SFB 649 Discussion Paper 2012-034

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

http://sfb649.wiwi.hu-berlin.de
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
Spandauer Straße 1, D-10178 Berlin

SFB 649 ECONOMIC RISK BERLIN

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May 11, 2012

Abstract

We introduce the notion of realized copula. Based on assumptions of the marginal distributions of daily stock returns and a copula family, realized copula is defined as the copula structure materialized in realized covariance estimated from within-day high-frequency data. Copula parameters are estimated in a method-of-moments type of fashion through Hoeffding's lemma. Applying this procedure day by day gives rise to a time series of copula parameters that is suitably approximated by an autoregressive time series model. This allows us to capture time-varying dependency in our framework. Studying a portfolio risk-management application, we find that time-varying realized copula is superior to standard benchmark models in the literature.

Keywords: realized variance, realized covariance, realized copula, multivariate dependence
JEL classification: G12, C13, C14, C22, C50

*Financial support from the Deutsche Forschungsgemeinschaft via SFB 649 "Ökonomisches Risiko", Humboldt-Universität zu Berlin, is gratefully acknowledged. The authors thank Francesco Audrino, Daniel Buncic, Umberto Cherubini, Roxana Halbleib, Wolfgang Härdle, Nikolaus Hautsch, Helmut Herwartz, Sabrina Mulinacci, Yarema Okhrin, Mark Podolskij and participants at the Statistics Seminar at the University of Augsburg, at ISI 2011 (Dublin), at the Statistische Woche 2011 (Leipzig), at the Risk Calibration Workshop 2011 at the Academia Sinica (Taipei), at the 14th Conference of the ASMDA International Society (Rome) and at the 3rd XMU-HUB workshop on 'Nonparametric and Nonstationary Econometrics' (Xiamen) for valuable comments and discussions. We remain responsible for errors and omissions.

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1 Introduction

Realized variance (RV) and realized covariance (RC) estimated from high-frequency intraday data have proved to be accurate ex-post measures for conditional variance and conditional covariance of daily returns. Being nonparametric in nature, RV and RC permit the econometrician to obtain proxies for financial (co)volatility without having to specify a priori an explicit (and potentially misspecified) model. An inherently latent variable, such as volatility, can thus be treated as an observable (Andersen, Bollerslev, Diebold and Ebens; 2001; Andersen, Bollerslev, Diebold and Labys; 2001). These insights spurred intensive research in the field and lead to widespread use of measures of RV and RC in numerous applications in finance, such asset pricing, portfolio optimization, risk management, and volatility forecasting.

The present article continues this agenda. We estimate RC matrices from high-frequency intraday data and take them as valid ex-post proxies for daily conditional covariance. Unlike previous studies, we complement these estimates by making assumptions on the marginal distributions of daily returns and the copula associated with their joint multivariate distribution. Based on these assumptions, we estimate the copula shape parameters by means of the covariance moment condition provided by Hoeffding's lemma. The procedure yields daily estimates of copula shape parameters as materialized in daily RC. We therefore call it *realized copula (RCop)* . The resulting time series of RC-implied copula shape parameters is subsequently modeled by standard time series techniques thereby allowing the dependence structure to be time-varying with the business cycle.

For risk-management purposes at the daily frequency, the benefits of using copulae to capture salient features of multivariate dependence, such as tail-dependence and other attributes of non-normality like skewness and fat-tailedness, are widely recognized (Jin; 2009, and references therein). Yet RV-based models often work with a (conditional) Gaussian structure. RCop allows to drop the rather restrictive Gaussian assumption and offers a more realistic description of the joint tails of the daily return distribution. It may therefore yield more accurate estimates of the quantiles of a portfolio's profit and loss distribution. Our empirical analysis confirms this expectation.

In this research, we combine two strands of literature. The first strand is a series of studies in the RV literature extending the univariate heterogeneous autoregressive (HAR) model to the multivariate level. The HAR model, originally suggested by Corsi (2009), is a stationary, restricted AR(22) model and captures long-range dependence in RV data by means of a cascade of volatility components that are interpreted as a daily, weekly and monthly volatility component. It nowadays is a standard benchmark model for modeling RV with unraveled forecasting performance.[1] A nontrivial challenge in constructing a multivariate HAR model is to ensure positive-definiteness of predicted covariance matrices. One therefore considers modeling nonlinear transformations of RC such as the Cholesky factorization (Chiriac and Voev; 2011) or the matrix log transformation (Bauer and Vorkink; 2010), or direct modeling by means of a Wishart autoregressive process (Gouriéroux et al.; 2009; Jin and Maheu; 2010; Bonato et al.; 2011). Our RCop approach is in the spirit of this research, since the copula parameter, which we imply from RC and subsequently describe by a HAR model, defines – together with the assumptions on the marginals – an entire

[1] See Corsi, Audrino and Renò (2012) for a review. As an alternative to HAR models pure long-memory models belonging to the ARFIMA class have been considered for modeling the variance processes, see e.g. Baillie (1996), Baillie et al. (1996), Andersen et al. (2003) among others. The forecasting performance of ARFIMA models for RV is very close to that of HAR-type models, but comes at the cost of a higher technical burden.

distribution and in consequence a well-posed covariance matrix.

