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The Natural Rate Hypothesis and Real Determinacy

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A Sticky-Information Perspective

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Abstract

The uniqueness of bounded local equilibria under interest rate rules is analyzed in a model with sticky information à la Mankiw and Reis (2002). The main results are tighter bounds on monetary policy than in sticky-price models, irrelevance of the degree of output-gap targeting for determinacy, independence of determinacy regions from parameters outside the interest-rate rule, and equivalence between real determinacy in models satisfying the natural rate hypothesis and nominal determinacy in the associated full-information, flex-price equivalent. The analysis follows from boundedness considerations on the nonautonomous recursion that describe the MA(∞) representation of variables' reaction to endogenous fluctuations.

Forty years have past since Friedman (1968, p. 11) quite succinctly stated, "there is always a temporary trade-off between inflation and employment; there is no permanent trade-off." Yet, at the foundation of current monetary policy analysis is a model of price setting that imposes a systematic relationship between inflation and output, stable even in the long run.[1]

The first central result of the analysis here is that the regions of determinacy associated with interest rate rules are tighter in a model without a permanent trade-off than in the literature standard sticky-price New Keynesian model. Though determinacy, at its core, is a long-run consideration, it has immediate relevance. Imposing boundedness, a dynamic path is uniquely determined if all but one of the paths consistent with equilibrium diverge. In the absence of determinacy, the economy is vulnerable to arbitrary and potentially welfare-reducing endogenous fluctuations, i.e. sunspots.(Carlstrom and Fuerst 2002, p. 79) The central result has, therefore, a pressing policy recommendation: if one is unwilling to accept a systematic, long-run relationship between inflation and output, the tighter parameter bounds for interest rate rules derived here ought to be heeded.

Mankiw and Reis (2002) (extended to general equilibrium by, e.g., Trabandt (2007)) have shown that several empirical and theoretical shortcomings in sticky-price models can be overcome by a sticky-information setup, in which firms optimally reset their prices each period constrained by a probabilistic Bayesian updating of information governed by a Poisson process. For the purposes here, the relevant improvement is "that it survives the McCallum critique" (Mankiw and Reis 2002, p. 1300); that is, it fulfills the strict version of the natural rate hypothesis.

The natural rate hypothesis states that "on average, and regardless of [the] monetary policy regime, output [...] should be equal to potential output" (Andrés, López-Salido, and Nelson 2005, p. 1027). A model that satisfies the natural rate hypothesis will have a

[1] cf. Woodford (2003b, p. 254)

vertical long-run Phillips curve and, thus, will exhibit no permanent trade-off.[2] The natural rate hypothesis being fulfilled, therefore, forces the output gap to converge *regardless* of monetary policy.

In the sticky-price model, a reaction of the nominal interest rate to the output gap serves as a substitute for a reaction to inflation, allowing the (direct) response to inflation to be less than one while still adhering to the Taylor principle.[3] This would be futile within the context of the sticky-information model examined here, as there is no long-run link between inflation and the output gap: the output gap cannot substitute for inflation insofar as determinacy is concerned.

This is the second result of the analysis here: the degree of output-gap targeting is irrelevant for determinacy. Simply put: via the natural rate hypothesis, the output gap must be zero asymptotically regardless of inflation and monetary policy; targeting the output gap will not yield a long-run change in the nominal interest rate. This, in turn, implies the third result: determinacy is independent of parameter values in the dynamic IS and Phillips curve equations. With the demand side defined by a dynamic IS curve, the convergence of the output gap implies convergence of the real interest rate regardless of monetary policy. Thus, determinacy via an interest rate rule rests on the determinacy of nominal variables through the Fisher equation, an equation with no relation to parameter values in the dynamic IS or aggregate supply equations.

As implied by the foregoing paragraph, these results are a consequence of the three equation reduced form (IS, Phillips curve, nominal interest rate rule) satisfying the natural rate hypothesis. The sticky-information model certainly fulfills this hypothesis, but the results of the analysis here extend to any model of the three equation reduced form with a short-run link between inflation and the output gap that satisfies the natural rate hypothesis. The uniqueness of a path for inflation occurs under the same conditions as would be obtained for the corresponding flex-price, full-information counterpart. In models with a short-run link between the nominal and real sides, the paths of the nominal side and real side are interdependent. Therefore, a monetary policy which ensures a unique path for inflation will ensure the uniqueness of the necessarily convergent path for the output gap and this will occur only if the monetary policy is consistent with nominal determinacy in the corresponding flex-price, full information equilibrium.

The specific results for the sticky-information model follow from the derivation of conditions for saddle-path stability in the system of nonautonomous homogenous linear difference equations that describe the dynamic response of the model to an endogenous fluctuation. For comparison with the sticky-price literature[4], inflation-forecast and contemporaneous inflation targeting rules in a pure and extended (with output-gap targeting and interest-rate smoothing), exogenous interest rate rules, and price-level targeting

[2]cf. Fischer (1977, p. 192)

[3]cf. Woodford (2003b, pp. 254-255), "... indeed, a large enough [response to] either [the output gap or inflation] suffices to guarantee determinacy."

[4]e.g. Woodford (2003b) or Lubik and Marzo (2007)

rules are examined. The Taylor Principle is a necessary condition for determinacy and a pure inflation-forecast rule is shown to be indeterminate everywhere, with interest-rate smoothing opening a small window for determinacy.

The results here extend Carlstrom and Fuerst's (2002)[5] analysis to models that satisfy Lucas's (1972) formulation of the natural rate hypothesis while failing to satisfy their yet stricter version: "the model's behavior becomes identical to flexible-price behavior in finite time" (Carlstrom and Fuerst 2002, p. 80). This distinction allows for a trade-off between inflation and the output gap at all finite horizons while still ensuring that "[t]he unconditional mean of the output gap cannot [...] be affect by any aspect of the monetary policy rule" (McCallum 1994, p. 259).

The rest of the paper is organized as follows: in Section 1, I shall discuss the basic sticky-price and sticky-information models. In Section 2, conditions for determinacy in the sticky-information model for various interest rate rules will be presented. Section 3 shows the equivalence of the bounds for determinacy to bounds on monetary policy for nominal determinacy in the full-information counterpart to motivate the extension of the results to a more general class of models. Section 4 examines specious determinacy arising from a common truncation method. Section 5 discusses the results and alternative equilibrium selections and Section 6 concludes.

1 A Sticky-Information Model

"[T]oday's near-canonical monetary policy model," the sticky-price model with Calvo (1983)-style overlapping contracts in general equilibrium, is composed of three structural equations determining the supply side, demand side, and monetary policy (McCallum 2003). Abstracting from exogenous driving processes, the New Keynesian model is given (in log-deviations) by [6]

$$y_t = E_t\left[y_{t+1}\right] - a_1 R_t + a_1 E_t\left[\pi_{t+1}\right] \tag{1}$$

$$\pi_t = \beta E_t\left[\pi_{t+1}\right] + \kappa y_t \tag{2}$$

where y_t is the output gap, π_t inflation , and R_t the nominal interest rate. Equation (1) is an expectational IS-curve derived from the first-order conditions of the household for intertemporal utility maximization and equation (2) is the New Keynesian Phillips curve derived from Dixit-Stiglitz aggregators of individual firms' intertemporal discounted profit maximization constrained by the probability that prices set today remain in effect into the future.

[5] Their differing timing convention serves only to alter the interpretation: their "backward-looking" rule corresponds to the contemporaneous inflation targeting rule here and their "current-looking" rule corresponds to the inflation-forecast targeting rule.

[6] cf. McCallum (2001, p. 152), equations (2.7) and (2.14); or Lubik and Marzo (2007, p. 21)

Both Woodford (2003b, pp. 243 & 245) and Lubik and Marzo (2007) restrict both κ and a_1 to be strictly positive and a positive a_1 is assumed here throughout. Lubik and Marzo (2007, p. 17) emphasize that the derivation of these parameters from first principles is absolutely essential due to "cross-equation restrictions". A main result of this paper is that the specific parameter values in the sticky-information model are irrelevant for determinacy.

In the sticky-information variant of the New Keynesian model, equation (2) is replaced by the sticky-information Phillips curve

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}\left[\pi_t + \xi\left(y_t - y_{t-1}\right)\right] \tag{3}$$

where ξ is Woodford's (2003b, pp. 160-161) measure of strategic complementarities, and $1-\lambda$ is the probability that a firm receives an information update. Equation (3) is due to Mankiw and Reis (2002) who derive this Phillips curve based on firms' pricing decisions being the expectation of the optimal price conditional on their (potentially) out-dated information set. A derivation of (1) and (3) based on first principles with exogenously given government expenditures analogous to Woodford (2003b, Ch. 4) can be found in Trabandt (2007). In any case, despite any similarities to the model examined by Carlstrom and Fuerst (2002), their model lacks the infinite regress in the information structure of the Phillips curve found in (3), making their model isomorphic to a flex-price model in finite time.

The dynamics of inflation as presented by Mankiw and Reis (2002) have been criticized by, e.g., Keen (2007) as the assumption of $0 < \xi < 1$ drives the results of the former and the latter find a specification larger than unity to be more plausible. I do not want to dwell on the importance of strategic complementarities here, as they play no role in the determinacy of equilibria under the interest-rate setting rules examined here. Thus, though the degree of strategic complementarities may be crucial for the dynamics of the model, it will be irrelevant for determinacy. So long as it can be accepted that, *ceteris paribus*, an increase in the deviation of aggregate output from its "natural" level induces firms to want to raise their prices ($\xi > 0$), no further restriction is necessary for the results that follow. This is certainly a mild assumption and covers the entire parameter space considered by Woodford (2003b, pp. 162-164).

Missing is a specification of monetary policy. Following Woodford (2003b) among many, I shall focus on interest-rate setting rules. "With the interest rate as the policy instrument, the central bank adjusts the money supply to hit the interest rate target" and, thus, "it is not necessary to specify a money market equilibrium condition" (Clarida, Galí, and Gertler 1999, p. 1667).

2 Indeterminacy and the Nominal Interest Rate

After introducing the methods of the analysis, the set of interest rate setting rules examined by Woodford (2003b, Ch. 4) will be examined in the context of equilibrium determinacy. The ordering therein will be roughly followed: examining first output-gap targeting rules (with exogenous rules presented as a special case), inflation targeting (both forecasted and, then, contemporaneous) and, finally, price-level targeting. With the exception of the examination of exogenous rules, the pattern will be to examine a pure targeting rule as a special case of an extended specification that includes both interest-rate smoothing and output-gap targeting. It will prove informative to examine the special case first, likewise following Woodford (2003b, Ch. 4).

2.1 Endogenous Fluctuations and Determinacy

Conspicuously absent from the preceding introduction to the sticky-price and sticky-information models are any exogenous driving forces. This may seem to be an omission of an important aspect of the system. However, following, e.g., Theorem 3.15 of Elaydi (2005, p. 130), the solution to a system of difference equations can be split into a particular and a homogenous solution. Only the homogenous solution of the system of difference equations is relevant for the examination of determinacy.[7] Following Taylor (1986), the bounded solution will be unique for any given bounded exogenous sequence of shocks if and only if the homogenous solution is uniquely determined by the boundedness conditions on the endogenous variables.[8]

By examining the infinite moving average representation of the model in response to endogenous fluctuations (i.e. to sunspot shocks), the system of difference equations originating from the model of sticky information yields a nonautonomous or time-variant system of homogenous difference equations. Appendix A provides the necessary theorems for the analysis in this paper and Appendix B shows the derivation of the system of difference equations that arise from the infinite moving average representation of the model's variables to sunspot shocks.

