)\mathbf{1}_0$$

with K, $\bar K>0$. Then,

$$G(s) \;=\; \left([p-K]^+\mathbf{1}_{\{q>\hat K\}}\right)\mathbf{1}_0 \quad\text{and}\quad G^{conc}(s) \;=\; [p-K]^+\left((q/\hat K)\wedge 1\right)\ ,$$

where $\hat K=\bar K/(1+\lambda^{10})$. For $\Delta\in I\!R$, we have

$$\begin{aligned} C(p;\Delta) &= \sup_{q\in(0,\infty)^d}\left\{\ [p-K]^+\left((q/\hat K)\wedge 1\right)-\Delta q\ \right\}\\ &= \begin{cases}\left([p-K]^+-\Delta\hat K\right)\mathbf{1}_{\{0\le\Delta\hat K\le[p-K]^+\}}\ , & \text{if } \Delta\ge 0\ ,\\ \infty & \text{otherwise.}\end{cases}\end{aligned}$$

Then, by Corollary 4.1,

$$\begin{aligned} p(0,S(0)) &= \min_{\Delta\ge 0}\left\{\mathbb{E}\left[C(P(T);\Delta)\right]+(1+\lambda^{01})\Delta Q(0)\right\}\\ &= \min_{\Delta\ge 0}\ \left\{\ \mathbb{E}\left[\left([P(T)-K]^+-\Delta\hat K\right)\mathbf{1}_{\{0\le\Delta\hat K\le[P(T)-K]^+\}}\right]\right.\\ &+\ \left.(1+\lambda^{01})\Delta Q(0)\ \right\}\\ &= \min_{\Delta\ge 0}\left\{\ \mathbb{E}\left[\left([P(T)-K-\Delta\hat K]^+\right)\right]+(1+\lambda^{01})\Delta Q(0)\ \right\}\ , \end{aligned} \tag{4.5}$$

where the expectation is convex in Δ. Then, if the optimal Δ is different from 0, it must satisfy the first order condition

$$-\hat{K}\ \mathbb{E}\left[\mathbf{1}_{\{P(T)-K\geq\Delta\hat{K}\}}\right]+(1+\lambda^{01})Q(0) \;=\; 0\ . \tag{4.6}$$

We consider two different cases.

1. If $\mathbb{P}\left[P(T)-K\geq 0\right]\leq(1+\lambda^{01})Q(0)/\hat{K}$, then, either the only solution of (4.6) is 0 or (4.6) has no solution. It follows that the optimum in (4.5) is achieved by $\hat{\Delta}=0$. Therefore

$$p(0,S(0)) \;=\; \mathbb{E}\left[[P(T)-K]^{+}\right]\ ,$$

and, by the Clark-Ocone's formula, the optimal hedging strategy (ϕ,L) is defined by $L=0$ and $\phi(t)=\mathbb{E}\left[P(T)\mathbf{1}_{\{P(T)\geq K\}}\mid\mathcal{F}_t\right]/P(t)$.

2. If $\mathbb{P}\left[P(T)-K\geq 0\right]>(1+\lambda^{01})Q(0)/\hat{K}$, then (4.6) has a unique solution $\hat{\Delta}>0$ which satisfies

$$p \;:=\; p(0,S(0)) \;=\; \mathbb{E}\left[[P(T)-K-\hat{\Delta}\hat{K}]^{+}\right]+(1+\lambda^{01})\hat{\Delta}Q(0)\ .$$

Observe that, in this model, $\hat{\Delta}$ can be computed explicitly in terms of the inverse of the cumulated distribution of the gaussian distribution. Let $\nu=(\phi,L)$ be defined by

$$L(t) \;=\; \hat{\Delta} \quad\text{and}\quad \phi(t) \;=\; \mathbb{E}\left[P(T)\mathbf{1}_{\{P(T)-K\geq\hat{\Delta}\hat{K}\}}\mid\mathcal{F}_t\right]/P(t) \quad \text{on } t\leq T\ .$$

By the Clark-Ocone's formula, we have

$$X^{\nu}_{p\mathbf{1}_0}(T) \;=\; \left([P(T)-K-\hat{\Delta}\hat{K}]^{+}\ ,\ \hat{\Delta}Q(T)\right)\ .$$

For ease of notations, let us define

$$\Psi := [P(T)-K]^{+}\mathbf{1}_{\{Q(T)\geq\bar{K}\}}\ .$$

On $\{\Psi\geq 0\}$, $P(T)\geq K$ and $Q(T)\geq\bar{K}$. If $P(T)-K\leq\hat{\Delta}\hat{K}$ then $X^{\nu}_{p\mathbf{1}_0}(T)=(0,\hat{\Delta}Q(T))$. Recalling the definition of $\hat{K}$, we then obtain

$$X^{\nu}_{p\mathbf{1}_0}(T) \;=\; (0,\hat{\Delta}Q(T)) \;\succeq\; (\hat{\Delta}\hat{K},0) \;\succeq\; (\Psi,0)\ .$$

If $P(T)-K>\hat{\Delta}\hat{K}$, then

$$X^{\nu}_{p\mathbf{1}_0}(T) \;=\; \left(P(T)-K-\hat{\Delta}\hat{K},\hat{\Delta}Q(T)\right) \;\succeq\; (P(T)-K,0) \;=\; (\Psi,0)\ .$$

On $\{\Psi=0\}$, we have $X^{\nu}_{p\mathbf{1}_0}(T)\succeq 0=(\Psi,0)$ since $\hat{\Delta}>0$.

5 Fictitious markets

In this section, we follow the arguments of Bouchard and Touzi (2000), i.e. we introduce an auxiliary control problem which can be interpreted as a super-replication problem in a fictitious market without transaction costs but were Q is replaced by a controlled process evolving in the "bid-ask" spreads associated to the transaction costs λ. This is obtained by introducing a controlled process $f(Y^{a,b})$, see below, which evolves in Λ. Then, the fictitious market is constructed by replacing S by $Z^{a,b} := (P, \mathrm{diag}[\underline{f}(Y^{a,b})]Q)$ and $g(S(T))$ by $f(Y^{a,b}(T)) \cdot g(S(T))$.

5.1 Parameterization of the fictitious markets

We first parameterize the compact set Λ. Since K^* is a polyhedral closed convex cone, we can find a family $e = (e_i)_{i \le n}$ in $(0,\infty)^{1+d}$, for some $n \ge 1$, such that, for all $\alpha \in I\!R^n_+$, $\sum_{i=1}^n \alpha^i e_i = 0$ implies $\alpha = 0$, and $K^* = \{\sum_{i=1}^n \alpha^i e_i \ , \ \alpha \in I\!R^n_+\}$. Then, we define the map f from $(0,\infty)^n$ into Λ by

$$f^j(y) \ := \ \left(\sum_{i=1}^n y^i e_i^j\right) / \left(\sum_{i=1}^n y^i e_i^0\right) \quad , \ y \in (0,\infty)^n \ , \ j = 0, \ldots, d \ .$$

In order to alleviate the notations, we define $\bar{f} = (\bar{f}^1, \ldots, \bar{f}^{m+d})$ by

$$\bar{f}^i = 1 \text{ for } i \le m \ , \ \ \bar{f}^{m+i} = f^i \text{ for } 1 \le i \le d \ \text{ and } \ \bar{F} := \mathrm{diag}[\bar{f}] \ , \ \underline{F} := \mathrm{diag}[\underline{f}] \ , \tag{5.1}$$

where $\underline{f} = (f^1, \ldots, f^d)$. Given a controlled process $Y_y^{a,b}$, to be defined later, we define the fictitious assets as

$$Z_y^{a,b} := \bar{F}(Y_y^{a,b})S = (P, R_y^{a,b}) \quad \text{where} \quad R_y^{a,b} := \underline{F}(Y_y^{a,b})Q \ . \tag{5.2}$$

By construction, the fictitious markets preserve the price process corresponding to the freely exchangeable assets P, and the new dynamics of the other assets satisfy

$$\pi^{ij-} \le R_y^{a,b,j}/R_y^{a,b,i} \le \pi^{ij+} \ \text{ and } \ \pi^{0j-} \le R_y^{a,b,j} \le \pi^{0j+} \quad \text{for } 1 \le i,j \le d \ , \tag{5.3}$$

see (4.1) and recall the definitions of Section 3.1.1. This means that the new exchange rates evolve in the bid-ask spreads of the original market.