The second stream of research our work is related to is the growing literature of dynamic copula models, such as Dias and Embrechts (2004) and Patton (2004, 2006), Chen and Fan (2006), Jondeau and Rockinger (2006), Giacomini et al. (2009), Jin (2009), Hafner and Manner (2010), Härdle et al. (2010), Christoffersen et al. (2011). All these approaches share in common the notion of a copula structure that has time-varying parameters driven by past realizations of the underlying data generating process or by additional exogenous variables, such as latent state factor. By exploiting intra-day data, we uncover a daily series of RCop parameters which we subsequently model by formulating a time series model. We thus obtain a dynamic copula model for daily returns, where time-variation is governed by the underlying dynamics of RC measures.

Remarkably, the literature using copulae to model dependency in the context of high-frequency data is scarce. To the authors' knowledge Breymann et al. (2003) and Dias and Embrechts (2004) appear to be the only work. In this study, however, the copula model is directly applied to analyze realized intraday returns. This is not the purpose of the present investigation. Our aim is to exploit intraday information as condensed in the RV measure to improve on modeling daily returns. In this sense we follow recent suggestions by Engle and Gallo (2006), Shephard and Sheppard (2010), Hansen, Huang and Shek (2011) and Hansen, Lunde and Voev (2011) that combine both low and high-frequency observations in a model framework at daily frequency.

The paper is organized as follows. In Section 2 we introduce the notion of RCop, discuss estimation and suggest a forecasting framework for RCop for risk-management purposes. The competitor models of RCop are presented in Section 3. In Section 4 we explore the empirical properties of RCop and its competitors on two portfolios of heavily traded NYSE stocks using two years of high-frequency data. Section 5 concludes.

2 Realized copula

2.1 Notion and estimation of realized copula

Copulae have emerged as a convenient way for constructing multivariate distributions since they allow to strictly separate the marginal distributions from cross sectional dependence, which is captured by the copula function, see Nelsen (2006) for an introduction on copulae. The main result due to Sklar (1959) states that if F is an arbitrary d-dimensional continuous distribution function of the random variables $X_1, \dots, X_d$, then the associated copula is unique and defined as a continuous function $C : [0,1]^d \to [0,1]$ satisfying the equality

$$C(u_1, \dots, u_d) = F\{F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\}, \quad u_1, \dots, u_d \in [0,1],$$

where $F_1^{-1}(\cdot), \dots, F_d^{-1}(\cdot)$ are the quantile functions of the corresponding marginal distributions $F_1(x_1), \dots, F_d(x_d)$. If F belongs to the class of elliptical distributions, this results in a so called elliptical copula. Most elliptical copulae, however, cannot be given explicitly, because the distribution function F and the inverse marginal distributions F_i usually have integral representations.

One class of copulae that overcomes this drawback is the class of Archimedean copulae

$$C(u_1, \dots, u_k) = \phi_\theta\{{\phi_\theta}^{-1}(u_1) + \dots + {\phi_\theta}^{-1}(u_d)\}, \quad u_1, \dots, u_d \in [0,1], \tag{1}$$

where $\phi_\theta : [0,\infty) \to [0,1]$, with $\phi_\theta(0) = 1$, $\phi_\theta(\infty) = 0$. The function ϕ_θ is called the generator of the copula and usually depends on a single parameter θ. The generator ϕ_θ is required to be d-monotone, i.e. differentiable up to the order $d-2$, with $(-1)^j {\phi_\theta}^{(j)}(x) \geq 0$, $j = 0,\dots,d-2$ for any $x \in [0,\infty)$ and with $(-1)^{d-2}{\phi_\theta}^{(d-2)}(x)$ being nondecreasing and convex on $[0,\infty)$, see McNeil and Nešlehová (2009). We give some examples of Archimedean copulae and their generators in Table 1, see Joe (1996) and Nelsen (2006) for more details.