2.2 Output-Gap Targeting and Exogenous Interest Rates

In this section, I shall examine interest-rate rules with feedback solely from the output gap. As a special case, a constant interest rate(i.e. no feedback) is considered.

Consider the model defined by (1) and (3) with an output-gap targeting interest-rate rule:

(4) $$R_t = \phi_y y_t, \ \ 0 \leq \phi_y < \infty$$

[7] cf. Lubik and Marzo (2007)

[8] Analogous conclusions can be found in, e.g., Woodford (2003b, p. 252, & p. 636) .

Lemma 2.1. *The uniqueness of the fundamental solution (i.e. the absence of sunspot equilibria) is determined by the existence of a unique bounded sequence* $\left\{\delta_i^y\right\}_{i=0}^{\infty}$ *that solves the following non-autonomous recursion:*

$$\left[\left(1-\lambda^{i+2}\right)\xi+\lambda^{i+2}\frac{1}{a_1}\right]\delta_{i+1}^y=\left[\left(1-\lambda^{i+1}\right)\lambda\xi+\lambda^{i+2}\frac{1+a_1\phi_y}{a_1}\right]\delta_i^y \qquad (5)$$
$$i=0,1,2,...,$$

Proof. See Appendix C.1.

Using Lemma 2.1, one looks for parameter spaces of monetary policy (ϕ_y) such that the boundedness condition in the Lemma provides an additional restriction on the recursion.

Proposition 2.2. *The model given by (1), (3), and (4) is indeterminate for all* $0 \leq \phi_y < \infty$.

Proof. See Appendix C.2.

Thus, contrary to Woodford (2003b, p. 254), if the feedback from endogenous variables is limited to the output gap, no degree of output-gap targeting will suffice to ensure real determinacy. This difference between the sticky-information and sticky-price models is due to the long-run slope of the Phillips curve. In the former it is vertical, while in the latter it is not.(Woodford 2003b, p. 254) Non-verticality allows monetary policy to substitute output-gap targeting for inflation targeting so as to satisfy the Taylor Principle, a possibility not available in a model without a systematic, long-rung relationship between inflation and output.[9]

With a bounded, exogenous interest rate, the system defined by (1) and (3) is extended by a bounded exogenous process for R_t. As determinacy is related solely to the homogenous part of the system of difference equations, the addition of any bounded stochastic process in the interest rate rule will not affect the results. Thus, determinacy with a bounded exogenous interest rate will be obtained under the same conditions as for a constant interest rate. Therefore, without loss of generality, the model is closed by the following interest rate rule:

$$R_t = 0 \qquad (6)$$

This is simply a special case of Proposition 2.2 and, thus, any constant or bounded exogenous interest rate rule is necessarily associated with indeterminacy. This corresponds to Woodford (2003b, p. 253) and confirms that a nominal interest rate rule must involve feedback from endogenous variables, if sunspot equilibria are to be avoided. This extends

[9]Fries (2007) notes the independence of determinacy from the degree of output-gap targeting in a sticky-information model, but this conclusion is based on Wang and Wen's (2006) supposition for finite lagged expectations and not directly applicable to the true, infinite specification of the sticky information model.

Sargent and Wallace's (1975, p. 251) conclusion in their model, where their "Phillips curve is not vertical [in the short run], but Wicksell's indeterminacy [i.e. indeterminacy with an exogenous interest rate] still arises," to hold in the model here where the Phillips curve is vertical only asymptotically.

2.3 Forward-Looking Inflation Targeting

Consider the model defined by (1) and (3), with an extended inflation-forecast Taylor-type rule:

$$R_t = \phi_R R_{t-1} + \phi_\pi E_t\left[\pi_{t+1}\right] + \phi_y y_t \qquad (7)$$
$$0 < \phi_\pi < \infty,\ 0 \le \phi_y < \infty,\ 0 \le \phi_R < 1$$

Lemma 2.3. *The uniqueness of the fundamental solution is determined by the existence of unique bounded sequences $\left\{\delta_i^y, \delta_i^R\right\}_{i=0}^{\infty}$ that solve the following non-autonomous recursion:*

$$\begin{bmatrix} \left(1-\lambda^{i+2}\right)\xi & -\frac{\lambda^{i+2}}{\phi_\pi} \\ 1 & -a_1\left(1-\frac{1}{\phi_\pi}\right) \end{bmatrix} \begin{bmatrix} \delta_{i+1}^y \\ \delta_i^R \end{bmatrix} = \begin{bmatrix} \left(1-\lambda^{i+1}\right)\lambda\xi-\lambda^{i+2}\frac{\phi_y}{\phi_\pi} & -\lambda^{i+2}\frac{\phi_R}{\phi_\pi} \\ 1+a_1\frac{\phi_y}{\phi_\pi} & a_1\frac{\phi_R}{\phi_\pi} \end{bmatrix} \begin{bmatrix} \delta_i^y \\ \delta_{i-1}^R \end{bmatrix}$$
$$i = 0, 1, 2, \ldots,$$
$$\delta_{-1}^R = 0 \qquad (8)$$

Proof. See Appendix D.1.

Using Lemma 2.3, one looks for parameter spaces of monetary policy (ϕ_π) such that the boundedness condition in the Lemma provides an additional restriction on the recursion.

Proposition 2.4. *The model given by (1), (3), and (7) is determinate if and only if* $1-\phi_R < \phi_\pi < 1+\phi_R$.[10]

Proof. See Appendix D.2.

It is instructive to begin with the special case $\phi_y = \phi_R = 0$, the case of pure inflation-forecast targeting.[11] Note that according to Proposition 2.4, the determinacy region collapses to an empty set: a pure inflation-forecast targeting rule is <u>necessarily</u> indeterminate.

[10] Here and in the following, it is to be understood that the analysis will be abstracting from cases where the relevant eigenvalues lie on the unit circle. Following Woodford (2003b, p. 254), in such a case, the linearized models examined here are insufficient to address the question of local determinacy. As shown by Klein (2000), the arbitrary initial condition associated with the stable manifold must be "translated" into the given initial condition. As is shown numerically in the Appendices, this would appear to be the case quite generally.

[11] It is also instructive to prove the special case first as can be found in Appendix D.2

Thus, despite the fact that the sticky-information does not satisfy Carlstrom and Fuerst's (2002) more stringent natural-rate hypothesis (i.e. the model is not isomorphic to its flexible-price equivalent in finite time), the same result (saving for the alternate timing-convention) for indeterminacy is obtained. Contrary to sticky-price models (cf. Lubik and Marzo (2007) or Woodford (2003b)), there is no region of determinacy for pure inflation-forecast targeting rules. In Carlstrom and Fuerst's (2002) world of finite stickiness, the model displays real determinacy only if the model possesses nominal determinacy. The latter is fulfilled only if the the inflation rate is uniquely determined at the dawn of the flexible-price world, which itself cannot hold if inflation forecasts are the sole feedback variable for nominal interest rate rules. Here, the flexible-price world dawns only at the end of time, yet the asymptotic vertically of the Phillips curve suffices to prevent determinacy under the rule considered here. This would certainly seem to be more consistent with Woodford's (2000) discussion of the non-optimality of purely forward-looking monetary policy rules than the analogous analysis in sticky-price models: the purely forward-looking rule considered here will always be indeterminate and, thus, opens the model to potentially welfare-reducing arbitrary fluctuations.

The driving force behind this result can be seen by first examining the sticky price model. Lubik and Marzo (2007, pp. 23-24) derive a lower bound for ϕ_π corresponding to the Taylor Principle, which rules out monotonic sunspot behavior, and an upper bound which rules out non-monotonic sunspot dynamics. A key insight from their analysis is: "that the determinacy region disappears as [...] prices become perfectly flexible."(Lubik and Marzo 2007, p. 23) As the Phillips curve becomes perfectly vertical in the long run, the upper bound converges to the lower bound.

The intuition behind the non-monotonic sunspots can be seen as follows. Consider the case: $\phi_\pi = 1 + \frac{\lambda}{1-\lambda}\frac{1}{\xi a_1}$,[12] and the sunspot belief structure $y_0 > 0$, $y_t = 0, \forall t \geq 1$ that disturbs the economy from its steady state: is this consistent with the equilibrium equations? From (3) and (1)

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + \left(1-\lambda^t\right)\pi_t + \left(1-\lambda^t\right)\xi\Delta y_t$$
$$y_t = y_{t+1} - a_1(\phi_\pi - 1)\pi_{t+1}$$

For $t = 0$, $\pi_0 = \frac{1-\lambda}{\lambda}\xi y_0$. From the demand equation,

$$y_1 = y_0 + a_1(\phi_\pi - 1)\pi_1 = y_0 + \frac{\lambda}{1-\lambda}\frac{1}{\xi}\pi_1$$

this should be zero without placing any restrictions on y_0 if the belief structure is a con-

[12]This particular parameter setting allows for the sunspots to converge after one period. For other values of ϕ_π, the sunspots converge only asymptotically and their analysis would unduly clutter the exposition here.

sistent sunspot. Examining the Phillips Curve of the next period

$$\pi_1 = \frac{1-\lambda}{\lambda}\xi y_1 + (1-\lambda)\pi_1 + (1-\lambda)\xi\Delta y_1 = \frac{1-\lambda}{\lambda}\xi\left(\frac{1+\lambda}{\lambda}y_1 - y_0\right)$$

Inserting this into the demand equation,

$$y_1 = y_0 + \frac{1+\lambda}{\lambda}y_1 - y_0 \Leftrightarrow y_1 = \frac{1+\lambda}{\lambda}y_1 = 0$$

leading to $\pi_1 = -\frac{1-\lambda}{\lambda}\xi y_0$. The sunspot belief in a positive deviation in the output gap leads to higher inflation through firms' monopolistic behavior. The expectation of a return to a closed output gap tomorrow implies negative growth in the output gap for tomorrow, the both of which lead to a fall in inflation tomorrow. This expected fall in inflation leads the central bank to lower the nominal interest rate today more than the expected fall in inflation tomorrow. This yields a fall in the real interest rate today, thereby confirming the sunspot increase in the output gap.

Like in sticky-price models, the sticky information models posits both a lower bound and an upper bound on the determinacy region. The lower bound requires the interest rate to follow the Taylor Principle, necessitating an active interest rate. The upper bound, however, requires that the interest rate not be overly aggressive, lest "the output gap and inflation [be] projected to converge back to the steady state *regardless* of their values in the current period." (Levin, Wieland, and Williams 2003, p. 628) The difference is that the two bounds collapse, meaning every interest rate rule of this type is either too aggressive or not aggressive enough.

