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Approximate Solutions to Dynamic Models – Linear Methods

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Abstract:

Linear Methods are often used to compute approximate solutions to dynamic models, as these models often cannot be solved analytically. Linear methods are very popular, as they can easily be implemented. Also, they provide a useful starting point for understanding more elaborate numerical methods. It shall be described here first for the example of a simple real business cycle model, including how to easily generate the log-linearized equations needed before solving the linear system. For a general framework, formulas are provided for calculating the recursive law of motion. The algorithm described here is implemented with the "toolkit" programs available per http://www.wiwi.hu-berlin.de/wpol/html/toolkit.htm .

0. Introduction

Linear Methods are often used to compute approximate solutions to dynamic models, as these models often cannot be solved analytically. While a plethora of advanced numerical methods exist, the most popular "bread-and-butter" method for solving them is linearization. It shall be described here first for the example of a simple real business cycle model, including how to easily generate the log-linearized equations needed before solving the linear system. The classic reference for solving linear difference models under rational expectations is Blanchard and Kahn (1980), while Kydland and Prescott (1982) is the origin of the modern approach of calculating numerically approximate solutions to dynamic stochastic models in order to obtain quantitative results. Much of the material here is taken from Uhlig (1999), which builds on the method of undetermined coefficients in King, Plosser and Rebelo (2002).

1. A basic example

As a basic example, consider a version of the real business cycle model of Hansen (1985). A social planner or representative agent chooses c_t, k_t, y_t, l_t and n_t to maximize the utility function

$$U = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)\right]$$

for some twice differentiable utility function u(.), satisfying the usual conditions, subject to the constraints

$$c_t + k_t = y_t + (1-\delta)k_{t-1}$$

$$y_t = \gamma_t f(k_{t-1}, n_t)$$

$$1 = n_t + l_t$$

as well as a given initial capital stock k_{-1},where c_t denotes consumption, k_t denotes capital, y_t denotes output, l_t denotes leisure, n_t denotes labor, f(k,n) denotes a twice differentiable production function, typically assumed to obey constant returns to scale, β is the discount factor and γ_t is total factor productivity, with

$$z_t = \log(\gamma_t) - \log(\gamma^*)$$

evolving according to

$$z_t = \rho z_{t-1} + \varepsilon_t$$

where

$$E_t[\varepsilon_{t+1}] = 0$$

for some values γ^* and ρ, with $-1 < \rho < 1$. A solution is a stochastic sequence (c_t, k_t, y_t, l_t, n_t),$t \geq 0$ where all variables dated t are independent of all ε_s for $s>t$ and satisfies all constraints, and which maximizes the utility function given above within the set of all such sequences.

The necessary first-order conditions for this problem are given by

$$u_c(c_t, l_t) = \lambda_t$$

$$u_l(c_t, l_t) = f_n(k_{t-1}, n_t)$$

$$\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]$$

$$R_t = f_k(k_{t-1}, n_t) + 1 - \delta$$

2. Linearization

The first step towards solving the model by linear approximation is to linearize all the constraints and necessary equations (possibly after substituting out some variables, if so desired). Linearization amounts to finding a first-order approximation to all equations. Formally, linearization amounts to replacing a set of equations

$$0 = g\left(x_t\right)$$

in a vector x_t of variables with its linearized counterpart around some point of approximation x^*,

$$0 = g\left(x^*\right) + g'\left(x^*\right)\tilde{x}_t$$

where

$$\tilde{x}_t = x_t - x^*$$

is the deviation of x_t from the approximation point x^* and where $G'(x^*)$ is the matrix of first derivatives of G(.). As point of approximation x^*, the nonstochastic steady state is often chosen, i.e. one solves the equations

$$0 = g\left(x^*\right)$$

under the assumption that all exogenous stochastic variables are constant (here: $\gamma_t=\gamma^*$ and all $\varepsilon_s=0$). Then, the remaining linearized system consists of

$$0 = g'\left(x^*\right)\tilde{x}_t$$

Since many economic variables are constrained to be positive, it is often more attractive to log-linearize the equations, rather than to linearize them. The difference between linearization and log-linearization is that entries in x_t denote the original variable (e.g. consumption c_t) in the case of linearization and the log of these variables (e.g. $\log(c_t)$) in the case of log-linearization. There is no need to choose either linearization or log-linearization for all entries in x_t. One may choose to linearize some and log-linearize others or take other transformations. Indeed, for variables such as trade balances, it is better to use linearization rather than log-linearization, if they can take negative values. Also, e.g. tax rates are often more appropriately linearized rather than log-linearized to provide a more useful interpretation.