In the following we will specialize our presentation to a setting with a single copula shape parameter such as the Archimedean copulae. We emphasize that the notion of RCop is not limited to this class of copulae: for instance the survival copula derived from an Archimedean copula by $C_{\text{rot}}(u,v) = C(1-u,1-v)+u+v-1$, such as the rotated Gumbel copula which we will use in our empirical part, is not Archimedean.

Suppose there are two random variables X_i and X_j with marginal distributions F_i and F_j and joint distribution F_{ij} and finite second moments. Hoeffding's lemma (Hoeffding; 1940) together with Sklar's theorem states that the covariance between X_i and X_j is a function in the copula parameter θ, the marginals and the joint distribution function:

$$
\begin{aligned}
\sigma_{ij}(\theta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big\{F_{i,j}(x_i,x_j,\theta) - F_i(x_i)F_j(x_j)\big\} dx_i dx_j \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big[C_\theta\{F_i(x_i),F_j(x_j)\} - F_i(x_i)F_j(x_j)\big] dx_i dx_j \,. \qquad (2)
\end{aligned}
$$

Usually this integral has no explicit form, but e.g. for the multivariate normal distribution, in which case one gets $\sigma_{ij} = \theta$. In other cases it can be approximated by numerical integration.

For our notion of RCop, we equate (2) with a measure of RV, i.e. we define the copula shape parameter θ implicitly through the equation

$$
h_{ij,t} = \mathrm{f}_{ij}(\theta_t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big[C_{\theta_t}\{F_{i,t}(x_i),F_{j,t}(x_j)\} - F_{i,t}(x_i)F_{j,t}(x_j)\big] dx_i dx_j \,, \qquad (3)
$$

where $h_{ij,t}$ denotes an element of the RC matrix measured at day t. We then exploit Hoeffding's lemma in a method-of-moments type of fashion to estimate θ_t.

Consider the case $d = 2$, with one off-diagonal element $h_{12,t}$ in the RC available. An estimate of θ_t is given by

$$
\hat{\theta}_t^{\text{MM}} = \mathrm{f}_{12}^{-1}(h_{12,t}) \,, \qquad (4)
$$

where f_{12}^{-1} denotes the inverse function of (2). In the general case for $d > 2$, define the moment condition $\mathrm{g}_{ij}(\theta) = h_{ij,t} - \mathrm{f}_{ij}(\theta)$, where $i < j$ and $i,j = 1,\dots,d$. Stacking all g_{ij} into a vector $\mathbf{g}$ of size $d(d-1)/2$, we define the estimator as

$$
\hat{\theta}_t^{\text{MM}} = \arg\min_\theta \ \mathbf{g}^\top(\theta)\mathbf{W}\mathbf{g}(\theta) \,, \qquad (5)
$$

where $\mathbf{W}$ is a positive definite weight matrix. A typical choice would be $\mathbf{W} = \mathbf{I}_n$ with $\mathbf{I}_n$ denoting the n-dimensional unit matrix and $n = d(d-1)/2$. For $d = 2$, (5) coincides with (4). We point out that these two estimators bear much similarity with method-of-moments approaches where the copula parameter of an Archimedean copula is estimated from Kendall's tau or Spearman's rho (Genest and Rivest; 1993).

Finally, we suggest an ad hoc estimator. This estimator is based on the transformation of the linear correlation coefficient of the normal distribution to Kendall's tau, and the consequent transformation of Kendall's tau to the copula parameter. Assuming a Gaussian setting, it is well known that the linear correlation coefficient ρ_{ij} translates into Kendall's tau by

$$\tau^{\mathrm{G}}_{ij,t} = \frac{2}{\pi} \arcsin \rho_{ij,t} \ . \tag{6}$$

As stated in Genest and Rivest (1993) Kendall's tau has the following representation f_τ in the terms of the generator function and the shape parameter of a two-dimensional Archimedean copula

$$\mathrm{f}_\tau(\theta) = 4 \int_0^1 {\phi_\theta}^{-1}(v)/({\phi_\theta}^{-1})'(v)\, dv + 1 \ .$$

For many Archimedean copulae this leads to an explicit and invertible relationship between Kendall's tau and their shape parameter, see Table 1. Then the ad hoc estimator is defined by

$$\hat{\theta}^{\text{ad hoc}}_t = \frac{2}{d(d-1)} \sum_{i<j} \mathrm{f}^{-1}_\tau(\hat{\tau}^{\mathrm{G}}_{ij,t}) \ . \tag{7}$$

Interestingly, despite being based on shaky theoretical grounds, the simulation results and our empirical findings show that for settings with small and moderate dependence this ad hoc estimator performs similarly to the estimator based on Hoeffding's lemma. It is, however, severely biased in situations with strong dependence.