We now turn to the construction of $Y_y^{a,b}$. Given some arbitrary parameter $\mu > 0$, we define for all $(y_0, s_0) \in (0,\infty)^n \times I\!R_+^{m+d}$ the continuous function α^{y_0,s_0} on $[0,T] \times I\!R_+^{m+d} \times (0,\infty)^n \times I\!M^{n,m+d} \times I\!R^n$ as

$$\alpha^{y_0,s_0}(t,s,y,a,b) := \begin{cases} A(t,s,y,a,b) & \text{if } \sum_{i=1}^{m+d} \sum_{j=1}^{n} \left(|s^i - s_0^i| + |\ln \frac{y^j}{y_0^j}| \right) < \mu \\ \text{constant} & \text{otherwise,} \end{cases} \tag{5.4}$$

where

$$\begin{aligned} A(t,s,y,a,b) &= \sigma(t,s)^{-1}\bar{F}(y)^{-1} \quad \Big\{ D\bar{f}(y)\text{diag}[y]b \\ &\qquad + \frac{1}{2}\text{Vect}\left[\text{Tr}\left(D^2\bar{f}^i(y)\text{diag}[y]aa'\text{diag}[y]\right)\right]_{i\le d} \\ &\qquad + \text{Vect}\left[\left(D\bar{f}(y)\text{diag}[y]a\sigma(t,s)'\right)_{ii}\right]_{i\le d} \Big\} \ . \end{aligned}$$

Let $\mathcal{D}$ be the set of all bounded progressively measurable processes $(a,b) = \{(a(t),b(t)), 0 \le t \le T\}$ where a and b are valued respectively in $I\!M^{n,m+d}$ and $I\!R^n$. For all y in $(0,\infty)^n$ and (a,b) in $\mathcal{D}$, we define the controlled process $Y_y^{a,b}$ as the solution on $[0,T]$ of the stochastic differential equation

$$\begin{aligned} dY(t) &= \text{diag}[Y(t)]\left[\left(b(t) + a(t)\alpha^{y,S(0)}(t,S(t),Y(t),a(t),b(t))\right)dt + a(t)dW(t)\right] \\ Y(0) &= y \ , \end{aligned} \tag{5.5}$$

It follows from our assumption on σ that $\alpha^{y,S(0)}(t,s,y',a,b)$ is a random Lipschitz function of y', so that the process $Y_y^{a,b}$ is well defined.

5.2 Super-replication in the fictitious markets

Let us fix y in $(0,\infty)^n$, (a,b) in $\mathcal{D}$ and let θ be a progressively measurable process valued in $I\!R^{m+d}$ satisfying

$$\sum_{i=1}^{m+d} \int_0^T |\theta^i(t)|^2 d\langle Z_y^{a,b,i}(t)\rangle < \infty \ . \tag{5.6}$$

Then, given $w \ge 0$, we introduce the process $W_{w,y}^{a,b,\theta}$ defined by

$$W_{w,y}^{a,b,\theta}(t) = w + \int_0^t \theta(s) \cdot dZ_y^{a,b}(s) \quad , \quad t \le T \tag{5.7}$$

and we denote by $\mathcal{B}^{a,b}(w,y)$ the set of all such processes θ satisfying the additional condition

$$W^{a,b,\theta}_{w,y}(t) \;\geq\; -c - \delta \cdot Z^{a,b}_y(t) \;\;,\;\; t \leq T \;,\;\; \text{for some } (c,\delta) \in I\!R \times I\!R^{m+d} \;. \tag{5.8}$$

We finally define the auxiliary stochastic control problems

$$\begin{aligned} u^{a,b}(0,y,\bar{F}(y)S(0)) \;\; &:= \;\; \inf \big\{ w \in I\!R \;:\; \exists \theta \in \mathcal{B}^{a,b}(w,y) \;, \\ & \qquad W^{a,b,\theta}_{w,y}(T) \geq f\left(Y^{a,b}_y(T)\right) \cdot g\left(S(T)\right) \big\} \;, \end{aligned} \tag{5.9}$$

and

$$u(0,y,\bar{F}(y)S(0)) := \sup_{(a,b)\in\mathcal{D}} u^{a,b}(0,y,\bar{F}(y)S(0)) \;. \tag{5.10}$$

The value function $u^{a,b}(0,y,\bar{F}(y)S(0))$ coincides with the super-replication price of the modified claim $f\left(Y^{a,b}_y(T)\right) \cdot g\left(S(T)\right)$ in the market formed by the assets $Z^{a,b}_y = (P, R^{a,b}_y)$ without transaction costs, recall (5.2).

The function $u(0,y,\bar{F}(y)S(0))$ is the upper-bound of these prices over all the "controlled" fictitious markets.

Recalling (5.3), these fictitious markets are constructed so as to be "cheaper" than the original one, in the sense that buying (resp. selling) the i-th asset in the fictitious market is always cheaper (resp. more profitable) than in the original market with transaction costs. This implies that the super-replication prices in the fictitious markets are always smaller than the one in the original model.

Proposition 5.1 *For all $y \in (0,\infty)^n$, we have $p(0,S(0)) \geq u(0,y,\bar{F}(y)S(0))$.*

The proof of this result follows line by line the one of Proposition 6.1 in Bouchard and Touzi (2000), up to obvious modifications. We therefore omit it.

5.3 The viscosity approach

In this section, we explain the viscosity approach followed by Bouchard and Touzi (2000) which turns out to be powerful in the case $m = 0$. This approach was initiated by Cvitanić, Pham and Touzi (1999) in the one dimensional case.

We first extend the value function u to general initial conditions. Given $(t,y,z) \in [0,T]\times(0,\infty)^n\times I\!R^{m+d}_+$, $u^{a,b}(t,y,z)$ is defined as in (5.9) with $(S_{t,s}, Y^{a,b}_{t,y,z}, Z^{a,b}_{t,y,z})$ defined as above but with initial conditions $(S_{t,s}(t), Y^{a,b}_{t,y,z}(t), Z^{a,b}_{t,y,z}(t)) = (s,y,z)$ where $s =$

$\bar{F}(y)^{-1}z$. Observe that the elements of Λ have positive components, see (4.2), so that s is well defined.

We then define the lower semicontinuous envelope of u on $[0,T]\times(0,\infty)^n\times I\!R_+^{m+d}$ by

$$u_*(t,y,z) \;:=\; \liminf_{\substack{(t',y',z')\to(t,y,z)\\ (t',y',z')\in[0,T)\times(0,\infty)^{n+m+d}}} u(t',y',z') \;.$$

Contrary to Bouchard and Touzi (2000), we need to extend the definition of u_* to the whole subspace $[0,T]\times(0,\infty)^n\times I\!R_+^{m+d}$ (in opposition to $[0,T]\times(0,\infty)^{n+m+d}$). Although, we are only interested by u_* on $[0,T)\times(0,\infty)^{n+m+d}$, since $S(0)\in(0,\infty)^{m+d}$, this extension will be useful to apply the comparison principle of Proposition 7.1 below.