Turning to the general case, the history dependence in the interest-rate rule induced by the motive of interest-rate smoothing is enough to open a window of determinacy for a forward-looking Taylor-type rule. That there exists an upper and a lower bound on the elasticity of the nominal interest rate with respect to expected inflation is consistent with sticky-price models as discussed above. The lower bound conforms to Woodford's (2003b, p. 96) inertial modification of the Taylor Principle: the cumulative response of the nominal interest rate must react more than one-to-one to a sustained deviation in inflation (saving for the irrelevance of output-gap targeting as discussed previously). Lubik and Marzo's (2007, p. 29) remark that the upper bound in their sticky-price model is "far above the range of reasonable inflation coefficients" for commonly encountered parameter values. This assurance is far from convincing in the sticky-information model. Indeed, for the inertial interest rate rule considered here, the coefficient must be less than two: a value which is certainly not far above the range of reasonable coefficients.

2.4 Contemporaneous Inflation Targeting

Consider again the model (1) and (3). If monetary policy pursues a extended inflation target, the model will be closed by the following Taylor-type rule:

(9)
$$R_t = \phi_R R_{t-1} + \phi_\pi \pi_t + \phi_y y_t$$
$$0 < \phi_\pi < \infty,\ 0 \leq \phi_y < \infty,\ 0 \leq \phi_R < 1$$

Lemma 2.5. *The uniqueness of the fundamental solution is determined by the existence of unique bounded sequences $\left\{\delta_i^y, \delta_i^R\right\}_{i=0}^{\infty}$ that solve the following non-autonomous recursion:*

$$\begin{bmatrix} \left(1-\lambda^{i+2}\right)\xi + \lambda^{i+2}\frac{\phi_y}{\phi_\pi} & -\frac{\lambda^{i+2}}{\phi_\pi} \\ 1 - a_1\frac{\phi_y}{\phi_{pi}} & \frac{a_1}{\phi_{pi}} \end{bmatrix} \begin{bmatrix} \delta_{i+1}^y \\ \delta_{i+1}^R \end{bmatrix} = \begin{bmatrix} \left(1-\lambda^{i+1}\right)\lambda\xi & -\lambda^{i+2}\frac{\phi_R}{\phi_\pi} \\ 1 & a_1\left(1+\frac{\phi_R}{\phi_\pi}\right) \end{bmatrix} \begin{bmatrix} \delta_i^y \\ \delta_i^R \end{bmatrix}$$
$$i = 0, 1, 2, \ldots,$$

(10)
$$\frac{\delta_0^R}{\phi_\pi} = \left(\frac{1-\lambda}{\lambda}\xi + \frac{\phi_y}{\phi_\pi}\right)\delta_0^y$$

Proof. See Appendix E.1.

Using Lemma 2.5, one looks for parameter spaces of (ϕ_π) such that the boundedness condition in the Lemma provides an additional restriction.

Proposition 2.6. *The model given by (1), (3), and (9) is determinate if and only if* $1-\phi_R < \phi_\pi$.

Proof. See Appendix E.2.

In the special case of pure inflation targeting $\phi_y = \phi_R = 0$. As with pure inflation-forecast targeting, although the sticky-information model does not satisfy Carlstrom and Fuerst's (2002) more stringent natural-rate hypothesis, the same result for indeterminacy is obtained. The celebrated Taylor Principle is seen to be a necessary (as discussed in the previous section) and, now, sufficient condition for determinacy.

Thus, contrary to Woodford (2003b, p. 255), determinacy is independent of the degree of output-gap targeting, as discussed in Section 2.2. Examining the case $\phi_R = 0$, this condition reduces to that of a pure inflation target, and, for $\phi_R \neq 0$, determinacy requires the nominal interest rate to move cumulatively more than one-to-one in response to a permanent change in inflation, Woodford's (2003b, pp. 95-96) "eventual" Taylor Principle.

The equivalence (up to cumulative effects) of the two interest-rate feedback rules examined here, reiterate the conclusion from previous sections that output-gap targeting is irrelevant for determinacy. The absence of parameters outside of monetary policy has the convenient attribute that determinacy can be evaluated solely on the merits of the interest rate rule. Thus, Woodford's (2003b, p. 255) slightly complicated interpretation of Taylor (2001), requiring parameter estimations of the sticky-price Phillips curve is not necessary in the sticky-information model. Indeed, if one is to take Woodford (2003b, p. 255) seriously, that monetary policy of the pre-Volker era implied indeterminacy is not necessarily due to too weak of a reaction to inflation, a higher reaction to the output gap would have sufficed; a conclusion which cannot be reached here.

2.5 Price-Level Targeting

Consider the model defined by (1) and (3). If monetary policy pursues a price-level target with feedback from the output gap, the model will be closed by the following rule:

$$R_t = \phi_p p_t + \phi_y y_t, \;\; 0 \le \phi_p, \phi_y < \infty \tag{11}$$

Lemma 2.7. *The uniqueness of the fundamental solution is determined by the existence of unique bounded sequences* $\left\{\delta_i^y, \delta_i^R\right\}_{i=0}^{\infty}$ *that solve the following non-autonomous recursion:*

$$\begin{bmatrix} \left(1-\lambda^{i+2}\right)\xi + \lambda^{i+2}\frac{\phi_y}{\phi_p} & -\frac{\lambda^{i+2}}{\phi_p} \\ 1 - a_1\frac{\phi_y}{\phi_p} & \frac{a_1}{\phi_p} \end{bmatrix} \begin{bmatrix} \delta_{i+1}^y \\ \delta_{i+1}^R \end{bmatrix} = \begin{bmatrix} \left(1-\lambda^{i+1}\right)\lambda\xi + \lambda^{i+2}\frac{\phi_y}{\phi_p} & -\lambda^{i+2}\frac{\phi_R}{\phi_p} \\ 1 - a_1\frac{\phi_y}{\phi_p} & a_1\left(1+\frac{1}{\phi_p}\right) \end{bmatrix} \begin{bmatrix} \delta_i^y \\ \delta_i^R \end{bmatrix}$$

$$i = 0, 1, 2, \ldots,$$

$$\frac{\delta_0^R}{\phi_p} = \left(\frac{1-\lambda}{\lambda}\xi + \frac{\phi_y}{\phi_p}\right)\delta_0^y \tag{12}$$

Proof. See Appendix F.1.

Using Lemma 2.7, one looks for parameter spaces of monetary policy (ϕ_π) such that the boundary condition in the Lemma provides an additional restriction on the recursion.

Proposition 2.8. *The model given by (1), (3), and (9) is determinate if and only if* $\phi_p > 0$.

Proof. See Appendix F.2.

With pure price-level targeting $\phi_y = 0$, any positive response to the price level will ensure determinacy. This corresponds to Woodford (2003b, p. 261) and follows from the Taylor Principle.

Note that the general result stands in contrast to that of Woodford (2003b, p. 261). This is an obvious consequence of the irrelevance of the degree of output-gap targeting that follows from the long-run verticality of the Phillips curve in the sticky-information model as discussed in the foregoing sections. Woodford's (2003b, p. 261) assessment of the attractiveness of a price-level target due to the independence of determinacy from the strength of the elasticity with respect to the output gap follows here and throughout this paper essentially tautologically as the question of determinacy is independent of output-gap targeting for all the rules considered here.

3 Real and Nominal Determinacy and the Natural Rate

In this section, I shall show that the results obtained heretofore are equivalent to nominal determinacy in the corresponding flex-price, full-information counterpart. Thereafter, I shall extend the results of previous sections to a broader class of models that share the natural-rate feature and the structure of the demand side and monetary policy.

The equivalence to nominal determinacy in the counterpart without nominal frictions provides a key insight for the extension to a broader class. In the full-information counterpart of the sticky-information model, there is no impediment to firms' setting the optimal, full-information price every period. It follows that the output gap is always zero and, thus, from (1),

$$R_t = E_t\left[\pi_{t+1}\right] \tag{13}$$

this is identical to the Fisher equation in Woodford's (2003b, Ch. 2) analysis of nominal (price-level) determinacy in a frictionless, flexible-price economy. The conditions for the existence of a unique, locally bounded solution for the nominal interest rate and inflation for each of the interest rate rules in the foregoing sections are identical to the conditions presented there. This is shown explicitly in Woodford (2003b, Ch. 2) for exogenous interest rate rules and contemporaneous inflation targeting with and without interest rate smoothing. It can easily be shown that this holds for the remaining rules examined here and will be excluded here for the sake of brevity.

Using the foregoing, the results in this paper can be extended to a broader class of models.

Proposition 3.1. *Consider a model with a demand side as given by (1) and monetary policy defined over control of the nominal interest rate. Let the supply side be described by any relationship between the output gap and inflation such that (I) the model conforms to Lucas's (1972) natural rate hypothesis and (II) for at least one horizon, there is a trade-off between inflation and the output gap. The bounds for real determinacy are identical to the bounds for nominal determinacy in the flex-price, full-information counterpart given by (13).*

Proof. If the model conforms to (I), then the unconditional expectation of the output gap must be equal to zero independent of monetary policy (see McCallum (1998, p. 359)). Taking the unconditional expectation of (1):

$$E\left[y_t - y_{t+1}\right] = a_1 E\left[\pi_{t+1} - R_t\right] \tag{14}$$

which posits a relationship between the unconditional expectation of the output gap with monetary policy (defined over the nominal interest rate R_t). One could certainly specify a process for the nominal interest rate such that the unconditional expectation of the output gap would be equal to zero, but the natural rate hypothesis requires that this hold regardless of monetary policy. Thus, that the unconditional expectation of the output gap is equal to zero must follow from the supply side equation and must hold independently of (1).

The natural rate hypothesis delivers, then, the existence but not the uniqueness of a bounded path for the output gap irrespective of the existence and uniqueness of bounded paths for inflation and the nominal interest rate. However, from (14) it must then be the case that the real interest rate $R_t - E_t[\pi_{t+1}]$ also converges. Furthermore, if the bounded path for the real interest rate is uniquely determined, then so is the bounded path for the output gap and vice-versa.

The uniqueness of a bounded path for inflation and the nominal interest rate is, thus, given by the rule for monetary policy and (13) as the real interest rate is some bounded process and can, for the purposes of determinacy,[13] be normalized to zero. Determinacy, therefore, corresponds to nominal determinacy in the flex-price, full-information counterpart.

Were it not for (II), there would be complete separation between the real and nominal sides of the economy and monetary policy through the nominal interest rate would serve only to establish nominal determinacy. That (II) holds by assumption links nominal and real determinacy: without a unique path for inflation, (II) implies that although every path for the output gap be bounded, a unique path for the output gap cannot be pinned down. If a unique path for inflation be determined by (13) and monetary policy, this path selects, through (II), a single path for the output gap.

Therefore, there is a unique convergent path for the output gap if and only if there is a unique convergent path for inflation and the nominal interest rate in the counterpart model (13). □

A few comments are in order here. Classical real business cycle models are generally of the type that (I) holds but not (II), as complete flexibility in prices is assumed.[14] This is simply the case studied at the beginning of this section and monetary policy serves only to establish nominal determinacy. In the sticky-price New Keynesian model, (II) holds while (I) does not. As a consequence of (I) not holding, the sticky-price model is not isomorphic to its flex-price equivalent even asymptotically, and there is no reason to expect a general equivalence between determinacy conditions in the two models. With there being a permanent link between the nominal and real side of the economy, nominal and real determinacy must be simultaneously ascertained.