Given the assumptions on the copula family and the marginal distributions, the structure

$$C_{\hat{\theta}_t}\{F_{1,t}(x_1), \dots, F_{d,t}(x_d)\} \ , \tag{8}$$

where $\hat{\theta}_t$ is any estimator presented above, fully characterizes the (ex-post) multivariate distribution as materialized in the RV measure in date t. We therefore call it *realized copula (RCop)*.

2.2 A forecasting framework for realized copula

For our portfolio risk management problem, we consider a model framework which combines daily and within-day modeling frequencies. The purpose is to exploit intra-day high-frequency data as an auxiliary source of information to improve on the 1-day ahead VaR forecasts. In this sense, our approach is close to the MIDAS approach by Ghysels et al. (2006), the multiplicative error model suggested by Engle and Gallo (2006) and Shephard and Sheppard (2010), and it can be embedded into the extensions thereof recently proposed by Hansen, Huang and Shek (2011) and Hansen, Lunde and Voev (2011), and Noureldin et al. (2011).

For the daily level, denote the log-prices of a d-dimensional vector of assets by $P = (P_1, \dots, P_d)^\top$ and the associated daily returns by $\Delta P_t = P_t - P_{t-1} = r_t$, $t = 1, \dots, T$. We assume that the conditional distribution of daily returns r_t can be approximated by

$$r_{t+1} \sim F_{r_{t+1}|\mathcal{F}_t}(\widehat{H}_{t+1|t}) \ ,$$

where $F_{r_{t+1}|\mathcal{F}_t}(\widehat{H}_{t+1|t})$ denotes a conditional distribution function parametrized by $\widehat{H}_{t+1|t}$ which is an $\mathcal{F}_t$-measureable forecast of the RC matrix of r_t. This forecast will be derived from a sequence of the RC matrices obtained from past within-day high-frequency data. When replacing $\widehat{H}_{t+1|t}$ by a known function of past daily returns, this framework is identical to the one formalized in standard volatility models of the multivariate GARCH type as suggested by Bollerslev (1990), Engle (2002), and Tse and Tsui (2002). However, rather than taking an a priori stand on an underlying model for H_t as a function of past daily returns, this approach relies on a finer information structure accumulated by intraday high-frequency returns for the VaR forecast.

By Sklar's theorem and following the notion of a conditional copula outlined in Patton (2006), we replace $F_{r_{t+1}|\mathcal{F}_t}$ by $F_{r_{t+1}|\mathcal{F}_t}(\widehat{H}_{t+1|t}) = C_{\hat{\theta}_{t+1|t}}\{F_{1,t}(\hat{h}_{1,t+1|t}), \ldots, F_{d,t}(\hat{h}_{d,t+1|t})\}$, where C_θ denotes a copula belonging to some parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$ which is specified in the following. Furthermore, $F_{j,t}(\hat{h}_{j,t+1|t})$, $j = 1, \ldots, d$, denote the marginal conditional distributions of daily returns depending on variance forecasts $\hat{h}_{j,t+1|t}$. As reported in Andersen, Bollerslev, Diebold and Ebens (2001), returns standardized by ex post RV are close to standard normal. We therefore assume that $F_{j,t}(\hat{h}_{j,t+1|t})$ is normal with variance $\hat{h}_{j,t+1|t}$, i.e. $\mathsf{N}(0, \hat{h}_{j,t+1|t})$. Finally, $\hat{\theta}_{t+1|t}$ is a forecast of the associated RCop parameter $\hat{\theta}_t$ which is estimated day by day from RC as outlined in Section 2.1.

We complete the model by specifying the forecasting rules:

$$\begin{pmatrix} \log \hat{h}_{1,t+1|t} \\ \vdots \\ \log \hat{h}_{d,t+1|t} \\ \hat{\theta}_{t+1|t} \end{pmatrix} = \mathsf{E}_t \begin{pmatrix} \log h_{1,t+1} \\ \vdots \\ \log h_{d,t+1} \\ \theta_{t+1} \end{pmatrix} = \begin{pmatrix} \beta_0^1 + \beta_{\mathrm{D}}^1 \log h_t^{\mathrm{D}} + \beta_{\mathrm{W}}^1 \log h_t^{\mathrm{W}} + \beta_{\mathrm{M}}^1 \log h_t^{\mathrm{M}} \\ \vdots \\ \beta_0^d + \beta_{\mathrm{D}}^d \log h_t^{\mathrm{D}} + \beta_{\mathrm{W}}^d \log h_t^{\mathrm{W}} + \beta_{\mathrm{M}}^d \log h_t^{\mathrm{M}} \\ \alpha_0 + \alpha_{\mathrm{D}} \theta_t^{\mathrm{D}} + \alpha_{\mathrm{W}} \theta_t^{\mathrm{W}} + \alpha_{\mathrm{M}} \theta_t^{\mathrm{M}} \end{pmatrix}, \tag{9}$$