Theorem 5.1 *Let* $(\mathbf{H}_\lambda)$ *and* $(\mathbf{H_g})$ *hold. Then* u_* *satisfies :*
(i) u_* *is independent of its variable* y*.*
(ii) u_* *is a viscosity supersolution on* $[0,T)\times(0,\infty)^n\times I\!R_+^{m+d}$ *of*

$$\inf_{a\in I\!M^{n,m+d}} -\mathcal{H}^a\varphi \geq 0 \;,$$

where, for a smooth function φ*,*

$$\mathcal{H}^a\varphi \;:=\; \frac{\partial\varphi}{\partial t}+\frac{1}{2}\mathrm{Tr}\left[\Gamma^{a'}D^2_{zz}\varphi\Gamma^a\right]$$

and

$$\Gamma^a(t,y,z) \;:=\; \mathrm{diag}[z]\left(\sigma(t,\bar{F}(y)^{-1}z)+\bar{F}(y)^{-1}D\bar{f}(y)\mathrm{diag}[y]a\right) \;.$$

(iv) For all $(y,z)\in(0,\infty)^n\times I\!R_+^{m+d}$

$$u_*(T,y,z) \;\geq\; G(z) \;.$$

This result is obtained by following line by line the arguments of Sections 6, 7 and 8 in Bouchard and Touzi (2000), see also Touzi (1999). Since its proof is rather long, we omit it.

In Bouchard and Touzi (2000), the above characterization was sufficient to solve the super-replication problem. Indeed, in the case where $m=0$, one can deduce from Theorem 5.1 that u_* is concave with respect to z and non-increasing in t. This turns out to be sufficient to show that $p := \sup_y u(0, y, \bar{F}(y)S(0))$, corresponds to the price of a buy-and-hold super-hedging strategy for the original market. Combined with Proposition 5.1, this implies that p is actually the super-replication price in the market with transaction costs.

In our context, where $m \geq 1$, we can only show that $u_*(t, y, z)$ is concave with respect to the last d components of z and there is no reason why it should be concave in z (in particular if $g(s)$ depends only on the first components of s). We therefore have to work a bit more.

6 A new interpretation for the super-hedging price and conclusion of the proof of Theorem 4.1

Up to now, we have followed the arguments of Bouchard and Touzi (2000), but as already explained this is not sufficient to conclude. In this section, we provide a new interpretation of the super-replication problem based on a reformulation of the PDE of Theorem 5.1. This is the key idea for solving our original problem.

As a first step, we rewrite the above PDE in a more tractable way. For all $(t, z) \in [0, T] \times \mathbb{R}_+^{m+d}$ and $\mu \in \mathbb{M}^{d,m+d}$, we define

$$\sigma^\mu(t, z) \quad := \quad \mathrm{diag}[z] \left[\sigma(t, z)^{ij} \mathbf{1}_{i \leq m} + \mu^{ij} 1_{i > m}\right]_{1 \leq i,j \leq m+d}$$

where, for real numbers (b^{ij}), $[b^{ij}]_{1 \leq i,j \leq m+d}$ denotes the square $(m + d)$-dimensional matrix M defined by $M^{ij} = b^{ij}$.

Since u_* does not depend on its y variable, see Theorem 5.1, we shall omit it from now on if not required by the context.

Corollary 6.1 *Let* $(\mathbf{H}_\lambda)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H_g})$ *hold. Then,*
(i) u_* *is a viscosity supersolution on* $[0, T) \times \mathbb{R}_+^{m+d}$ *of*

$$\inf_{\mu \in \mathbb{M}^{d,m+d}} -\mathcal{G}^\mu \varphi \geq 0 \ , \tag{6.1}$$

where, for a smooth function φ *and* $\mu \in \mathbb{M}^{d,m+d}$,

$$\mathcal{G}^\mu \varphi(t, z) \quad = \quad \frac{\partial \varphi}{\partial t} + \frac{1}{2} \mathrm{Tr} \left[\sigma^\mu(t, z)' D^2_{zz} \varphi(t, z) \sigma^\mu(t, z)\right] \ .$$

(ii) For each $(t, p) \in [0, T) \times (0, \infty)^m$, *the map* $r \in (0, \infty)^d \mapsto u_*(t, p, r)$ *is concave.*

(iii) For all $(p,r) \in \mathbb{R}_+^{m+d}$

$$u_*(T,p,r) \quad \geq \quad G^{conc}(p,r) \;, \tag{6.2}$$

where we recall that G^{conc} is the concave envelope of G with respect to its last variable r.

The concavity property and (6.1) are obtained by playing with the controls a in the PDE of Theorem 5.1. The complete proof of is reported in Section 7.

Before to enter into the technicalities, observe that, formally, the Hamilton-Jacobi-Equation of Corollary 6.1 coincides with the control problem

$$v(0,z) \quad := \quad \sup_{\mu \in \mathcal{U}} \mathbb{E}\left[G^{conc}(\hat{Z}_z^\mu(T))\right]$$

where, for $z = (p,r) \in (0,\infty)^{m+d}$, $\hat{Z}_z^\mu$ is defined by

$$\hat{Z}_z^\mu(t) = (P_p(t), \hat{R}_r^\mu(t)) \quad \text{with} \quad \hat{R}_r^\mu(t) = r + \int_0^t \mathrm{diag}[\hat{R}_r^\mu(s)]\mu(s)dW(s) \tag{6.3}$$

and, by $(\mathbf{H}_\sigma)$, solves

$$\hat{Z}(t) \quad = \quad z + \int_0^t \sigma^{\mu(s)}(s, \hat{Z}(s))dW(s) \quad t \leq T \;. \tag{6.4}$$

Here, $\mathcal{U}$ is the collection of all $\mathbb{M}^{d,m+d}$-valued square integrable predictable processes μ such that $\hat{R}_r^\mu$ is a martingale for $r \in (0,\infty)^d$.

If we could show that v is a (viscosity) subsolution of (6.1)-(6.2), then a comparison principle would imply $u \geq v$. Also, the above statement is not correct, because the boundary condition (6.2) is not satisfied in general by v, we will show in Section 7 that the inequality $u \geq v$ actually holds.

Proposition 6.1 *Let $(\mathbf{H}_\lambda)$, $(\mathbf{H}_\sigma)$ and $(\mathbf{H_g})$ hold. Then, for all $z \in (0,\infty)^{m+d}$,*

$$u_*(0,z) \quad \geq \quad \sup_{\mu \in \mathcal{U}} \mathbb{E}\left[G^{conc}(\hat{Z}_z^\mu(T))\right] \;.$$

Proof. See Section 7 $\square$

Using an approximation argument combined with the martingale property of the $\hat{R}^{\mu}_{r}$'s, (6.3) and the concavity of u_* with respect to its last d components, recall Corollary 6.1, this implies that, for all $r \in (0,\infty)^d$ and $\Delta \in \partial_r u_*(0, P(0), r)$, we have

$$u_*(0, P(0), r) \geq \mathbb{E}\left[\sup_{\tilde{r}\in(0,\infty)^d} \{G^{conc}(P(T), \tilde{r}) - \Delta \cdot \tilde{r}\}\right] + \Delta \cdot r \ , \qquad (6.5)$$

where, $\partial_r u_*(0, P(0), r)$ is the subgradient of the concave mapping $r \mapsto u_*(0, P(0), r)$.