When both (I) and (II) hold, (II) provides the link between nominal and real determinacy as in standard sticky-price models. Condition (I), however, ensures that this link

[13] See Section (2.1). As the real rate converges independently of inflation and the nominal interest rate, it can be omitted from the homogenous solution of the latter for the purpose of determinacy, leaving the homogenous solution to (13) and the monetary policy rule to establish the determinacy of the nominal variables.

[14] cf. Woodford (2003b, p. 6)

dissolves such that conditions necessary to determine this determinacy are identical to the conditions for nominal determinacy that would prevail were Condition (II) not present. This conceptual link between nominal determinacy in RBC models and both real and nominal determinacy in real-rate models with nominal frictions is more than just a curiosity. It provides for a simple means to establish nominal and real determinacy: one need only to examine the conditions for nominal determinacy in the corresponding flex-price, full-information equivalent, i.e. the "corresponding RBC model." This is generally a much simpler task and, in the class of models examined above, is independent of the parameters in the demand and supply equation.

4 Truncation and Specious Determinacy

Here, I shall examine the consequence of the truncation used by Trabandt (2007) and Andrés, López-Salido, and Nelson (2005) under a pure-inflation forecast targeting rule for the nominal interest rate. Their truncation eliminates the tail end of the distribution of lagged expectations in the sticky-information Phillips curve and, thereby, induces a form of permanent rigidity causing the model to violate the natural rate hypothesis. As such, it is not surprising that this will lead to a specious determinacy region for the otherwise indeterminate monetary policy rule considered in Section (2.3).

Equation (3) is replaced with

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + (1-\lambda)\sum_{i=0}^{I-1}\lambda^i E_{t-i-1}\left[\pi_t + \xi\Delta y_t\right] \tag{15}$$

for some $I < \infty$. And monetary policy is given by $R_t = \phi_\pi E_t\left[\pi_{t+1}\right],\ 0 < \phi_\pi < \infty$.

The system, augmented additionally by (1), can be written in matrix form as:

$$0 = \sum_{i=0}^{I} A_i E_{t-i}\left[Y_{t+1}\right] + \sum_{i=0}^{I} B_i E_{t-i}\left[Y_t\right] + \sum_{i=0}^{I} C_i E_{t-i}\left[Y_{t-1}\right]$$

where $Y_t = \begin{bmatrix}\pi_t & y_t & R_t\end{bmatrix}'$. This is the canonical form of Meyer-Gohde (2007) and the uniqueness of a bounded solution can be determined by examining the eigenvalues (Γ) of the matrix pencil

$$\begin{bmatrix}\sum_{i=0}^{I} C_i & \sum_{i=0}^{I} B_i \\ 0 & I\end{bmatrix} - \Gamma\begin{bmatrix}0 & -\sum_{i=0}^{I} A_i \\ I & 0\end{bmatrix} \tag{16}$$

There will, in general, be six generalized eigenvalues and associated eigenvectors. At most four finite solutions to the associated problem exist, three of which are equal to zero as can be seen in Appendix G. Uniqueness can be characterized by the uniqueness of exactly three Γ_i associated with the foregoing problem, where $|\Gamma_i| < 1$.

As shown in Appendix G, there will exist a unique bounded solution if the interest-rate rule satisfies the Taylor Principle and does not react "too strongly"to expected inflation:

$$1 < \phi_\pi < 1 + \frac{2\lambda^{I+1}}{a_1\xi\left(1+\lambda-2\lambda^{I+1}\right)} \tag{17}$$

A simple explanation for the (re)emergence of a determinacy region compared with Section (2.3) is: the truncation scheme (as the tail end of the distribution of lagged expectations *never* adjusts) causes the natural rate hypothesis to be violated.

The right-hand condition of (17) approaches 1 as I approaches infinity:

$$\lim_{I\to\infty}\left(1+\frac{2\lambda^{I+1}}{a_1\xi\left(1+\lambda-2\lambda^{I+1}\right)}\right)=1$$

Thus, as the truncated version approaches its intended form, the conditions for a unique equilibrium as determined by monetary policy are given by $1 < \phi_\pi < 1$; or, indeed, that no value for the response of the nominal interest rate to expected inflation is consistent with a unique equilibrium. Thereby replicating the result of Section (2.3). Thus, in addition to potentially spurious dynamics as shown by Meyer-Gohde (2007), truncation in a sticky-information model can also lead to specious determinacy. This deficiency is only fully overcome in the limit with this truncation scheme, as, for any finite I, the truncated model violates the natural rate hypothesis.

Alternatively, one could truncate to ensure that the natural rate hypothesis is still fulfilled, e.g.

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + \frac{(1-\lambda)}{(1-\lambda^I)}\sum_{i=0}^{I-1}\lambda^i E_{t-i-1}\left[\pi_t+\xi\Delta y_t\right] \tag{18}$$

for some $I < \infty$. As is confirmed in Appendix G, the equilibrium is necessarily indeterminate by repeating the foregoing exercise. The point here is that the weights in the probability distribution of information updating must sum to one ($\frac{(1-\lambda)}{(1-\lambda^I)}\sum_{i=0}^{I-1}\lambda^i = 1$).

5 Discussion

Woodford's (2003b, pp. 252-259) conclusion that an interest-rate setting rule which does not directly (i.e. with respect to inflation only) satisfy the Taylor principal need not be associated with indeterminacy does not carry over to the sticky-information model. Whereas the "near-canonical" sticky-price model exhibits a non-vertical long-run Phillips curve that allows for both "substitution" of output-gap targeting for inflation targeting and a pure inflation-forecast target to ensure uniqueness, the sticky-information model's vertical long-run Phillips curve precludes both policies. This yields stricter bounds (lower and, in the case of forward-looking policy rules, upper) on the coefficients of interest-rate setting rules for the sticky-information model. The bounds on interest-rate rules as derived here are juxtaposed in Table 1 with the bounds derived in Woodford (2003b, Ch. 4) for the sticky-price model.

These tighter bounds have one obvious advantage over the looser ones derived by Woodford (2003b): the are independent of model-specific parameter values. Regardless of the calibration, the Taylor principle is only satisfied if the direct (cumulative) reaction of the nominal interest rate to a (permanent) deviation in inflation is greater than one: $\phi_\pi > 1 - \phi_R$ is necessary under sticky information. This corresponds neatly to the conclusions of Taylor (2001) with regards to the pre-Volcker and the Volcker-Greenspan eras that the elasticity of the nominal interest rate with respect to inflation was less than one in the former implying an instability in inflation and greater than one in the latter implying stable inflation.

Perhaps discomforting is the conclusion that the upper bound, present under a forward-looking interest-rate setting rule, is significantly lower than would be concluded from a sticky-price model. Indeed, without interest-rate smoothing, the model is necessarily indeterminate. The estimated values from Clarida, Galí, and Gertler (2000), however, are such that the period 1982-96 would be associated with a unique equilibrium in the sticky-information model, as in a sticky-price model of Woodford (2003b, p. 260). The analysis here serves to strengthen and clarify conclusions drawn from empirical examinations of American monetary policy, as determinacy is a direct consequence of the form and parameter values of monetary policy with the coefficients in the IS and Phillips curves being irrelevant.

The upper bound on determinacy with forward-looking rules has been criticized by, e.g., McCallum (2001) as not being consistent with his MSV solution approach or with E-stability. As the sticky-information model has a state vector of infinite dimension, E-stability would seem difficult to ascertain. The MSV approach proposes to select a solution using the minimum number of state variables, but this is necessarily infinite in this case; thus, having no "advantage" over the infinite state solution in the form of an MA(∞) solution as per Muth (1961) and Taylor (1986). The MSV solution has, however, a second requirement: "the MSV solution involves a procedure that makes it unique by construction."(McCallum 1999, p. 627) The bubble-free solution in the model considered here would be the trivial solution zero for all variables, as no exogenous forces were postulated. That this solution is readily identifiable here is not very useful when confronted with a model containing such exogenous forces.

Woodford's (2003b, p. 258) Figure 4.1 shows, for his parameter set, that the upper bound in sticky-price models is so high that the discussio of, e.g., McCallum (2003) and Woodford (2003a) regarding this upper bound is almost purely academic. Yet, this discussion is of pressing importance in the sticky-information model, as its upper bound $1 + \phi_R$ is not far from the range of relevant parameter values for ϕ_π. Should the upper bound be found to not be "of dubious merit" (McCallum 2003, p. 1154), a pure inflation-

Table 1: Determinacy Regions: Comparison of Sticky Information and Sticky Prices

Interest Rate Rule	Sticky Prices	Sticky Information
	Non-Price-Related Feedback	
$R_t = 0$	$\emptyset$	$\emptyset$
$R_t = \phi_y y_t$	$\phi_y > \frac{\kappa}{1-\beta}$	$\emptyset$
	Inflation-Forecast Feedback	
$R_t = \phi_\pi E_t\left[\pi_{t+1}\right]$	$1 < \phi_\pi < 1 + 2\frac{1+\beta}{a_1 \kappa}$	$\emptyset$
$R_t = \phi_R R_{t-1} + \phi_\pi E_t\left[\pi_{t+1}\right] + \phi_y y_t$	$\phi_\pi > 1 - \phi_R - \frac{1-\beta}{\kappa}\phi_y$ $\phi_\pi < 1 + \phi_R + \frac{1+\beta}{\kappa}\left(\phi_y + 2\frac{1+\phi_R}{a_1}\right)$	$\phi_\pi > 1 - \phi_R$ $\phi_\pi < 1 + \phi_R$
	Contemporaneous Inflation Feedback	
$R_t = \phi_\pi \pi_t$	$\phi_\pi > 1$	$\phi_\pi > 1$
$R_t = \phi_R R_{t-1} + \phi_\pi \pi_t + \phi_y y_t$	$\phi_\pi > 1 - \phi_R - \frac{1-\beta}{\kappa}\phi_y$	$\phi_\pi > 1 - \phi_R$
	Price-Level Feedback	
$R_t = \phi_p p_t$	$\phi_p > 0$	$\phi_p > 0$
$R_t = \phi_p p_t + \phi_y y_t$	$\phi_p > 0$ or $\phi_p = 0,\ \phi_y > \frac{\kappa}{1-\beta}$	$\phi_p > 0$

forecast rule should be avoided rather generally by monetary policy and implemented only with great caution when it be imbued with some form of history dependence such as the interest-rate smoothing examined here. Even then, the lower bound derived here would still prescribe a more stringent interpretation of the Taylor Principle than in sticky-price models due to the irrelevance of output-gap targeting for determinacy.

6 Conclusion

Inflation targeting does work, but some conclusions reached thus far on the basis of a model that violates the natural rate hypothesis are premature. The generalization of the results for the prototypical sticky-information model presents some comfort: Lucas's (1972) natural rate hypothesis suffices to induce an inexorable link between nominal determinacy in the associated full-information, flexible-price counterpart and both real and nominal determinacy in the nominally rigid model, the former being a generally simple exercise to determine. This equivalence, furthermore, highlights the link in terms of determinacy that the natural rate hypothesis provides between standard RBC models and models with nominal rigidities.

As argued in Sargent (1973, p. 480), "right or wrong, the long-run natural rate [hypo]thesis has immediate relevance because it says something important about the impact of systematic and predictable changes on the economic system." One aspect of this relevance is that interest rate rules induce multiple equilibria in broader parameter spaces than previously thought.