where $\beta^j = (\beta_0^j, \beta_{\mathrm{D}}^j, \beta_{\mathrm{W}}^j, \beta_{\mathrm{M}}^j)^\top$, for $j = 1, \ldots, d$, and $\alpha = (\alpha_0, \alpha_{\mathrm{D}}, \alpha_{\mathrm{W}}, \alpha_{\mathrm{M}})^\top$ are parameter vectors, and $x_t^{\mathrm{D}} = x_t$ are daily, $x_t^{\mathrm{W}} = \frac{1}{5}\sum_{i=0}^{4} x_{t-i}$ weekly, and $x_t^{\mathrm{M}} = \frac{1}{21}\sum_{i=0}^{20} x_{t-i}$ monthly averages of past realizations of x_t. This forecasting rule, which is motivated from the idea of heterogeneous agents with differing investment horizons, is due to Corsi (2009). It has found wide application in the RV literature as it approximately captures the long-memory patterns typically observed in RV data.[2] We extend this idea here also for the copula parameter θ. This extension, together with the assumptions on the marginals, allows us to predict the entire multivariate distribution. Moreover, since the copula parameter parametrizes in some sense the covariance matrix, this setting (implicitly) provides well-defined covariance matrices. From this perspective, it is similar in spirit to Bauer and Vorkink (2010) and Chiriac and Voev (2011) who subject RC to nonlinear transformations, such as the matrix logarithm or the Cholesky decomposition, to ensure positive-definiteness of the predicted RC, see Section 3 for further details. Our modeling approach can therefore be interpreted as another multivariate extension of the univariate HAR model.

As is discussed in Bauer and Vorkink (2010) and Chiriac and Voev (2011), an unbiased prediction of the variables parameterizing the covariance matrix, will generally not yield unbiased forecasts of covariance when the transformation between both is nonlinear. This issue also applies to the present estimator, since the relationship between the copula parameter and covariance as presented by Hoeffding's lemma is nonlinear. However, since we consider 1-day VaR forecasts, only, we

[2]Further refinements of this base line model have been suggested by Andersen et al. (2007), Corsi et al. (2008), Bollerslev et al. (2009), Corsi et al. (2010), and Audrino and Hu (2011), see Corsi, Audrino and Renò (2012) for an overview.

conjecture these biases to be small (see also Halbleib and Voev (2011) for corroborative evidence). As in Chiriac and Voev (2011), we therefore renounce on a bias adjustment.

2.3 Simulation Study

In order evaluate the performance of the moment based estimator we subject it to the following simulation study. Given an assumption on a copula family (e.g. Clayton, Gumbel), we draw 1000 vector-valued random variates from the copula based on standard normal margins (we consider dimensions $d = 2$ and $d = 3$). From these draws, the sample covariance matrix is estimated by the unbiased covariance estimator. Afterwards the method-of-moment estimators outlined in Section 2.1 are applied. This procedure is repeated 1000 times.

In Figure 1, we present the differences of the estimates from the true parameter value along with the mean (red) and the median (blue) difference as functions of the underlying Gumbel copula parameter.[3] We also contrast the results with a maximum likelihood (ML) estimator, namely with the method of inference functions for margins due to Joe and Xu (1996). The shaded areas are the 95% pointwise confidence intervals computed from the 1000 repetitions of the exercise. As is apparent from Figure 1, the moment-based estimators are unbiased and only slightly less efficient than the ML estimator. The linear correlation estimator is strongly biased in settings of strong dependence. For instance, in the two-dimensional Gumbel case for copula parameters larger than three, which corresponds to Kendall's tau larger than 2/3, the estimates start to be severely downward biased.

2.4 Portfolio risk-management and backtesting

Computing risk measures for portfolios of stocks followed by a subsequent backtesting analysis are standard procedures in applied risk management, see e.g., Berkowitz and O'Brien (2002), Giacomini et al. (2009), Jin (2009), Berkowitz et al. (2010) among others. Closest to our research is Giot and Laurent (2004) who appear to be among the first to simultaneously include both low and high-frequency data for such an analysis.