This makes no difference as far as the linearized solution is concerned. More generally, differentiable and differentiable invertible transformations (i.e. homeomorphisms) of the variables (e.g. taking ratios of variables, etc.) make no difference to the properties of the linearized solution. The differences only always lies in the recalculation of the original variables, where one may want to take into account the nonlinearities originally inherent in the model. To see more generally, that any homeomorphism (i.e. differentiable and differentiably invertible transformation)

$$y_t = h(x_t)$$

of the variables makes no difference to remaining calculations, note that the equations can be restated as

$$0 = g\left(h^{-1}\left(y_t\right)\right)$$

The linearized version is now

$$0 = g\left(h^{-1}\left(y^*\right)\right) + g'\left(x^*\right)\left(f^{-1}\right)'\left(y^*\right)\tilde{y}_t$$

which coincides with the previous linearization if $y^*=F(x^*)$, noting that

$$\hat{y} = f'\left(x^*\right)\hat{x}_t$$

as well as

$$I = f'\left(x^*\right)\left(f^{-1}\right)'\left(y^*\right)$$

While linearization can be performed numerically or with the usual rules of calculus, one can often "read" the log-linearized version of an equation from its original form, exploiting

$$x_t = \exp\left(y_t\right) \approx x^* + x^*\tilde{y}_t$$

where now $y_t = \log(x_t)$. Write $\hat{x}_t$ Instead of $\tilde{y}_t$ for the loglinear deviation.

For log-linearization, the following useful "rules" can easily be derived. Let a_t, b_t, c_t be three variables, with $c_t=h(a_t)$ for some monotone and differentiable function h(.), and let B be some constant. Then,

$$a_t + Bb_t \approx \left(a^* + Bb^*\right) + \left(a^*\hat{a}_t + Bb^*\hat{b}_t\right)$$

$$Ba_tb_t \approx \left(Ba^*b^*\right) + \left(Ba^*b^*\right)\left(\hat{a}_t + \hat{b}_t\right)$$

$$\hat{c}_t \approx \frac{h'(a^*)a^*}{h(a^*)}\hat{a}_t$$

Either with these rules or directly, the equations in the example log-linearize to

$$c^* \hat{c}_t + k^* \hat{k}_t = y^* \hat{y}_t + (1-\delta) k^* \hat{k}_{t-1}$$

$$\hat{y}_t = z_t + \frac{f_k k^*}{f} \hat{k}_{t-1} + \frac{f_n n^*}{f} \hat{n}_t$$

$$0 = n^* \hat{n}_t + \left(1 - n^*\right) \hat{l}_t$$

$$\hat{\lambda}_t = \frac{u_{cc} c^*}{u_c} \hat{c}_t + \frac{u_{cl} l^*}{u_c} \hat{l}_t$$

$$\frac{u_{cl} l^*}{u_l} \hat{c}_t + \frac{u_{ll} l^*}{u_l} \hat{l}_t = \frac{f_{nk} k^*}{f_n} \hat{k}_{t-1} + \frac{f_{nn} n^*}{f_n} \hat{n}_t$$

$$\hat{\lambda}_t = E_t \left[\hat{\lambda}_{t+1} + \hat{R}_{t+1} \right]$$

$$R^* \hat{R}_t = \frac{f_{kk} k^*}{f_k} \hat{k}_{t-1} + \frac{f_{kn} n^*}{f_k} \hat{n}_t$$

3. Solving for the recursive law of motion

With some further algebra, one can turn this system into a second-order one-dimensional difference equation,

$$0 = E_t \left[F x_{t+1} + L z_{t+1} \right] + G x_t + M z_t + H x_{t-1}$$

plus the evolution of the exogenous state,

$$z_t = N z_{t-1} + O \varepsilon_t$$

where $x_t=k_t$ is the capital stock, and F, L, G, M, H, N and O are real numbers (here, with N= and O=1). Alternatively, use the system of equations above directly (or with some variables substituted out) and stack all variables into a vector x_t to reformulate it in this form, where now F, L, G, M and H are matrices of coefficients. Indeed, if there is more than one predetermined variable like k_{t-1} in the system of equations, one will need to use such a matrix restatement of the equations anyways. More generally, z_t may also be a vector, and N and O matrices.

Anderson et al (1996) as well as Binder and Pesaran (1997) contain detailed and general results for solving linearized systems. In most cases, the system has a solution in the form of a recursive law of motion,

$$x_t = Px_{t-1} + Qz_t$$

for some coefficient matrices P and Q. Most models require the solution to be stable, i.e. all eigenvalues of P to be less than unity in absolute value. Often, one also allows for roots equal to unity in absolute value, as this arises easily e.g. in models of international trade or with multiple agents: one may then want to think of the linear approximation as a local solution. In many models, this uniquely determines the matrix P and usually also Q.

The solutions can be found by substituting the recursive law of motion in for x_{t+1} and again for all x_t into the second-order difference equation above, exploiting

$$Nz_t = E_t\left[z_{t+1}\right]$$

so that only x_{t-1} and z_t and some coefficient matrices remain.

Examine first the equation by matching coefficients on x_{t-1}. One obtains the equation

$$0 = FP^2 + GP + H$$

for P. In case of a one-dimensional difference equation (as can be obtained for the example above and $x_t=k_t$), this is a quadratic equation in the feedback coefficient P, which has two solutions. The system is said to be saddle-path stable, if only one of the two roots is smaller than unity in absolute value. Thus, if a stable solution is desired, this is the unique solution for P.