Corollary 6.2 *Let the conditions* $(\mathbf{H}_\lambda)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H}_\mathbf{g})$ *hold. Then, for all* $r \in (0,\infty)^d$ *and* $\Delta \in \partial_r u_*(0, P(0), r)$, *the inequality* (6.5) *holds.*

Proof. By definition of Δ and Corollary 6.1, we have

$$u_*(0, P(0), r) \ = \ \sup_{\tilde{r}\in(0,\infty)^d} \{u_*(0, P(0), \tilde{r}) - \Delta \cdot (\tilde{r} - r)\} \ .$$

Since, for each $\tilde{r} \in (0,\infty)^d$ and $\mu \in \mathcal{U}$, $\mathbb{E}\left[\hat{R}^{\mu}_{\tilde{r}}(T)\right] = \tilde{r}$, it follows from Proposition 6.1 that

$$u_*(0, P(0), r) \ \geq \ \sup_{\tilde{r}\in(0,\infty)^d} \sup_{\mu\in\mathcal{U}} \mathbb{E}\left[G^{conc}\left(P(T), \hat{R}^{\mu}_{\tilde{r}}(T)\right) - \Delta \cdot \hat{R}^{\mu}_{\tilde{r}}(T)\right] + \Delta \cdot r \ .$$

Since G^{conc} is continuous, we deduce from the representation theorem and (6.3) that

$$\begin{aligned} u_*(0, P(0), r) \ &\geq \ \sup_{\xi\in L^\infty(\mathcal{B}_\kappa;\mathcal{F}_T)} \mathbb{E}\left[G^{conc}(P(T), \xi) - \Delta \cdot \xi\right] + \Delta \cdot r \\ &\geq \ \mathbb{E}\left[\max_{\tilde{r}\in\mathcal{B}_\kappa} \{ \ G^{conc}(P(T), \tilde{r}) - \Delta \cdot \tilde{r} \ \}\right] + \Delta \cdot r \ , \end{aligned}$$

where $\mathcal{B}_\kappa := \{\alpha \in (0,\infty)^d \ : \ \ |\ln(\alpha^i)| \leq \kappa \ \ , \ i \leq d\}$. The result then follows from monotone convergence. $\square$

We can now conclude the proof of our main result.

Proof of Theorem 4.1. In view of Proposition 5.1 and (5.1), we have

$$p(0, S(0)) \ \geq \ \sup_{\xi\in\Lambda} u_*(0, P(0), \mathrm{diag}[\underline{\xi}]Q(0)) \ . \qquad (6.6)$$

1. Recalling that Λ is compact and u_* is concave in its last d variables, there is some $\hat{\xi} \in \Lambda$ which attains the optimum in the above inequality. Moreover, by standard

arguments of calculus of variations, we can find some $\hat{\Delta} \in \partial_r u_*(0, P(0), \mathrm{diag}[\hat{\underline{\xi}}]Q(0))$ such that

$$(\mathrm{diag}[\hat{\Delta}]Q(0)) \cdot (\hat{\underline{\xi}} - \underline{\xi}) \;\; \geq \;\; 0 \text{ for all } \xi \in \Lambda \; . \tag{6.7}$$

From (4.3), we deduce that $(\mathrm{diag}[\hat{\Delta}]Q(0) \cdot \hat{\underline{\xi}}, 0) \succeq (0, \mathrm{diag}[\hat{\Delta}]Q(0))$.

2. Set

$$\delta \; := \; \mathrm{diag}[\hat{\Delta}]Q(0) \cdot \hat{\underline{\xi}} \quad \text{and} \quad \hat{C}(P(T)) \; := \; \sup_{\tilde{r} \in (0,\infty)^d} G^{conc}(P(T), \tilde{r}) - \hat{\Delta} \cdot \tilde{r} \;\; ,$$

so that, by (6.6) and Corollary 6.2,

$$p(0, S(0)) \;\; \geq \;\; p \; := \; \mathbb{E}\left[\hat{C}(P(T))\right] + \delta \; . \tag{6.8}$$

Since by $(\mathbf{H}_\sigma)$ the dynamics of P depends only on P, it follows that there is some $I\!R^m$-valued predictable process ϕ satisfying $\int_0^T |\phi(t)|^2 dt < \infty$ such that

$$X^0(\cdot) \; := \; p - \delta + \int_0^{\cdot} \phi(t) \cdot dP(t) \text{ is a martingale and } X^0(T) \; = \; \hat{C}(P(T)) \; . \tag{6.9}$$

3. By combining 1. and 2., we deduce that there is some strategy $\nu = (\phi, L)$ such that $L(t) = L(0)$, $X^{\nu}_{p\mathbf{1}_0}(0) = (p - \delta, \mathrm{diag}[\hat{\Delta}]Q(0))$ and

$$X^{\nu}_{p\mathbf{1}_0}(T) \;\; = \;\; \left(X^0(T), \mathrm{diag}[\hat{\Delta}]Q(T)\right) \; \succeq \; \left(\hat{C}(P(T)), \mathrm{diag}[\hat{\Delta}]Q(T)\right) \; .$$

Using (6.9), (4.3) and the definition of $\hat{C}$ this implies that

$$\tilde{\xi} \cdot X^{\nu}_{p\mathbf{1}_0}(T) - \xi \cdot g\left(P(T), \mathrm{diag}[\underline{\xi}]^{-1}\mathrm{diag}[\tilde{\underline{\xi}}]Q(T)\right) \;\; \geq \;\; 0 \quad \text{for all } \; \tilde{\xi}, \, \xi \in \Lambda \;\; .$$

Considering the case where $\xi = \tilde{\xi}$ and using (4.3) leads to

$$X^{\nu}_{p\mathbf{1}_0}(T) \;\; \succeq \;\; g(S(T)) \; .$$

In view of (6.8) it remains to check that $\nu \in \mathcal{A}$, but this readily follows from (6.9) and assumption (3.5). □

Proof of Corollary 4.1. In view of Theorem 4.1, we only have to show that

$$p(0, S(0)) \;\; \leq \;\; \inf_{\Delta \in I\!R^d} \left\{ \;\; \mathbb{E}\left[C(P(T); \Delta)\right] + \sup_{\xi \in \Lambda} \underline{\xi} \cdot \mathrm{diag}[\Delta]Q(0) \;\; \right\} \; .$$

To see this, fix some $\Delta \in I\!R^d$ such that

$$\tilde{p} \;:=\; \mathbb{E}\left[C(P(T);\Delta)\right] + \sup_{\xi\in\Lambda} \underline{\xi} \cdot \mathrm{diag}[\Delta]Q(0) \;<\; \infty \;,$$

which is possible by Theorem 4.1. Then, by the same argument as in the proof of Theorem 4.1, see above, we obtain that there exists some $\nu = (\phi, L) \in \mathcal{A}^{BH}$, with $L(t) = \Delta$ on $t \le T$, such that $X^{\nu}_{\tilde{p}\mathbf{1}_0} \succeq g(S(T))$. This proves the required inequality as well as the last statement of the Corollary. □

Remark 6.1 A close look at the PDE (6.1) and the above arguments shows that, in the limiting case where $\lambda = 0+$, everything behaves as if the volatility of the non-freely exchangeable assets was stochastic, taking any possible values in $I\!M^{d,m+d}$. Indeed, in the case $\lambda = 0+$, $\Lambda = \{1, \ldots, 1\}$. Combining Proposition 5.1, Proposition 6.1 and the above arguments, we obtain that

$$p(0, S(0)) \;=\; \sup_{\mu\in\mathcal{U}} \mathbb{E}\left[G^{conc}(\hat{Z}^{\mu}_{S(0)}(T))\right]$$

where the value function associated to the right hand-side term formally solves

$$\inf_{\mu\in I\!M^{d,m+d}} -\mathcal{G}^{\mu}\varphi \;=\; 0 \;\text{ on }\; [0,T)\times(0,\infty)^{m+d} \quad,\quad \varphi(T,\cdot) \;=\; G^{conc}(\cdot) \;\text{ on }\; (0,\infty)^{m+d} \;.$$

This equation can be viewed as a Black-Scholes-Barenblatt equation where the volatility matrix of the last d assets can take any values in $I\!M^{d,m+d}$. This is the kind of equation we obtain in stochastic volatility models, see e.g. Cvitanic et al. (1999).