For portfolio risk-management, the aggregate portfolio and loss (P&L) distribution must be determined. Consider a portfolio, where $a_t = \{a_{1,t}, \ldots, a_{d,t}\}$ with $a_{i,t} \in \mathbb{R}^d$ denoting the number of shares in the portfolio. The market value V_t of this portfolio is given by

$$V_t = \sum_{j=1}^{d} a_{j,t} S_{j,t} \ , \tag{10}$$

where $S_{j,t}$ is the asset price. In this study, we will consider only portfolios which are equally weighted in terms of wealth allocation. This implies that $a_{j,t} = w_j V_t / S_{j,t}$ where $w_j = 1/d, \ j = 1, \ldots, d$. Hence absolute portfolio weights are adjusted on a daily basis in order to keep the relative contributions constant.

[3] Simulation results for the Clayton copula are similar and are therefore not reported.

The daily trading P&L on this portfolio is given by

$$L_{t+1} = (V_{t+1} - V_t) = \sum_{j=1}^{d} a_j S_{j,t} \left\{\exp(r_{j,t+1}) - 1\right\} \ , \tag{11}$$

where $r_{j,t}$ denotes the log-return on asset j. Denote the conditional distribution function of L by

$$F_{L_{t+1}|\mathcal{F}_t}(x) = \mathrm{P}(L_{t+1} \leq x|\mathcal{F}_t) \ . \tag{12}$$

As the practically most important risk measure, we employ the Value-at-Risk (VaR) at level α defined as the α-quantile of $F_{L_{t+1}|\mathcal{F}_t}$:

$$\mathrm{VaR}_{t+1|\mathcal{F}_t}(\alpha) = F^{-1}_{L_{t+1}|\mathcal{F}_t}(\alpha) \ . \tag{13}$$

It follows that $F_{L_{t+1}|\mathcal{F}_t}$ is determined by the d-dimensional distribution of log-returns $F_{r_{t+1}|\mathcal{F}_t}$ described by the general framework in Section 2.2. The accuracy of the VaR estimates therefore depends on how well the RCop model and the alternative approaches presented in Section 3 capture the unknown multi-dimensional conditional distribution of daily returns.

A variety statistical criteria have been suggested in the literature to measure the quality of estimated VaR, see e.g. Campbell (2006) and Christoffersen (2009) for overviews. Let $\{l_t\}$ be the true realizations of the respective P&L distribution. Unconditional coverage testing focuses on the exceedances ratio of the respective VaR. The exceedances ratio $\hat{\alpha}$ is defined by

$$\hat{\alpha} = \frac{N}{T}$$
$$N = \sum_{t=1}^{T} \mathbf{I}\{l_t < \widehat{\mathrm{VaR}}_t(\alpha)\}$$

where N denotes the number of observed exceedances. A natural likelihood ratio test based on binomial theory for $H_0 : \hat{\alpha} = \alpha$ is

$$LR_{uc} = 2 \log \frac{\hat{\alpha}^N (1-\hat{\alpha})^{T-N}}{\alpha^N (1-\alpha)^{T-N}}$$

which has asymptotically a $\chi^2(1)$ distribution under H_0. This test is due to Kupiec (1995). We also considered a simple t-test based on the normal approximation for the binomial distribution and independence testing as suggested by Berkowitz et al. (2010). Both alternative tests did not yield additional insights, which is why these results will not be reported.

3 Competitor models

As competitor models, we choose four classical representatives. As models, which only exploit daily data, we consider a naïve rolling window approach and a locally adaptive estimation algorithm to capture time-varying dependency. As alternatives for RV models, which make use of high-frequency data, we employ another two approaches. Similarly to the RCop approach, both methods use linear time series models of nonlinear transformations of RC: the matrix logarithm and the Cholesky decomposition, respectively.

3.1 Rolling window and local change point detection

The rolling window approach estimates the time-varying copula parameter on a fixed window of size w, while the locally adaptive change point (LCP) detection algorithm[4] allows for a time-varying window width. We sketch the LCP algorithm here. Corresponding theory and further applications in volatility modeling and risk management may be found in Spokoiny (1998), Mercurio and Spokoiny (2004), Chen et al. (2008), Čížek et al. (2009), Giacomini et al. (2009), Spokoiny (2009), Chen et al. (2010), and Härdle et al. (2010).