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A Time-Variant Difference Equations

In this appendix, I shall present the necessary theorems to establish the existence of (locally) unique bounded solutions to the models presented in the text. The systems of difference equations that describe the solutions to the sticky-information Phillips curve are nonautonomous (time-variant) difference equations (Meyer-Gohde 2007, p. 4). Unfortunately, following Elaydi (2005, p. 191), "eigenvalues do not generally provide any information about the stability of nonautonomous difference equations." Yet, both Wang and Wen (2006) and Meyer-Gohde (2007) work with the eigenvalues of the system described by the limiting coefficients. While the former provides no support for the conjecture in the infinite case, the latter imposes the requirement that the matrices of coefficients converge element-wise, ruling out the periodic coefficients responsible for, e.g., Edwards and Ford's (2002, p. 286) counterexample.

This convergence will prove sufficient to determine the number of backward-looking variables, i.e. the familiar eigenvalue accounting of Blanchard and Kahn (1980). As in, e.g., Klein (2000), even if there are exactly as many backward-looking variables as given initial conditions, it need not hold that the given conditions can be "translated" to the backward-looking variables. With potentially singular time-variant coefficient matrices being involved, a definitive answer to the translatability cannot be given analytically. Thus, although regions of indeterminacy smaller than those in standard sticky-price models are analytically proven to exist, regions of determinacy are contingent upon translatability, confirmed for broad parameter spaces numerically.

A.1 Stability of Nearly Time-Invariant Systems

Here, I shall present necessary conditions for the stability (and therefore boundedness)[15] of nearly time-invariant linear systems by repeating Theorem 3-29 in Ludyk (1985, p. 61).

Consider the system given by $x_{k+1} = [C + D(k)]\,x_k$

Theorem A.1. *If*

1. *the time-invariant equivalent* $y_{k+1} = Cy_k$ *is stable*
2. *and* $\sum_{k=k_0}^{\infty} ||D(k)|| < \infty$[16]

then $x_{k+1} = [C + D(k)]\,x_k$ *is stable.*

Proof. See Ludyk (1985, pp. 61-62).[17]

[15] cf. Theorem 3-12 of Ludyk (1985, p. 39)

[16] Edwards and Ford (2002, p. 286) note that the matrix 2-norm provides the suitable measure for nonautonomous systems and following, e.g., Golub and van Loan (1989, p. 57), the 2-norm is bounded above by the Frobenius norm, establishing the appropriateness of the latter in the analysis that follows here.

[17] or Satz 4 and 5 "stability" [Stabilität] in Perron (1929, pp. 45-47), the full linearity of the models

A.2 Asymptotically Constant Systems of Difference Equations

Here, I shall present Theorem 8.25 of Elaydi (2005, pp. 360-361) as it applies to this paper.

Consider the system given by $x_{k+1} = A(k)x_k$, where $A(k) = C + D(k)$

Theorem A.2. *If*

1. *C contains as many distinct, non-zero, hyperbolic eigenvalues and as many linearly independent eigenvectors as its dimension*
2. *and* $\sum_{k=k_0}^{\infty} ||D(k)|| < \infty$

then

1. $x_{k+1} = A(k)x_k$ *can be written as* $\Theta_{k+1} = \Lambda\Theta_k + \tilde{D}(k)\Theta_k$ *where* $\Lambda = P^{-1}CP$, $\tilde{D}(k) = P^{-1}D(k)P$, *and* $\Theta_{k+1} = P^{-1}x_{k+1}$
2. *there exists a one-to-one correspondence between bounded solutions of* $\Theta_{k+1} = \Lambda\Theta_k + \tilde{D}(k)\Theta_k$ *and* $\Xi_{k+1} = \Lambda\Xi_k$ *given by*

$$\Theta_k = \Xi_k + \sum_{j=k_0}^{k-1} \Phi_1(k)\Phi^{-1}(j+1)\tilde{D}(j)\Theta_j - \sum_{j=k}^{\infty} \Phi_2(k)\Phi^{-1}(j+1)\tilde{D}(j)\Theta_j \tag{A-1}$$

where $\Phi(k)$ *is the solution map of* Ξ_k, $\Phi_2(k)$ *is the solution map of* Ξ_k *with regards to unstable eigenvalues, and* $\Phi_1(k)$ *is the solution map of* Ξ_k *with regards to stable eigenvalues.*

Proof. See Theorem 8.19 in Elaydi (2005, pp. 351-355).[18] □

A.3 Initial Conditions on the Stable Manifold for Asymptotically Constant Time-Variant 2x2 Systems of Difference Equations

Now, I shall apply Theorem A.2 to two-dimensional systems of time-variant systems of difference equations to determine initial conditions that ensure the system lie on the stable manifold.

considered here allows his conditions 𝔄 and ℭ to be fulfilled when the second assumption of the foregoing is fulfilled (Perron 1929, pp. 62-63).

[18] or Satz 8 and 9 “conditional stability” [bedingte Stabilität] in Perron (1929, pp. 49-53), the full linearity of the models considered here allows his conditions 𝔄 and ℭ to be fulfilled when the second assumption of the foregoing is fulfilled.

Proposition A.3. *If $x_{k+1} = A(k)x_k$ is a two-dimensional system of time-variant difference equations that satisfy the assumptions of Theorem A.2 and the first eigenvalue is inside and the second outside the unit circle, then the solution x_k is bounded if and only if*

$$[P^{-1}]_2 \left(I_{2x2} + \sum_{j=0}^{\infty} z_2^{-(j+1)} D(j) \prod_{i=0}^{j-1} A(j-i-1) \right) x_0 = 0 \tag{A-2}$$

where $|z_2| > 1$ is the second eigenvalue of C and P_2 is the second row of the matrix of (right) eigenvectors of C.

Proof. See the construction of the initial conditions below

Let $A(k)$ satisfy the assumptions of Theorem A.2. Then C can be diagonalized to yield the system $\Theta_{k+1} = \Lambda\Theta_k + \tilde{D}(k)\Theta_k$ with constant-coefficient diagonal counterpart $\Xi_{k+1} = \Lambda\Xi_k$. Assume, without loss of generality, the second eigenvalue of C (and thus the lower right entry in Λ) lies outside the unit circle. The only bounded solution to Ξ_k^2 (the second element of Ξ_k) is $\Xi_k^2 = 0 \forall k$, as Ξ_k^2 would grow without bound otherwise. Using (A-1),

$$\Theta_0 = \Xi_0 - \sum_{j=0}^{\infty} diag(z_1^0, z_2^0) diag(0, z_2^{-(j+1)}) \tilde{D}(j)\Theta_j \tag{A-3}$$

thus

$$\Theta_0 = \Xi_0 - \sum_{j=0}^{\infty} diag(0, z_2^{-(j+1)}) \tilde{D}(j)\Theta_j \tag{A-4}$$

as the solution of Ξ_k^1 (the first element of Ξ_k) was stable by construction, Ξ_k^1 is indeterminate with respect to boundedness, and, therefore, does not provide a restriction. Thus

$$\Theta_0 = \begin{bmatrix} c \\ 0 \end{bmatrix} - \sum_{j=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & z_2^{-(j+1)} \end{bmatrix} \tilde{D}(j)\Theta_j \tag{A-5}$$

for some initial condition c. With the definitions $P\Theta_k = x_k$ and $x_{k+1} = A(k)x_k$,

$$P^{-1}x_0 = \begin{bmatrix} c \\ 0 \end{bmatrix} - \sum_{j=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & z_2^{-(j+1)} \end{bmatrix} P^{-1} D(j) \prod_{i=0}^{j-1} A(j-i-1) x_0 \tag{A-6}$$

The second row of which is

$$[P^{-1}]_2 \left(I_{2x2} + \sum_{j=0}^{\infty} z_2^{-(j+1)} D(j) \prod_{i=0}^{j-1} A(j-i-1) \right) x_0 = 0 \tag{A-7}$$

B Model Appendix

In this appendix, I derive the nonautonomous system of difference equations that characterizes the response of (1) and (3) to a sunspot. Assume without loss of generality that a sunspot shock occurs at time t and denote with δ_i^x the response i periods after the sunspot of the variable x,[19]

Equation (1) can be rewritten as

$$\delta_i^y = \delta_{i+1}^y - a_1\delta_i^R + a_1\delta_{i+1}^\pi \tag{B-8}$$

As the system's response to a perturbation at time t from equilibrium is being examined, the response of all variables and all expectations dated before t are equal to zero (i.e. the model is starting from the equilibrium steady-state solution).[20] Equation (3) can thus be rewritten as

$$\delta_i^\pi = \frac{1-\lambda}{\lambda}\xi\delta_i^y + (1-\lambda)\sum_{j=0}^{i-1}\lambda^j\left[\delta_i^\pi + \xi\left(\delta_i^y - \delta_{i-1}^y\right)\right]$$

collecting terms

$$\lambda^i\delta_i^\pi = \left(\frac{1-\lambda}{\lambda} + \left(1-\lambda^i\right)\right)\xi\delta_i^y - \xi\left(1-\lambda^i\right)\delta_{i-1}^y$$

or

$$\lambda^{i+1}\delta_i^\pi = \left(1-\lambda^{i+1}\right))\xi\delta_i^y - \xi\lambda\left(1-\lambda^i\right)\delta_{i-1}^y \tag{B-9}$$

C Proofs from Section 2.2

C.1 Proof of Lemma 2.1

Proof. With the nominal interest rate being a feedback rule dependant on the output gap, the system defined by (B-8) and (B-9) is extended by the equation $\delta_i^R = \phi_y\delta_i^y$ to yield the dynamical system

$$\begin{aligned}
\lambda^{i+1}\delta_i^\pi &= \left(1-\lambda^{i+1}\right)\xi\delta_i^y - \xi\lambda\left(1-\lambda^i\right)\delta_{i-1}^y \\
\delta_i^y &= \delta_{i+1}^y - a_1\delta_i^R + a_1\delta_{i+1}^\pi \\
\delta_i^R &= \phi_y\delta_i^y
\end{aligned}$$

[19] cf. Taylor's (1986) method for solving for the infinite moving average coefficients of endogenous variables

[20] cf. Mankiw and Reis's (2002) Appendix

(C-10) $i = 0, 1, \ldots$

Substituting the third into the second and the second, then, into the first,

$$\lambda \delta_0^{\pi} = (1-\lambda)\xi \delta_0^y$$

$$\left[\left(1-\lambda^{i+2}\right)\xi + \lambda^{i+2}\frac{1}{a_1}\right]\delta_{i+1}^y = \left[\left(1-\lambda^{i+1}\right)\lambda\xi + \lambda^{i+2}\frac{1+a_1\phi_y}{a_1}\right]\delta_i^y$$

(C-11) $i = 0, 1, 2, \ldots$

The first equation places no restriction on the recursion described by the second (determining only the initial response of the inflation rate given the initial response of the output gap), and, thus, the dynamical system is given by (5)

C.2 Proof of Proposition 2.2

Proof. The difference equation in (5) can be inverted (a_1 and ξ were assumed to be both positive and finite and $0 < \lambda < 1$) and rewritten as

$$\delta_{i+1}^y = \left(\lambda + \lambda^{i+2}\frac{(1-\lambda)\left(\frac{1}{a_1}-\xi\right)+\phi_y}{\xi+\lambda^{i+2}\left(\frac{1}{a_1}-\xi\right)}\right)\delta_i^y$$

(C-12) $i = 0, 1, 2, \ldots$

Defining $A = \lambda$ and $D(k) = \lambda^{i+2}\frac{(1-\lambda)\left(\frac{1}{a_1}-\xi\right)+\phi_y}{\xi+\lambda^{i+2}\left(\frac{1}{a_1}-\xi\right)}$, it is trivial ($0 < \lambda < 1$) to see that A is stable. The second assumption of Theorem (A.1) requires

$$\sum_{i=0}^{\infty}||D(i)|| < \infty$$

$$\sum_{i=0}^{\infty}|\lambda^{i+2}\frac{(1-\lambda)\left(\frac{1}{a_1}-\xi\right)+\phi_y}{\xi+\lambda^{i+2}\left(\frac{1}{a_1}-\xi\right)}| \leq |\sum_{i=0}^{\infty}\frac{\lambda^{i+2}}{\xi+\lambda^{i+2}\left(\frac{1}{a_1}-\xi\right)}||\left((1-\lambda)\left(\frac{1}{a_1}-\xi\right)+\phi_y\right)|$$

$$= |\left((1-\lambda)\left(\frac{1}{a_1}-\xi\right)+\phi_y\right)||\sum_{i=0}^{\infty}\frac{\lambda^{i+2}}{\xi+\lambda^{i+2}\left(\frac{1}{a_1}-\xi\right)}|$$

(C-13)
$$= |\left((1-\lambda)\left(\frac{1}{a_1}-\xi\right)+\phi_y\right)|M < \infty$$

where the last line comes from the absolute convergence of the infinite series which follows from the ratio test.