This comforts the usual intuition that transaction costs "increase" the effective volatility.

7 Proofs

7.1 Proof of Corollary 6.1

We first state the following Lemma which easy proof can be found in Bouchard and Touzi (2000).

Lemma 7.1 *Let* $(\mathbf{H}_\lambda)$ *hold. Then,*
(i) There is some $\delta > 0$ *such that* $0 < \xi^i + \frac{1}{\xi^i} \le \delta$ *for each* $\xi \in \Lambda$ *and* $i = 0, \ldots, d$.
(ii) On $(0,\infty)^n$*, the rank of the Jacobian matrix* Df *of* f *is* d.

Proof of Corollary 6.1. (i). Recall from Lemma 7.1 that the rank of $Df(y)$ is d whenever $y \in (0,\infty)^n$. Since $\bar{f}^i = 1$ for $i \leq m$, we deduce from $(\mathbf{H}_\sigma)$ that, for each $\mu \in I\!M^{d,m+d}$, we can find some $a \in I\!M^{n,m+d}$ such that $\Gamma^a(t,y,z) = \sigma^\mu(t,z)$. Then, the first result follows from Theorem 5.1.
(ii). For φ satisfying (6.1) we must have, for all $(t,p,r) \in [0,T) \times (0,\infty)^{m+d}$, $-\mathrm{Tr}[\mu' D^2_{rr}\varphi(t,p,r)\mu] \geq 0$ for all $\mu \in I\!M^{d,m+d}$ since otherwise we would get a contradiction of (6.1) by considering $\delta\mu$ and sending δ to infinity. Then, the concavity property follows from the same argument as in Lemma 8.1 of Bouchard and Touzi (2000).
(iii). In view of the boundary condition of Theorem 5.1, it suffices to show that $u_*(T,p,r)$ is concave with respect to r. This readily follows from (ii) by passing to the limit (inf) on t. □

7.2 Proof of Proposition (6.1)

As explained in Section 6, the value function v is not, in general, a viscosity solution of (6.1)-(6.2), at least in the usual sense. This is due to the fact that the boundary condition is in general not satisfied. To overcome this problem, we define a similar control problem but with bounded controls and then take the limit as the bound goes to infinity. The bound on the control will also play an important role in the proof of the comparison principle of Proposition 7.1 below.

Given $\kappa \geq 0$, we define U_κ as the set of all elements M of $I\!M^{d,m+d}$ such that $|M| \leq \kappa$, and we denote by $\mathcal{U}_\kappa$ the collection of all U_κ-valued predictable processes.

We first show that the auxiliary control problems

$$v_\kappa(t,z) := \sup_{\mu \in \mathcal{U}_\kappa} \mathbb{E}\left[G^{conc}\left(\hat{Z}^\mu_{t,z}(T)\right)\right] \qquad (t,z,\kappa) \in [0,T] \times (0,\infty)^{m+d} \times (0,\infty)\ , \quad (7.1)$$

recall (6.4) and (6.3), satisfy $u_* \geq \sup_{\kappa>0} v^*_\kappa$, where v^*_κ is the upper-semicontinuous function defined on $[0,T] \times I\!R^{m+d}_+$ by

$$v^*_\kappa(t,z) \quad := \quad \limsup_{\substack{(t',z') \to (t,z) \\ (t',z') \in [0,T) \times (0,\infty)^{m+d}}} \quad v_\kappa(t',z')\ .$$

This will be done by means of a comparison argument on the PDE defined by (6.1)-(6.2) with U_κ substituted to $I\!M^{d,m+d}$.

7.2.1 Viscosity properties of v_κ

We start with the subsolution property in the domain. The proof is rather standard now but, as it is short, we provide it for completeness.

Lemma 7.2 *For each $\kappa > 0$, v_κ^* is a viscosity subsolution on $[0,T) \times \mathbb{R}_+^{m+d}$ of*

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi \;\; \leq \;\; 0 \; .$$

Proof. Let $\varphi \in C^2([0,T] \times \mathbb{R}^{m+d})$ and (t_0, z_0) be a strict global maximizer of $v_\kappa^* - \varphi$ on $[0,T) \times \mathbb{R}_+^{m+d}$ such that $(v_\kappa^* - \varphi)(t_0, z_0) = 0$. We assume that

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi(t_0, z_0) \;\; > \;\; 0 \; , \tag{7.2}$$

and work towards a contradiction. If (7.2) holds, then it follows from our continuity assumptions on σ that there exists some $t_0 < \eta < T - t_0$ such that

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi(t,z) \;\; \geq \;\; 0 \quad \text{for all } (t,z) \in B_0 := B((t_0,z_0),\eta) \; , \tag{7.3}$$

where $B((t_0,z_0),\eta)$ is the open ball of radius η centered on (t_0,z_0). Let $(t_n,z_n)_{n\geq 0}$ be a sequence in $B_0 \cap ([0,T) \times (0,\infty)^{m+d})$ such that

$$(t_n, z_n) \;\longrightarrow\; (t_0, z_0) \quad \text{and} \quad v_\kappa(t_n,z_n) \;\longrightarrow\; v_\kappa^*(t_0,z_0)$$

and notice that

$$v_\kappa(t_n,z_n) - \varphi(t_n,z_n) \;\longrightarrow\; 0 \; . \tag{7.4}$$

Next, define the stopping times

$$\theta_n^\mu \;\; := \;\; T \wedge \inf\left\{ s > t_n \;:\; (s, \hat{Z}_n^\mu(s)) \notin B_0 \right\}$$

where μ is any element of $\mathcal{U}_\kappa$ and $\hat{Z}_n^\mu := \hat{Z}_{t_n,z_n}^\mu$. Let $\partial_p B_0 = [t_0, t_0+\eta] \times \partial B(z_0,\eta) \cup \{t_0+\eta\} \times B(z_0,\eta)$ denote the parabolic boundary of B_0 and observe that

$$0 \; > \; -\zeta \;\; := \sup_{(t,z) \in \partial_p B_0 \cap ([0,T] \times \mathbb{R}_+^{m+d})} (v_\kappa^* - \varphi)(t,z)$$

since (t_0,z_0) is a strict maximizer of $v_\kappa^* - \varphi$. Then, for a fixed $\mu \in \mathcal{U}_\kappa$, we deduce from Itô's Lemma and (7.3) that

$$\varphi(t_n,z_n) \;\; \geq \;\; \mathbb{E}\left[\varphi(\theta_n^\mu, \hat{Z}_n^\mu(\theta_n^\mu))\right] \geq \mathbb{E}\left[v_\kappa^*\left(\theta_n^\mu, \hat{Z}_n^\mu(\theta_n^\mu)\right) + \zeta\right] \geq \zeta + \mathbb{E}\left[G^{conc}(\hat{Z}_n^\mu(T))\right] \; ,$$

where we used the fact that $\varphi \geq v_\kappa^* \geq v_\kappa$ and

$$v_\kappa\left(\theta_n^\mu, \hat{Z}_n^\mu(\theta_n^\mu)\right) \quad \geq \quad \mathbb{E}\left[G^{conc}(\hat{Z}_n^\mu(T)) \mid \mathcal{F}_{\theta_n^\mu}\right] \ .$$

By arbitrariness of $\mu \in \mathcal{U}_\kappa$, it follows from the previous inequality that

$$\varphi(t_n, z_n) \quad \geq \quad \zeta + v_\kappa(t_n, z_n) \ .$$

In view of (7.4), this leads to a contradiction since $\zeta > 0$. $\square$

We now turn to the boundary condition.

Lemma 7.3 *Under* $(\mathbf{H_g})$, *for each* $\kappa > 0$ *and* $z \in I\!R_+^{m+d}$, $v_\kappa^*(T, z) \leq G^{conc}(z)$.