In both cases, in the rolling window case and for LCP detection, the estimator is the maximum likelihood estimator

$$\widetilde{\theta}_t = \arg\max_{\theta} \mathcal{L}(\theta) = \arg\max_{\theta} \sum_{i=1}^{n_t} \log \left[c\{F_{1,t}(x_{1,i}), \ldots, F_{d,t}(x_{d,i}); \theta_t\} \prod_{j=1}^{d} f_{j,t}(x_{j,i}) \right], \tag{14}$$

where n_t denotes the sample size of the respective window width, on which estimation is carried out, $c\{\cdot;\theta_t\}$ the copula density and $f_{j,t}(x)$, $j = 1, \ldots, d$ the marginal densities. The marginal densities are assumed to be $\mathsf{N}(0, \hat{\sigma}_t^2)$, where $\hat{\sigma}_t^2$ is the variance estimated from the (daily) returns of respective homogeneous time interval. The estimator can be obtained by exact maximum-likelihood estimation, i.e. directly by a one-step maximization of (14), or by a two-step procedure, the method of inference functions for margins (IFM) due to Joe and Xu (1996). In the latter case one first estimates the parameters of the marginals and – given these estimated parameters – those of the copula function. Through all this work we will use the less efficient, but computationally more benign IFM-method, see Härdle et al. (2009) for an comprehensive discussion of alternative estimation strategies for copula-based models.

In what follows, let θ_t denote the time varying but otherwise unknown copula parameter. Locally adaptive estimation selects for each time point t_0 an interval I during which θ_t is reasonably well approximated by a constant θ^*. A possible measure of discrepancy between two copulae $C(\cdot;\theta)$ and $C(\cdot;\theta')$ is the Kullback-Leibler divergence $\mathcal{K}\{C(\cdot;\theta'), C(\cdot;\theta)\} = \mathsf{E}_{\theta'} \log\{c(\cdot;\theta')/c(\cdot;\theta)\}$, where $c(\cdot)$ is the copula density. The aim is to select I as close as possible to the so-called "oracle" choice interval I_{k^*}, defined as the largest interval $I = [t_0 - m_{k^*}; t_0]$, for which the small modeling bias condition

$$\triangle_I(\theta) = \sum_{t \in I} \mathcal{K}\{C(\cdot;\theta_t), C(\cdot;\theta)\} \leq \triangle, \text{ for some } \triangle \geq 0,\ \theta\ , \tag{15}$$

is fulfilled. The LCP is based on sequentially testing the hypotheses of homogeneity on intervals I_k. We select I_k with $k = -1, 0, 1, \ldots$ as the sequence of intervals $I_k \subset I_{k+1}$, starting with $k = 1$. If there are no change points in $\mathcal{T}_k \subset I_k \setminus I_{k-1}$, we accept I_k as an interval with a constant copula structure. At the next step we take $\mathcal{T}_{k+1}$ and test it for homogeneity. We repeat these steps until rejection or until the largest possible interval I_K is accepted, leading to an interval $I_{\hat{k}}$.

Testing for local homogeneity works as follows. Fix some t_0 and let $I = [t_0 - m, t_0]$ be an interval candidate and $\mathcal{T}_I$ be a set of interval points within I. We estimate the copula parameter θ by the ML estimator from observations in I, assuming a homogeneous model within I. Thus the H_0

[4] Alternative change point methods for copulae have been developed by Dias and Embrechts (2004) and Guégan and Zhang (2010).

hypothesis and H_1 alternative can be formulated as:

$$\begin{aligned} H_0 &: \forall \tau \in \mathcal{T}_I,\ \theta_t = \theta,\ \forall t \in I = J \cup J^c = [\tau, t_0] \cup [t_0 - m, \tau) \\ H_1 &: \exists \tau \in \mathcal{T}_I,\ \theta_t = \theta_1,\ \forall t \in J = [\tau, t_0],\ \text{and } \theta_t = \theta_2 \neq \theta_1,\ \forall t \in J^c = [t_0 - m, \tau). \end{aligned}$$

Denote by $\mathcal{L}_I(\theta)$ and $\mathcal{L}_J(\theta_1) + \mathcal{L}_{J^c}(\theta_2)$ the log-likelihood functions corresponding to H_0 and H_1, respectively. Then the likelihood ratio test for the single change point with known fixed location τ is given by

$$T_{I,\tau} = \max_{\theta_1,\theta_2}\{\mathcal{L}_J(\theta_1) + \mathcal{L}_{J^c}(\theta_2)\} - \max_{\theta} \mathcal{L}_I(\theta).$$

Since the point τ is unknown, one defines the test statistic:

$$T_I = \max_{\tau \in \mathcal{T}_I} T_{I,\tau}\ .$$

T_I tests the homogeneity hypothesis in I against a change point alternative with unknown location τ (in the set $\mathcal{T}_I$). The decision rule of the test requires to compare T_I with the critical value $\mathfrak{z}_I$. The critical value depends on the interval I, the dimension and the parameter of the copula. We reject the hypothesis of homogeneity if $T_I > \mathfrak{z}_I$.