Thus, both the assumptions of Theorem (A.1) are fulfilled. The recursion is stable and, therefore, any finite sunspot deviation of the output gap is consistent with a bounded solution of the recursion: the equilibrium is indeterminate.

D Proofs from Section 2.3

D.1 Proof of Lemma 2.3

Proof. With the nominal interest rate now feeding back on its own lagged value, tomorrow's expected inflation, and the current output gap, the system defined by (B-8) and (B-9) is extended by the equation $\delta_i^R = \phi_\pi \delta_{i+1}^\pi$ to yield the dynamical system

$$\delta_i^R = \phi_R \delta_{i-1}^R + \phi_\pi \delta_{i+1}^\pi + \phi_y \delta_i^y \tag{D-14}$$

which defines the dynamical system

$$\begin{aligned}
\lambda^{i+1}\delta_i^\pi &= \left(1-\lambda^{i+1}\right)\xi\delta_i^y - \xi\lambda\left(1-\lambda^i\right)\delta_{i-1}^y \\
\delta_i^y &= \delta_{i+1}^y - a_1\delta_i^R + a_1\delta_{i+1}^\pi \\
\delta_i^R &= \phi_R\delta_{i-1}^R + \phi_\pi\delta_{i+1}^\pi + \phi_y\delta_i^y \\
i &= 0,1,2,\ldots \\
\delta_{-1}^R &= 0
\end{aligned}$$

Inserting the third into the first and the third into the second gives

$$\begin{aligned}
\left(1-\lambda^{i+2}\right)\xi\delta_{i+1}^y - \frac{\lambda^{i+2}}{\phi_\pi}\delta_i^R &= \left[\xi\lambda\left(1-\lambda^{i+1}\right) - \lambda^{i+2}\frac{\phi_y}{\phi_\pi}\right]\delta_i^y - \lambda^{i+2}\frac{\phi_R}{\phi_\pi}\delta_{i-1}^R \\
\delta_{i+1}^y - a_1\left(1-\frac{1}{\phi_\pi}\right)\delta_i^R &= \left(1+a_1\frac{\phi_y}{\phi_\pi}\right)\delta_i^y + a_1\frac{\phi_R}{\phi_\pi}\delta_{i-1}^R \\
i &= 0,1,2,\ldots \\
\delta_{-1}^R &= 0 \\
\lambda\delta_0^\pi &= (1-\lambda)\xi\delta_0^y
\end{aligned} \tag{D-15}$$

As in previous sections, the last equation places no restriction on the recursion; thusly, the dynamical system can be written in matrix form as (8).

D.2 Proof of Proposition 2.4

It is instructive to begin with the special case of pure inflation-forecast targeting, as the system can be reduced to a scalar system and direct methods can be applied.

Special Case: $\phi_y = \phi_R = 0$

With pure inflation-forecast targeting, the system can be rewritten as

$$\lambda\delta_0^\pi = (1-\lambda)\xi\delta_0^y$$

$$\begin{aligned}\left[\left(1-\lambda^{i+2}\right)\xi a_1(\phi_\pi-1)-\lambda^{i+2}\right]\delta^y_{i+1} &= \left[\left(1-\lambda^{i+1}\right)\lambda\xi a_1(\phi_\pi-1)-\lambda^{i+2}\right]\delta^y_i \\ i &= 0,1,2,\ldots, \end{aligned} \tag{D-16}$$

As in Section (C.1), the first equation places no restriction on the recursion described by the second (determining only the initial response of the inflation rate given the initial response of the output gap), and, thus, the dynamical system is given by

$$\begin{gathered}\left[\left(1-\lambda^{i+2}\right)\xi a_1(\phi_\pi-1)-\lambda^{i+2}\right]\delta^y_{i+1} = \left[\left(1-\lambda^{i+1}\right)\lambda\xi a_1(\phi_\pi-1)-\lambda^{i+2}\right]\delta^y_i \\ i = 0,1,2,\ldots, \end{gathered} \tag{D-17}$$

Proof of Special Case: $\phi_y = \phi_R = 0$

Proof. The difference equation in (D-17) can be inverted to yield

$$\delta^y_{i+1} = a(i)\delta^y_i \tag{D-18}$$

$$a(i) = \frac{\lambda\left[\left(1-\lambda^{i+1}\right)\xi a_1(\phi_\pi-1)-\lambda^{i+1}\right]}{\left[(1-\lambda^{i+2})\xi a_1(\phi_\pi-1)-\lambda^{i+2}\right]} \tag{D-19}$$

so long as $\phi_\pi \neq 1+\frac{\lambda^{i+2}}{\left(1-\lambda^{i+2}\right)\xi a_1}$ for some i.

The solution map is given by

$$\delta^y_{i+1} = \left(\prod_{n=0}^{i} a(n)\right)\delta^y_0 \tag{D-20}$$

Simple calculations reveal that

$$\prod_{n=l}^{k} a(n) = \lambda^{k-l}\frac{\lambda\left[\left(1-\lambda^{l+1}\right)\xi a_1(\phi_\pi-1)-\lambda^{l+1}\right]}{\left[\left(1-\lambda^{k+2}\right)\xi a_1(\phi_\pi-1)-\lambda^{k+2}\right]} \tag{D-21}$$

and thus

$$\begin{aligned}\delta^y_{i+1} &= \prod_{j=0}^{i}\alpha(j)\delta^y_0 \\ &= \lambda^i\frac{\lambda\left[(1-\lambda)\xi a_1(\phi_\pi-1)-\lambda\right]}{\left[(1-\lambda^{i+2})\xi a_1(\phi_\pi-1)-\lambda^{i+2}\right]}\delta^y_0\end{aligned} \tag{D-22}$$

the limit of which as $i \to \infty$ is zero for any finite $\delta^{j,y}_0$ as λ is necessarily within the unit circle. Thus, the stability of the difference equation alone satisfies the boundedness condition and no other condition is present to pin down the initial conditions of the recursion necessary for a unique solution.

Should $\phi_\pi = 1 + \frac{\lambda^{i+2}}{(1-\lambda^{i+2})\xi a_1}$ for some i (say i_τ), then the system does not fulfill the second assumption of Theorem (A.2) for $k_0 = 0$. This is not, however, problematic, but does require a closer inspection. For all $i > i_\tau$

$$\begin{aligned} \delta^y_{i+1} &= (\lambda + b(i))\,\delta^y_i \\ b(i) &= -\lambda^{i+2}\frac{(1-\lambda)(1+\xi a_1(\phi_\pi - 1))}{\xi a_1(\phi_\pi - 1) - \lambda^{i+2}(1+\xi a_1(\phi_\pi - 1))} \end{aligned} \tag{D-23}$$

but $b(i_\tau + 1) = -\lambda$ and, thus, $\delta^y_{i_\tau+2} = 0$. As $(\lambda + b(i)) \neq 0$, $\forall i > i_\tau + 1$, the foregoing requires $\delta^y_i = 0$, $\forall i \geq i_\tau + 2$. From (D-17), if $\phi_\pi = 1 + \frac{\lambda^{i_\tau+2}}{(1-\lambda^{i_\tau+2})\xi a_1}$, $\delta^y_{i_\tau} = 0$. Whence $\delta^y_i = 0$, $i = 0, 1, \ldots i_\tau$. Thus, the bounded solution is given by $\delta^y_i = 0$ for all $i \geq 0$ except $i_\tau + 1$ for any $|\delta^y_{i_\tau+1}| < \infty$.

This has served merely to shift the initial point of the recursion and the conclusion of indeterminacy still holds. For the foregoing cases (and indeed whenever $\phi_\pi \leq 1 + \frac{\lambda^2}{(1-\lambda^2)\xi a_1}$) the difference equation falls into the category "stable in the further sense" (Perron 1929, pp. 41-42)[21], as the replacement of $i = 0$ with $i \geq i_\tau + 1$ as the initial point of the recursion produces a recursion which is stable in the sense of Theorem (A.1) or Perron's (1929, pp. 45-46) Satz 4.

Thus, there is necessarily indeterminacy in the case of pure inflation-forecast targeting.

Proof of General Case: $0 \leq \phi_y < \infty$, $0 \leq \phi_R < 1$

Proof. The system of difference equations in (8) can be inverted to yield

$$\begin{aligned} \begin{bmatrix} \delta^y_{i+1} \\ \delta^R_i \end{bmatrix} &= (C + D(i))\begin{bmatrix} \delta^y_i \\ \delta^R_{i-1} \end{bmatrix} \\ C &= \begin{bmatrix} \lambda & 0 \\ -\frac{a_1\xi\phi_y + (1-\lambda)\xi\phi_\pi}{a_1\xi(\phi_\pi - 1)} & -\frac{\phi_R}{\phi_\pi - 1} \end{bmatrix} \\ D(i) &= \frac{\lambda^{i+2}}{\lambda^{i+2}(1 + a_1\xi(\phi_\pi - 1)) - a_1\xi(\phi_\pi - 1)} \\ &\quad * \begin{bmatrix} (1-\lambda)(1+\xi a_1(\phi_\pi - 1)) + a_1\frac{\phi_y}{\phi_\pi} & \phi_R \\ \phi_y\frac{\phi_\pi}{\phi_\pi - 1} + \xi(1-\lambda)\left(1 + \frac{1}{a_1(\phi_\pi - 1)}\right) & \phi_R\frac{\phi_\pi}{\phi_\pi - 1} \end{bmatrix} \end{aligned} \tag{D-24}$$

so long as $\phi_\pi \neq 1 + \frac{\lambda^{i+2}}{(1-\lambda^{i+2})\xi a_1}$, $\forall i \geq 0$.