Proof. Let $(t_n, z_n)_n$ be a sequence in $[0, T) \times (0, \infty)^{m+d}$ such that $(t_n, z_n) \to (T, z)$ and $v_\kappa(t_n, z_n) \to v_\kappa^*(T, z)$. Let $(\mu_n)_n$ be a sequence in $\mathcal{U}_\kappa$ such that

$$v_\kappa(t_n, z_n) \quad \leq \quad \mathbb{E}\left[G^{conc}(\hat{Z}_{t_n,z_n}^{\mu_n}(T))\right] + n^{-1} \ \ , \ n \geq 1 \ .$$

Recalling that $(\mu_n)_n$ is uniformly bounded, it follows from standard arguments that $\hat{Z}_{t_n,z_n}^{\mu_n}(T) \to z$ $\mathbb{P}-$a.s. as $n \to \infty$, recall (6.3). Moreover, one easily checks that $(\hat{Z}_{t_n,z_n}^{\mu_n}(T))_n$ is bounded in any L^p. Since G^{conc} is continuous with linear growth, it then follows from the dominated convergence theorem and the above inequality that

$$\lim_{n\to\infty} v_\kappa(t_n, z_n) \quad \leq \quad \mathbb{E}\left[\lim_{n\to\infty} G^{conc}(\hat{Z}_{t_n,z_n}^{\mu_n}(T))\right] \quad = \quad G^{conc}(z) \ .$$

By definition of $(t_n, z_n)_n$, this leads to the required result. $\square$

7.3 The comparison principle

In order to show that $u_* \geq v_\kappa^*$, we need to prove a comparison principle for (6.1)-(6.2) with U_κ in place of $I\!M^{d,m+d}$. We adapt the arguments of Barles et al. (2003) to our context.

Proposition 7.1 *Let* V *be an upper semicontinuous viscosity subsolution and* U *be a lower semicontinuous viscosity supersolution on* $[0, T) \times I\!R_+^{m+d}$ *of*

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi \quad = \quad 0 \ .$$

Assume that V and U satisfy the linear growth condition

$$|V(t,z)| + |U(t,z)| \leq K\ (1+|z|) \quad , \ (t,z) \in [0,T) \times I\!R_+^{m+d} \ , \quad \textit{for some} \ \ K > 0 \ .$$

Then,

$$V(T,.) \ \leq \ U(T,.) \quad \textit{implies} \quad V \ \leq \ U \quad \textit{on} \ [0,T] \times I\!R_+^{m+d} \ .$$

Proof. 1. Let λ be some positive parameter and consider the functions

$$u(t,z) \ := \ e^{\lambda t} U(t,z) \quad \text{and} \quad v(t,z) \ := \ e^{\lambda t} V(t,z) \ .$$

It is easy to verify that the functions u and v are, respectively, a lower semicontinuous viscosity supersolution and an upper semicontinuous viscosity subsolution on $[0,T) \times I\!R_+^{m+d}$ of

$$\lambda \varphi - \frac{\partial \varphi}{\partial t} - \sup_{\mu \in U_\kappa} \operatorname{Tr}\left[\sigma^{\mu\prime} D^2_{zz}\varphi \sigma^\mu\right] \ = \ 0 \ . \tag{7.5}$$

Moreover u and v satisfy

$$u(T,z) \ \geq \ v(T,z) \quad \text{for all} \ \ z \ \in \ I\!R_+^{m+d} \ ,$$

as well as the linear growth condition

$$|v(t,z)| + |u(t,z)| \leq A\ (1+|z|) \quad , \ (t,z) \in [0,T) \times I\!R_+^{m+d} \ , \quad \text{for some} \ \ A > 0 \ . \tag{7.6}$$

Through the following steps of the proof we are going to show that $u \geq v$ on the entire domain $[0,T] \times I\!R_+^{m+d}$, which is equivalent to $U \geq V$ on $[0,T] \times I\!R_+^{m+d}$.

We argue by contradiction, and assume that for some (t_0, z_0) in $[0,T] \times I\!R_+^{m+d}$

$$0 \ < \ \delta \ := v(t_0,z_0) - u(t_0,z_0) \ .$$

2. Following Barles et al. (2003), we introduce the following functions. For some positive parameter α, we set

$$\phi_\alpha(z,z') \ = \ \left[1+|z|^2\right]\left[\varepsilon + \alpha|z'|^2\right] \quad \text{and} \quad \Phi_\alpha(t,z,z') \ = \ e^{L(T-t)}\phi_\alpha(z+z', z-z') \ .$$

Here, L and ε are positive constants to be chosen later and we don't write the dependence of ϕ_α, Φ_α and Ψ_α with respect to them.

By the linear growth condition (7.6), the upper semicontinuous function Ψ_α defined by

$$\Psi_\alpha(t,z,z') \ := \ v(t,z) - u(t,z') - \Phi_\alpha(t,z,z')$$

is such that for all (t,z,z') in $[0,T] \times I\!R_+^{2(m+d)}$

$$\begin{aligned} \Psi_\alpha(t,z,z') \ &\leq \ A\left(1+|z|+|z'|\right) - \min\{\varepsilon,\alpha\}\left(|z-z'|^2+|z+z'|^2+1\right) \\ &\leq \ A\left(1+|z|+|z'|\right) - \min\{\varepsilon,\alpha\}\left(|z|^2+|z'|^2\right) \ . \end{aligned}$$

We deduce that Ψ_α attains its maximum at some $(t_\alpha, z_\alpha, z'_\alpha)$ in $[0,T]\times I\!R_+^{2(m+d)}$. The inequality $\Psi_\alpha(t_0,z_0,z_0) \leq \Psi_\alpha(t_\alpha,z_\alpha,z'_\alpha)$ reads

$$\Psi_\alpha(t_\alpha,z_\alpha,z'_\alpha) \ \geq \ \delta - \varepsilon\left(1+4|z_0|^2\right)e^{LT} \ .$$

Hence, ε can be chosen sufficiently small (depending on L and $|z_0|$) so that

$$v(t_\alpha,z_\alpha) - u(t_\alpha,z'_\alpha) \ \geq \ \Psi_\alpha(t_\alpha,z_\alpha,z'_\alpha) \ \geq \ \delta - \varepsilon\left(1+4|z_0|^2\right)e^{LT} \ > \ 0 \ . \qquad (7.7)$$

From (7.7) and (7.6), we get

$$\begin{aligned} 0 \ \leq \ \frac{\alpha}{2}|z_\alpha - z'_\alpha|^2 + \frac{\varepsilon}{2}|z_\alpha+z'_\alpha|^2 \ &\leq \ v(t_\alpha,z_\alpha) - u(t_\alpha,z'_\alpha) - \frac{\varepsilon}{2}|z_\alpha+z'_\alpha|^2 - \frac{\alpha}{2}|z_\alpha-z'_\alpha|^2 \\ &\leq \ A\left(1+|z_\alpha|+|z'_\alpha|\right) - \min\left\{\frac{\varepsilon}{2},\frac{\alpha}{2}\right\}\left(|z_\alpha|^2+|z'_\alpha|^2\right) \ . \end{aligned}$$

We deduce that $\{\alpha|z_\alpha - z'_\alpha|\}_{\alpha>0}$ as well as $\{(z_\alpha,z'_\alpha)\}_{\alpha>0}$ are bounded. Therefore, after possibly passing to a subsequence, we can find $(\bar{t},\bar{z}) \in [0,T]\times I\!R_+^{m+d}$ such that

$$(t_\alpha,z_\alpha,z'_\alpha) \to (\bar{t},\bar{z},\bar{z}) \quad \text{as} \quad \alpha\to\infty \ .$$

Since $v - u$ is upper semicontinuous, it follows from (7.7) that

$$v(\bar{t},\bar{z}) - u(\bar{t},\bar{z}) \ \geq \ \limsup_{\alpha\to\infty} v(t_\alpha,z_\alpha) - u(t_\alpha,z'_\alpha) \ \geq \ \delta - \varepsilon\left(1+4|z_0|^2\right)e^{LT} \ > \ 0 \ .$$

Since $u(T,.) \geq v(T,.)$ on $I\!R_+^{m+d}$, $\bar{t}$ is in $[0,T)$, hence for α sufficiently large t_α is in $[0,T)$.