Generally, the equation above is a matrix quadratic equation, which can be solved per computing generalized eigenvalues or by QZ-decomposition as follows. Let m be the dimensionality of x_t. Define the matrices

$$A = \begin{bmatrix} -G & -H \\ I_m & 0_m \end{bmatrix}, B = \begin{bmatrix} F & 0_m \\ 0_m & I_m \end{bmatrix}$$

where I_m is the m-by-m identity matrix and 0_m the m-by-m matrices of only zeros.

Recall that a generalized eigenvector s with eigenvalue λ for the matrices A and B is defined as satisfying

$$\lambda Bs = As$$

The generalized eigenvector problem reduces to the standard eigenvector problem of $B^{-1}A$, if B is invertible. If s is a generalized eigenvector with eigenvalue λ for the matrices A and B above, it can be written as s'=[λ x',x'] for some m-dimensional vector x. If there are m generalized eigenvalues $\lambda_1,...,\lambda_m$ together with generalized eigenvectors

$$s_i = \left[\lambda_i x_i', x_i'\right]$$

such that

$$C = \left[x_1, \ldots, x_m\right]$$

is of full rank, then

$$P = C\Lambda C^{-1}$$

is a solution to the matrix quadratic equation, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_m \end{bmatrix}$$

is the diagonal matrix of the eigenvalues for the generalized eigenvectors used as well as of P. The system is said to be saddle-path stable, if there are exactly m generalized eigenvalues smaller than unity in absolute value. In that case, the matrix P is unique, if one requires all eigenvalues of P to be stable. If there are fewer than m eigenvalues smaller than (or equal to) unity in absolute value, then there is no solution, such that the difference equation $x_t=Px_{t-1}$ remains bounded for all x_0. In that case, the set of bounded solution is characterized by $e'x_0 = 0$ as well as $e'Qz_t = 0$ for all t for all eigenvectors e of P corresponding to explosive eigenvalues. The second of these two constraints may impose restrictions on the exogenous shock process. If there are more than m eigenvalues smaller than (or equal to) unity in absolute value, then sunspot solutions may arise, i.e., there are additional solutions. In the one-dimensional case and if F is nonzero, the general solution is now given by the original equation, i.e. as

$$x_t = -F^{-1}Gx_{t-1} - F^{-1}Hx_{t-2} - F^{-1}(LN + M)z_{t-1} + \nu_t$$

where ν_t is any stochastic process with $E_t\left[\nu_{t+1}\right] = 0$ and which is independent of all ε_s for s>t, but not necessarily independent of ε_t. Note that the recursive law of motion now includes an additional lag of the state variable, as well as the possibility for additional random influences ("sunpots") via ν_t, which are not part of the original system of equations. Farmer (1999) provides a detailed treatment of sunspots in linearized solutions.

Equivalently, consider the stacked variable $s_t'=[\, x_t', x_{t-1}']$, and note that the second half of this vector is "predetermined", i.e. must be independent of all ε_s for s>t-1. The linearized system can be rewritten as

$$BE_t\left[s_{t+1}\right] = As_t + \begin{bmatrix} -M - LN \\ 0 \end{bmatrix} z_t$$

If B is invertible, the solutions can now be characterized in terms of the eigenvalues and eigenvectors of $B^{-1}A$. This is the approach taken in the classic reference of Blanchard and Kahn (1980).

Alternatively, find the QZ-decomposition (or generalized Schur decomposition) of A and B, see Sims (2002), i.e., find unitary matrices U and V as well as upper triangular matrices K and L such that

$$A = U'LV$$

$$B = U'KV$$

(and recall that a matrix is unitary, if the product with its complex conjugate transpose is the identity matrix). Such a Schur decomposition always exists, although it may not be unique. Partition U and V into m-by-m submatrices,

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

If U_{21} and V_{21} are invertible, then

$$P = -V_{21}^{-1}V_{22}$$

solves the matrix quadratic equation. Suppose furthermore, that the QZ-decomposition has been chosen so that the ratios $\left|L_{ii} / K_{ii}\right|$ are in ascending order. Furthermore, suppose $\left|L_{mm} / K_{mm}\right| < 1$. Then P is stable.

To solve for Q, given a solution to P, compare the coefficients on z_t to find

$$V\ \mathrm{vec}(Q) = -\mathrm{vec}(LN + M)$$

where vec(.) denotes columnwise vectorization and where

$$V = N' \otimes F + I_k \otimes (FP + G)$$

with k the dimensionality of z_t. If V is invertible, the solution is unique.

Many links for codes for solving dynamic stochastic models are available per http://dge.repec.org/codes.html. The procedure outlined above has been used in particular in the toolkit programs of H. Uhlig, see http://www.wiwi.hu-berlin.de/wpol/html/toolkit.htm. For a discussion of the accuracy of linearized solution, see e.g. Taylor and Uhlig (1990) and Aruoba et al (2005).

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