[21] "*im weiteren Sinne* stabil"

Using the Frobenius norm $||D(i)|| = \nu(i)M$, where $\nu(i) = |\frac{\lambda^{i+2}}{\lambda^{i+2}(1+a_1\xi(\phi_\pi-1))-a_1\xi(\phi_\pi-1)}|$. Applying the ratio test to $\nu(i)$ delivers $\sum_{i=0}^{\infty}\nu(i) < \infty$ and, thus, $D(i)$ satisfies the second assumption of Theorems A.1 and A.2.

Examining the eigenvalues of C, $z_1 = \lambda$, $z_2 = \frac{\phi_R}{1-\phi_\pi}$, the first of which is necessarily stable. Thus, the system will be inherently stable if $|\frac{\phi_R}{1-\phi_\pi}| < 1$ and will allow for a unique stable solution $\delta_i^y, \delta_i^R, \delta_i^\pi = 0$ following from the initial condition $\delta_{-1}^R = 0$ and Proposition A.3 if $|\frac{\phi_R}{1-\phi_\pi}| > 1$. Thus, in the former case, the boundedness condition will be insufficient to rule out sunspot equilibria (indeterminacy), whereas, in the latter, only the trivial, sunspot-free equilibrium (determinacy) will satisfy the boundedness condition.

Should $\phi_\pi = 1 + \frac{\lambda^{i+2}}{(1-\lambda^{i+2})\xi a_1}$ for some i (say i_τ), then the system does not fulfill the second assumption of Theorem (A.2) for $k_0 = 0$. This is analogous to the situation in the proof of the special case. The singularity of the coefficient matrix combined with the initial condition $\delta_{-1}^R = 0$ implies that $\delta_{i-1}^R = \delta_i^y = 0$, $i \leq i_\tau$. Using either of the two equations in the recursion then yields a new initial condition $\left(1-\lambda^{i_\tau+2}\right)\xi\delta_{i_\tau+1}^y = \frac{\lambda^{i_\tau+2}}{\phi_\pi}\delta_{i_\tau}^R$, which then yields a non-singular recursion for $i = i_\tau+1, i_\tau+2, \ldots$, with the same stability characteristics as in the recursion without the singularity. The linear relationship between $\delta_{i_\tau}^R$ and $\delta_{i_\tau+1}^y$ is supplemented by the condition supplied by Proposition A.3. Both conditions run through the origin and, thus, the unique solution is $\delta_{i_\tau+1}^y = \delta_{i_\tau}^R = 0$, unless both conditions define the same linear relationship. As the condition provided by Proposition A.3 cannot, in general, be solved for analytically, numerical investigations must be relied upon. Numerical calculations confirm, for a wide range of parameter values, that the supplemental initial condition provides a linearly independent relationship between $\delta_{i_\tau+1}^y$ and $\delta_{i_\tau}^R$. Figure 1 plots the initial condition $\left(1-\lambda^{i_\tau+2}\right)\xi\delta_{i_\tau+1}^y = \frac{\lambda^{i_\tau+2}}{\phi_\pi}\delta_{i_\tau}^R$ and the initial condition from Proposition A.3 for a wide parameter range normalized for the former to have a slope of one. Therefore, the system is stable (or "stable in the further sense" (Perron 1929, pp. 41-42)) when $|\frac{\phi_R}{1-\phi_\pi}| < 1$ and, thusly, indeterminate and the system is saddle-path stable (or "conditionally stable in the further sense" (Perron 1929, pp. 41-42,53)) when $|\frac{\phi_R}{1-\phi_\pi}| > 1$

E Proofs from Section 2.4

E.1 Proof of Lemma 2.5

Proof. The system defined by (B-8) and (B-9) is now closed with an extended contemporaneous-inflation targeting rule of the form

(E-25) $$\delta_i^R = \phi_R\delta_{i-1}^R + \phi_\pi\delta_i^\pi + \phi_y\delta_i^y$$

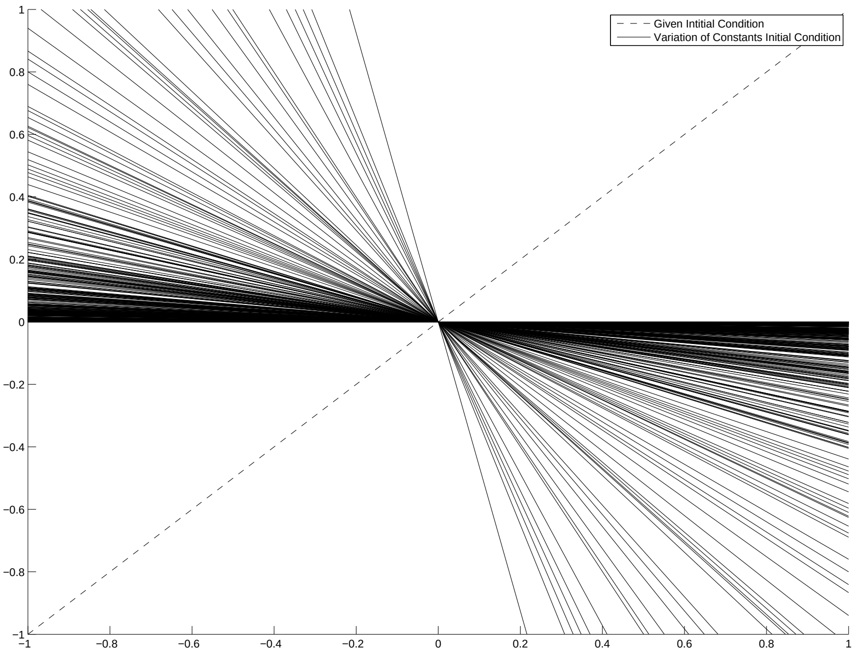

Figure 1: Linear Independence of Initial Conditions for Proposition D.2
$\xi, a_1 \in \{0.01 : 0.1 : 2, 1 : 1 : 10\}$, $i_\tau \in \{0, 1, 5, 10, 20\}$, $\phi_y \in \{0, 0.5, 1, 2\}$, $\phi_R \in \{0.25, 0.5, 0.9\}$, $\lambda \in \{0.25, 0.5, 0.75\}$

The dynamical system is given by

$$\begin{aligned}
\lambda^{i+1}\delta_i^\pi &= \left(1-\lambda^{i+1}\right)\xi\delta_i^y - \xi\lambda\left(1-\lambda^i\right)\delta_{i-1}^y \\
\delta_i^y &= \delta_{i+1}^y - a_1\delta_i^R + a_1\delta_{i+1}^\pi \\
\delta_i^R &= \phi_R\delta_{i-1}^R + \phi_\pi\delta_i^\pi + \phi_y\delta_i^y \\
i &= 0,1,2,\dots \\
\delta_{-1}^R &= 0
\end{aligned} \tag{E-26}$$

Combining inserting the third into the first and into the second of the foregoing yields

$$\begin{aligned}
\left[\left(1-\lambda^{i+2}\right)\xi + \lambda^{i+2}\frac{\phi_y}{\phi_\pi}\right]\delta_{i+1}^y - \frac{\lambda^{i+2}}{\phi_\pi}\delta_{i+1}^R &= \left(1-\lambda^{i+1}\right)\lambda\xi\delta_i^y - \lambda^{i+2}\frac{\phi_R}{\phi_\pi}\delta_i^R \\
\left(1-a_1\frac{\phi_y}{\phi_{pi}}\right)\delta_{i+1}^y + \frac{a_1}{\phi_\pi}\delta_{i+1}^R &= \delta_i^y + a_1\left(1+\frac{\phi_R}{\phi_\pi}\right)\delta_i^R \\
i &= 0,1,2,\dots,
\end{aligned}$$

$$
\begin{aligned}
\lambda\delta_0^R &= ((1-\lambda)\xi\phi_\pi + \lambda\phi_y)\,\delta_0^y \\
\delta_{-1}^R &= 0
\end{aligned} \tag{E-27}
$$

the last of which is irrelevant for the stability considerations of the recursion, yielding (10).

E.2 Proof of Proposition 2.6

Proof. The system of difference equations in (10)can be inverted to yield

$$
\begin{aligned}
\begin{bmatrix}\delta_{i+1}^y \\ \delta_{i+1}^R\end{bmatrix} &= (C+D(i))\begin{bmatrix}\delta_i^y \\ \delta_i^R\end{bmatrix} \\
C &= \begin{bmatrix} \lambda & 0 \\ \frac{\phi_\pi}{a_1}\left(a_1\frac{\phi_y}{\phi_\pi}+(1-\lambda)\right) & \phi_\pi\left(1+\frac{\phi_R}{\phi_\pi}\right)\end{bmatrix} \\
D(i) &= \frac{\lambda^{i+2}}{\lambda^{i+2}(1-a_1\xi)+a_1\xi}\begin{bmatrix}(1-\lambda)(1-a_1\xi) & a_1 \\ -\phi_\pi\frac{(1-a_1\xi)(1-\lambda)}{a_1} & -\phi_\pi\left(1-a_1\frac{\phi_y}{\phi_\pi}\right)\end{bmatrix}
\end{aligned} \tag{E-28}
$$

as long as $\frac{1}{\phi_\pi}\left(\lambda^{i+2}+a_1\xi\left(1-\lambda^{i+2}\right)\right)\neq 0$, which, as both a_1 and ξ are finite and positive and $0<\lambda<1$, will be well defined and will hold for $0<\phi_\pi<\infty$.

Using the Frobenius norm, $||D(i)|| = \nu(i)M$, where $\nu(i) = |\frac{\lambda^{i+2}}{\lambda^{i+2}(1-a_1\xi)+a_1\xi}|$. Applying the ratio test to $\nu(i)$ confirms that $\sum_{i=0}^{\infty}\nu(i)$ is finite and, thus, $D(i)$ satisfies the second assumption of Theorems A.1 and A.2.

The eigenvalues of C, z_1 and z_2, are given by $z_1=\lambda$ and $z_2=\phi_\pi\left(1+\frac{\phi_R}{\phi_\pi}\right)=\phi_\pi+\phi_R$. The first of which is necessarily inside the unit circle. Thus, if $|\phi_\pi+\phi_R|<1$, the system is inherently stable and, therefore, indeterminate following Theorem A.1.

If $|\phi_\pi|>1$, Proposition A.3 will provide an initial condition to ensure that the recursion remains bounded. Unfortunately, the formula cannot be evaluated analytically. Numerical calculations reveal, however, that the initial conditions require a negative relationship between δ_0^y and δ_0^R running through the origin. This, in conjunction with the additional initial condition $\frac{\delta_0^R}{\phi_\pi}=\left(\frac{1-\lambda}{\lambda}\xi+\frac{\phi_y}{\phi_\pi}\right)\delta_0^y$ which posits a positive relationship between the two variables running through the origin, has as its unique bounded solution $\delta_i^y=\delta_i^R=0$, $\forall i\geq 0$; thereby ruling out sunspot reactions of endogenous variables. Figure 2 plots the two initial conditions for a wide range of parameter values normalized for the former to have a slope of one. This demonstrates that the two conditions are, at very least, not generally linearly dependant.