3. Let α be sufficiently large so that

$$|z_\alpha - z'_\alpha| \ < \ 1 \quad \text{and} \quad t_\alpha \ \in \ [0,T) \ .$$

Since $(t_\alpha, z_\alpha, z'_\alpha)$ is a maximum point of Ψ_α, by the fundamental result in the User's Guide to Viscosity Solutions (Theorem 8.3 in Crandall et al. 1993), for each $\eta > 0$, there are numbers a_1^η , a_2^η in $I\!R$, and symmetric matrices X^η and Y^η in $I\!M^{m+d}$ such that

$$
\begin{aligned}
(a_1^\eta, D_z\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha), X^\eta) &\in \bar{\mathcal{P}}^{2,+}(v)(t_\alpha, z_\alpha) , \\
(a_2^\eta, -D_{z'}\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha), Y^\eta) &\in \bar{\mathcal{P}}^{2,+}(u)(t_\alpha, z'_\alpha) ,
\end{aligned}
$$

with

$$
a_1^\eta - a_2^\eta \;=\; \frac{\partial \Phi_\alpha}{\partial t}(t_\alpha, z_\alpha, z'_\alpha) \;=\; -L\,\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) ,
$$

and

$$
\left(\begin{array}{cc} X^\eta & 0 \\ 0 & -Y^\eta \end{array} \right) \;\leq\; M + \eta M^2 , \quad \text{where} \quad M \;:=\; D^2_{(z,z')}\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) . \tag{7.8}
$$

Since v is a viscosity subsolution and u is a viscosity supersolution of (7.5) on $[0,T) \times I\!R_+^{m+d}$, we must have

$$
\begin{aligned}
\lambda v(t_\alpha, z_\alpha) - a_1^\eta - \frac{1}{2}\sup_{\mu \in U_\kappa} \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)\right] &\leq 0 , \\
\lambda u(t_\alpha, z_\alpha) - a_2^\eta - \frac{1}{2}\sup_{\mu \in U_\kappa} \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)\right] &\geq 0 .
\end{aligned}
$$

Taking the difference we get

$$
\begin{aligned}
& \lambda\left(v(t_\alpha, z_\alpha) - u(t_\alpha, z'_\alpha)\right) + L\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \\
\leq\; & \frac{1}{2}\sup_{\mu \in U_\kappa} \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)\right] - \frac{1}{2}\sup_{\mu \in U_\kappa} \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)\right] \\
\leq\; & \frac{1}{2}\sup_{\mu \in U_\kappa} \left\{\mathrm{Tr}\left[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)\right] - \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)\right]\right\} .
\end{aligned} \tag{7.9}
$$

Let $(e_i, i = 1, ..., m+d)$ be an orthonormal basis of $I\!R^{m+d}$, and for each μ in U_κ set

$$
\xi_i^\mu \;:=\; \sigma^\mu(t_\alpha, z_\alpha)e_i \quad \text{and} \quad \zeta_i^\mu \;:=\; \sigma^\mu(t_\alpha, z'_\alpha)e_i
$$

so that

$$
\begin{aligned}
& \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)\right] - \mathrm{Tr}\left[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)\right] \\
=\; & \sum_{i=1}^{m+d} X^\eta \xi_i^\mu \cdot \xi_i^\mu - Y^\eta \zeta_i^\mu \cdot \zeta_i^\mu
\end{aligned}
$$

and, by (7.8),

$$\mathrm{Tr}\left[\sigma^\mu(t_\alpha,z_\alpha)'X^\eta\sigma^\mu(t_\alpha,z_\alpha)\right] - \mathrm{Tr}\left[\sigma^\mu(t_\alpha,z'_\alpha)'Y^\eta\sigma^\mu(t_\alpha,z'_\alpha)\right] \;\le\; \sum_{i=1}^{m+d}(M+\eta M^2)\beta_i^\mu\cdot\beta_i^\mu \;,$$

where β_i^μ is the $2(m+d)$-dimensional column vector defined by : $\beta_i^\mu := (\xi_i^{\mu\prime},\zeta_i^{\mu\prime})'$. Letting η go to zero, and using (7.9), we get

$$\lambda\left(v(t_\alpha,z_\alpha)-u(t_\alpha,z'_\alpha)\right)+L\Phi_\alpha(t_\alpha,z_\alpha,z'_\alpha) \;\le\; \frac{1}{2}\sup_{\mu\in U_\kappa}\left\{\sum_{i=1}^{m+d}M\beta_i^\mu\cdot\beta_i^\mu\right\}\;. \tag{7.10}$$

4. In this last step, we are going to see that, for a convenient choice of the positive constant L, inequality (7.10) leads to a contradiction to (7.7).
Notice that

$$M \;=\; e^{L(T-t_\alpha)}\begin{pmatrix} D_{zz}\phi_\alpha + D_{z'z'}\phi_\alpha + 2D_{zz'}\phi_\alpha & D_{zz}\phi_\alpha - D_{z'z'}\phi_\alpha \\ D_{zz}\phi_\alpha - D_{z'z'}\phi_\alpha & D_{zz}\phi_\alpha + D_{z'z'}\phi_\alpha - 2D_{zz'}\phi_\alpha \end{pmatrix}$$

the (partial) Hessian matrices of ϕ_α being taken at the point $(z_\alpha+z'_\alpha, z_\alpha-z'_\alpha)$. Then, for μ in U_κ

$$\begin{aligned}\sum_{i=1}^{m+d}M\beta_i^\mu\cdot\beta_i^\mu \;&=\; e^{L(T-t_\alpha)}\sum_{i=1}^{m+d}D_{zz}\phi_\alpha\left(\xi_i^\mu+\zeta_i^\mu\right)\cdot\left(\xi_i^\mu+\zeta_i^\mu\right)\\ &\quad + 2D_{zz'}\phi_\alpha\left(\xi_i^\mu+\zeta_i^\mu\right)\cdot\left(\xi_i^\mu-\zeta_i^\mu\right)\\ &\quad + D_{z'z'}\phi_\alpha\left(\xi_i^\mu-\zeta_i^\mu\right)\cdot\left(\xi_i^\mu-\zeta_i^\mu\right)\;.\end{aligned} \tag{7.11}$$

Since for each μ in U_κ, $|\mu|$ is bounded by κ, we deduce from the Lipschitz property of the function $z\mapsto \mathrm{diag}[z]\sigma(t,z)$ that, for some positive constant C

$$|\sigma^\mu(t,z)-\sigma^\mu(t,z')| \;\le\; C|z-z'| \quad\text{and}\quad |\sigma^\mu(t,z)| \;\le\; C\left(1+|z|\right)\;,$$

for each z, z' in $I\!R_+^{m+d}$ and t in $[0,T]$.

In the following C denotes a positive constant (independent of α, ε and L) which value may change from line to line.
4.1. Since α satisfies $|z_\alpha - z'_\alpha|\le 1$, for $i=1,..,m+d$,

$$D_{zz}\phi_\alpha\left(\xi_i^\mu+\zeta_i^\mu\right)\cdot\left(\xi_i^\mu+\zeta_i^\mu\right) \;\le\; C|D_{zz}\phi_\alpha|\left[(1+|z_\alpha|)^2+(1+|z'_\alpha|)^2\right]\;.$$

From $|D_{zz}\phi_\alpha| \leq 2(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)$ and the previous estimate, we deduce that

$$D_{zz}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu + \zeta_i^\mu) \;\; \leq \;\; C(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;. \quad (7.12)$$

4.2. For $i = 1, .., m+d$

$$D_{zz'}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\leq\; C|D_{zz'}\phi_\alpha|\,[(1+|z_\alpha|) + (1+|z'_\alpha|)]\,|z_\alpha - z'_\alpha| \;.$$

Since $|D_{zz'}\phi_\alpha| \leq 4\alpha|z_\alpha - z'_\alpha||z_\alpha + z'_\alpha|$ and $|z_\alpha - z'_\alpha| \leq 1$, we deduce that

$$D_{zz'}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\; \leq \;\; C(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;. \quad (7.13)$$

4.3. For $i = 1, .., m+d$

$$D_{z'z'}\phi_\alpha\,(\xi_i^\mu - \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\leq\; C|D_{z'z'}\phi_\alpha||z_\alpha - z'_\alpha|^2$$

and since $|D_{z'z'}\phi_\alpha| \leq 2\alpha\,(1 + |z_\alpha + z'_\alpha|^2)$, we get

$$D_{z'z'}\phi_\alpha\,(\xi_i^\mu - \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\; \leq \;\; C(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;. \quad (7.14)$$

Finally, collecting the estimates (7.12), (7.13) and (7.14), we deduce from (7.11) that for some positive constant $\tilde{C}$ (independent of L, ε and α)

$$\sum_{i=1}^{m+d} M\beta_i^\mu\cdot\beta_i^\mu \;\; \leq \;\; \tilde{C}e^{L(t-t_\alpha)}(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;\; = \;\; \tilde{C}\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \;.$$

Hence, if we take $L \geq \frac{\tilde{C}}{2}$, then (7.10) reads

$$\lambda(v(t_\alpha, z_\alpha) - u(t_\alpha, z'_\alpha)) \;\leq\; (\frac{\tilde{C}}{2} - L)\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \;\leq\; 0$$

which is in contradiction with (7.7). □

7.3.1 Proof of Proposition 6.1

We first make use of Proposition 7.1 to obtain the intermediary inequality $u_* \geq \sup_{\kappa>0} v^*_\kappa$.

Corollary 7.1 *Under* $(\mathbf{H}_\lambda)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H_g})$, *for each* $\kappa > 0$, *we have* $u_* \geq v^*_\kappa$ *on* $[0, T] \times I\!R_+^{m+d}$.

Proof. In view of Lemmas 7.2, 7.3, Corollary 6.1 and Proposition 7.1, it suffices to show that u_* and v^*_κ have linear growth. To check this condition for v^*_κ, it suffices to recall that $\hat{Z}^\mu$ is a martingale and use assumption $(\mathbf{H_g})$. We now consider u_*. First, recall from $(\mathbf{H_g})$ that G^{conc} has linear growth. Using Lemma 7.1, we deduce that, for each $(a,b) \in \mathcal{D}$ and $(t,y,z) \in [0,T] \times (0,\infty)^{n+m+d}$, we have

$$f\left(Y^{a,b}_{t,y,z}(T)\right) \cdot g\left(S_{t,s}(T)\right) \;\leq\; G^{conc}\left(Z^{a,b}_{t,y,z}(T)\right) \;\leq\; \delta\left(1 + \sum_{i=1}^{m+d} Z^{a,b,i}_{t,y,z}(T)\right)$$

where $s = \bar{F}(y)^{-1}z$ and δ is some positive constant, recall (5.1). It follows from the definition of $u^{a,b}$ in (5.9) that

$$u(t,y,z) \;=\; \sup_{(a,b)\in\mathcal{D}} u^{a,b}(t,y,z) \leq \delta\left(1 + \sum_{i=1}^{m+d} z^i\right) . \tag{7.15}$$

Now, observe that, for $(a,b) = (0,0)$, $Y^{(0,0)}$ is constant so that $Z^{(0,0),i}$ coincides with S^i up to a multiplicative constant, $i \leq m+d$. Hence, $Z^{(0,0)}$ is a $\mathbb{P}$-martingale and it follows from the definition of $u^{(0,0)}$ that

$$u(t,y,z) \;\geq\; u^{(0,0)}(t,y,z) \;\geq\; \mathbb{E}\left[f\left(y\right) \cdot g\left(S_{t,s}(T)\right)\right] \;\; , \; s = \bar{F}(y)^{-1}z \; .$$

Using Lemma 7.1 and (3.5), we then deduce as above that

$$u(t,y,z) \;\geq\; u^{(0,0)}(t,y,z) \;\geq\; -\hat{\delta}\left(1 + \sum_{i=1}^{m+d} z^i\right)$$

for some positive constant $\hat{\delta}$. Combining the last inequality with (7.15) shows that u_* has linear growth. □

Proof of Proposition 6.1. 1. We first show that $\{\hat{Z}^\mu_z \;,\; \mu \in \cup_{\kappa\geq 0}\mathcal{U}_\kappa\}$ is dense in probability in $\{\hat{Z}^\mu_z \;,\; \mu \in \mathcal{U}\}$. To see this, take $\mu \in \mathcal{U}$ and consider the sequence defined by $\mu_\kappa := \mu \mathbf{1}_{|\mu|\leq\kappa} \in \mathcal{U}_\kappa$, $\kappa \in I\!N$. Recalling (6.3), we deduce from the Itô's isometry that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{m+d} |\ln(\hat{Z}^{\mu,i}_z(T)) - \ln(\hat{Z}^{\mu_\kappa,i}_z(T))|^2\right] \\
&\leq \delta\, \mathbb{E}\left[\int_0^T |\mu(t) - \mu_\kappa(t)|^2 + ||\mu(t)|^2 - |\mu_\kappa(t)|^2|\, dt\right]
\end{aligned}$$

for some $\delta > 0$. Since $\mu_\kappa \to \mu$ dt $\times$ d$\mathbb{P}$-a.e. and, by definition of $\mathcal{U}$, μ is square integrable, we deduce from the dominated convergence theorem that $\ln(\hat{Z}_z^{\mu_\kappa,i}(T))$ goes to $\ln(\hat{Z}_z^{\mu,i}(T))$ in L^2, $i \le m+d$. It follows that the convergence holds $\mathbb{P}-$a.s. along some subsequence.

2. By Corollary 7.1, we have

$$u_*(0,z) \quad \ge \quad \sup_{\kappa>0} \sup_{\mu \in \mathcal{U}_\kappa} \mathbb{E}\left[G^{conc}(\hat{Z}_z^\mu(T))\right] \ .$$

Since G^{conc} has linear growth, see $(\mathbf{H_g})$, there is some $(c,\Delta) \in I\!R \times I\!R^d$ such that $G^{conc}(\hat{Z}_z^\mu(T)) + \Delta \cdot \hat{Z}_z^\mu(T) \ge -c$. Since $\hat{Z}_z^\mu$ is a martingale, it follows that

$$u_*(0,z) \quad \ge \quad \sup_{\kappa>0} \sup_{\mu \in \mathcal{U}_\kappa} \mathbb{E}\left[G^{conc}(\hat{Z}_z^\mu(T)) + \Delta \cdot \hat{Z}_z^\mu(T)\right] - \Delta \cdot z \ .$$

Using 1. and Fatou's Lemma, we then deduce that

$$u_*(0,z) \quad \ge \quad \sup_{\mu \in \mathcal{U}} \mathbb{E}\left[G^{conc}(\hat{Z}_z^\mu(T)) + \Delta \cdot \hat{Z}_z^\mu(T)\right] - \Delta \cdot z \ .$$

Since for $\mu \in \mathcal{U}$, $\hat{Z}_z^\mu$ is also a martingale, the result follows. □

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