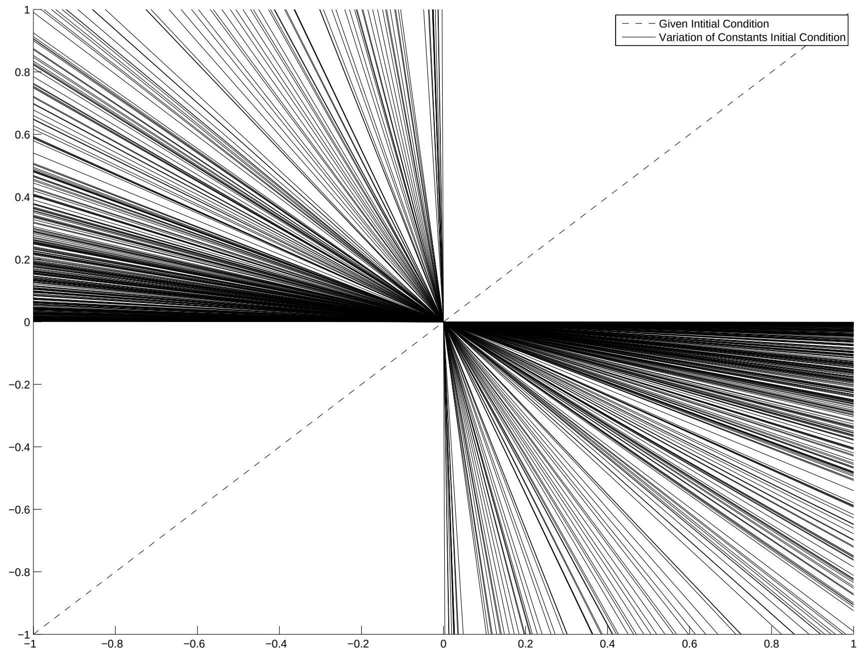

Figure 2: Linear Independence of Initial Conditions for Proposition E.2
$\xi, a_1 \in \{0.01:0.1:1, 2:1:10\}, \phi_\pi \in \{1.001:0.1:2, 4:2:10\}, \phi_y \in \{0, 0.5, 1, 2\}, \phi_R \in \{0.25, 0.5, 0.9\}, \lambda \in \{0.25, 0.5, 0.75\}$

F Proofs from Section 2.5

F.1 Proof of Lemma 2.7

Proof. The system defined by (B-8) and (B-9) is now closed by a monetary policy described by

$$\delta_i^R = \phi_p \delta_i^p + \phi_y \delta_i^y \tag{F-29}$$

Combining the foregoing with equations (B-9) and (B-8), the dynamical system is given by

$$\begin{aligned} \lambda^{i+1}\left(\delta_i^p - \delta_{i-1}^p\right) &= \left(1-\lambda^{i+1}\right)\xi\delta_i^y - \xi\lambda\left(1-\lambda^i\right)\delta_{i-1}^y \\ \delta_i^y &= \delta_{i+1}^y - a_1\delta_i^R + a_1\left(\delta_{i+1}^p - \delta_i^p\right) \\ \delta_i^R &= \phi_p\delta_i^p + \phi_y\delta_i^y \\ i &= 0, 1, \ldots \end{aligned}$$

(F-30) $$\delta^y_{-1} = \delta^p_{-1} = 0$$

As before, the final restriction can be omitted from the recursion. Substituting the third into the second and the third into the first (and lagging forward once) yields (12).

F.2 Proof of Proposition 2.8

Proof. The system of difference equations in (12) can be inverted to yield

$$\begin{aligned}
\begin{bmatrix}\delta^y_{i+1}\\ \delta^R_{i+1}\end{bmatrix} &= (C+D(i))\begin{bmatrix}\delta^y_{i}\\ \delta^R_{i}\end{bmatrix}\\
C &= \begin{bmatrix}\lambda & 0\\ (1-\lambda)\left(\frac{\phi_p}{a_1}-\phi_y\right) & 1+\phi_p\end{bmatrix}\\
D(i) &= \frac{\lambda^{i+2}}{\lambda^{i+2}(1-a_1\xi)+a_1\xi}\begin{bmatrix}(1-\lambda)(1-a_1\xi) & a_1\\ -\frac{(1-a_1\xi)\left(\phi_p-a_1\phi_y\right)(1-\lambda)}{a_1} & -(\phi_p-a_1\phi_y)\end{bmatrix}
\end{aligned} \tag{F-31}$$

as long as $\frac{1}{\phi_p}\left(\lambda^{i+2}+a_1\xi\left(1-\lambda^{i+2}\right)\right)\neq 0$, which, as both a_1 and ξ are finite and positive and $0<\lambda<1$, will be well defined and will hold for $0<\phi_p<\infty$.

Using the Frobenius norm, $||D(i)|| = \nu(i)M$, where $\nu(i) = |\frac{\lambda^{i+2}}{\lambda^{i+2}(1-a_1\xi)+a_1\xi}|$. Applying the ratio test to $\nu(i)$ confirms that $\sum_{i=0}^{\infty}\nu(i)$ is finite and, thus, that $D(i)$ satisfies the second assumption of Theorems A.1 and A.2.

Examining the eigenvalues of C, z_1 and z_2, it is trivial to see that $z_1=\lambda$, $z_2=1+\phi_p$. Thus, according to Theorem A.2, the system is stable (and, therefore, indeterminate) if both eigenvalues are within the unit circle. As λ necessarily is, then, if $|1+\phi_p|<1$, the system is inherently stable.

If $|1+\phi_p|>1$, Proposition A.3 will provide an initial condition to ensure that the recursion remains bounded. Unfortunately, the formula cannot be evaluated analytically. Numerical calculations reveal, however, that the initial conditions require a relationship between δ^y_0 and δ^R_0 running through the origin. This, in conjunction with the additional initial condition $\frac{\delta^R_0}{\phi_p} = \left(\frac{1-\lambda}{\lambda}\xi+\frac{\phi_y}{\phi_p}\right)\delta^y_0$ which posits a positive relationship (for $\phi_p>0$) between the two variables running through the origin, has as its unique bounded solution $\delta^y_i=\delta^R_i=0$, $\forall i\geq 0$; thereby ruling out sunspot reactions of endogenous variables. Figure 3 plots the two initial conditions for a wide range of parameter values normalized for the former to have a slope of one. Note, that although the addition of output-gap targeting can bring the additional initial condition arbitrarily close to the original condition, the two never cross. This demonstrates that the two conditions are, at very least, not generally linearly dependant.

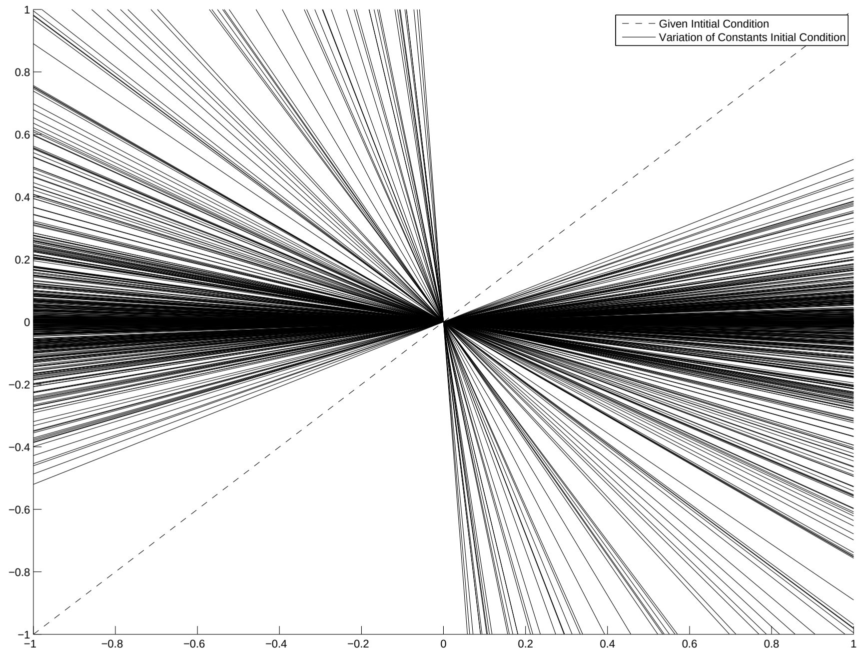

Figure 3: Linear Independence of Initial Conditions for Proposition F.2
$\xi, a_1 \in \{0.01 : 0.1 : 1, 2 : 1 : 10\}, \phi_p \in \{0.001 : 0.1 : 2, 4 : 2 : 10\}, \phi_y \in \{0, 0.5, 1, 2\}, \lambda \in \{0.25, 0.5, 0.75\}$

G Proofs from Section 4

The pencil (16) can be written as

$$\text{(G-32)} \quad \begin{bmatrix} 0 & 0 & -\xi\left(\lambda-\lambda^{I+1}\right) & -\lambda^{I+1} & 0 & \xi\left(1-\lambda^{I+1}\right) \\ 0 & 0 & 0 & a_1\Gamma & -a_1 & \Gamma-1 \\ 0 & 0 & 0 & \phi_\pi\Gamma & 1 & 0 \\ -\Gamma & 0 & 0 & 1 & 0 & 0 \\ 0 & -\Gamma & 0 & 0 & 1 & 0 \\ 0 & 0 & -\Gamma & 0 & 0 & 1 \end{bmatrix}$$

the determinate of which gives the polynomial

$$\text{(G-33)} \quad \left[\left(\lambda^{I+1}-\left(\lambda-\lambda^{I+1}\right)a_1\xi\left(\phi_\pi-1\right)\right)-\Gamma\left(\lambda^{I+1}-\left(1-\lambda^{I+1}\right)a_1\xi\left(\phi_\pi-1\right)\right)\right]\Gamma^3=0$$

The two “missing” eigenvalues are called “infinite” following Klein’s (2000) abuse of language. Of the remaining four eigenvalues, it is trivial to see that three are equal to zero. Thus, determinacy will rest upon the final eigenvalue being outside the unit circle:

(G-34)
$$\Gamma = \frac{\lambda^{I+1} - \left(\lambda - \lambda^{I+1}\right) a_1 \xi \left(\phi_\pi - 1\right)}{\lambda^{I+1} - \left(1 - \lambda^{I+1}\right) a_1 \xi \left(\phi_\pi - 1\right)}$$

If (15) is replaced with the truncation (18), the pencil (16) can be rewritten as

(G-35)
$$\begin{bmatrix} 0 & 0 & -\lambda & 0 & 0 & 1 \\ 0 & 0 & 0 & a_1\Gamma & -a_1 & \Gamma - 1 \\ 0 & 0 & 0 & \phi_\pi\Gamma & 1 & 0 \\ -\Gamma & 0 & 0 & 1 & 0 & 0 \\ 0 & -\Gamma & 0 & 0 & 1 & 0 \\ 0 & 0 & -\Gamma & 0 & 0 & 1 \end{bmatrix}$$

the determinate of which gives the polynomial

(G-36)
$$\left[\lambda \left(\phi_\pi - 1\right) - \Gamma \left(\phi_\pi - 1\right)\right] a_1 \Gamma^3 = 0$$

Excepting for the knife-edge case $\phi_\pi = 1$ following Woodford (2003b), the two "missing" eigenvalues are infinite, three eigenvalues are trivially zero, and the remaining eigenvalue is $\Gamma = \lambda$ which is necessarily inside the unit circle.

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