For running the tests, several parameters have to be specified. This includes the choice of the interval candidates I_k and internal points $\mathcal{T}_{I_k}$ for each of these intervals and the choice of the critical values $\mathfrak{z}_{I_k}$. One possible example of the implementation is based on the choice of the interval candidates I_k in form of a geometric grid. We fix m_0, which is the smallest possible interval of homogeneity, and then define $m_k = [m_0 c^{k-1}]$ for $k = 1, 2, \ldots, K$ and $c > 1$, where $[x]$ means the integer part of x. Furthermore, we set $I_k = [t_0 - m_k, t_0]$ and $\mathcal{T}_k = [t_0 - m_{k-1}, t_0 - m_{k-2}]$ for $k = 1, 2, \ldots, K$. For the empirical results these parameters are set to $c = 1.25$, $m_0 = 40$, $K = 10$ which corresponds to the settings found in Giacomini et al. (2009) and Härdle et al. (2010).

In this work, we use the sequential choice of critical values $\mathfrak{z}_k$ discussed in Spokoiny (2009). Considering the situation after k steps of the algorithm, we may distinguish two cases. In case one, a change point has been detected at some step $\ell \leq k$; in the second case, no change point has been detected. Following notation in Spokoiny (2009), let $\mathcal{B}_\ell = \{T_1 \leq \mathfrak{z}_1, \ldots, T_{\ell-1} \leq \mathfrak{z}_{\ell-1}, T_\ell > \mathfrak{z}_\ell\}$ be the event meaning the rejection of the null hypothesis at step ℓ and $(\hat{\theta}_k) = (\tilde{\theta}_{\ell-1})$ on $\mathcal{B}_\ell$ for $\ell = 1, \ldots, k$. By Monte-Carlo simulations from fixed parametric models, we sequentially find a minimal value of $\mathfrak{z}_l$ which ensures the inequality

$$\max_{k=l,\ldots,K} \mathsf{E}_{\theta^*} |\mathcal{L}(\tilde{\theta}_k) - \mathcal{L}(\tilde{\theta}_{\ell-1})|^{1/2} \mathbf{I}(\mathcal{B}_\ell) \leq \rho\, \mathfrak{R}(\theta^*)\, k/(K-1)\ ,$$

where $\mathbf{I}$ is the indicator function and $\mathfrak{R}(\theta^*) = \max_{k=l,\ldots,K} |\mathcal{L}(\tilde{\theta}_k) - \mathcal{L}(\theta^*)|^{1/2}$. For $\ell = 1$ this inequality depends only on $\mathfrak{z}_1$ in $\mathcal{B}_1 = \{T_1 > \mathfrak{z}_1\}$. For every $\ell \geq 2$ we take $\mathfrak{z}_1, \ldots, \mathfrak{z}_{\ell-1}$ being fixed from previous steps, which means that $\mathcal{B}_\ell$ is controlled by $\mathfrak{z}_\ell$, only. The parameter ρ plays the role of the level of significance and influences the sensitivity of the procedure to inhomogeneity. For large values of ρ, small critical values are obtained which makes the procedure more sensitive; decreasing ρ makes the procedure more conservative. We set $\rho = 0.5$, following the detailed robustness analysis for various choices of ρ in Giacomini et al. (2009).

To obtain forecasts of the estimated parameters in the rolling window and LCP approach we do not apply a forecasting rule as for the RV models. We simply extrapolate the current estimates to the following day (i.e. hold them constant). The logic of this "degenerated prediction" is that both approaches assume that the parameters involved are estimated on a local interval of homogeneity. It therefore appears natural to assume that this interval of homogeneity continues to hold at the following day. As another benefit we avoid fitting time series models on estimates obtained from overlapping return data which is likely to invalidate the statistical analysis.

3.2 Realized variance models

While at the univariate level the HAR formulation as described in Section 2.2 has emerged as an undisputed base-line model (Corsi, Audrino and Renò; 2012), the literature has yet not found agreement on its most competitive multi-variate extension. This is because at the multi-variate level, it has remained challenging to maintain positive-definiteness of predicted covariance matrices. Two recent contributions addressing this issue are the matrix logarithm model due to Bauer and Vorkink (2010) and the Cholesky decomposition model due to Chiriac and Voev (2011).

In Bauer and Vorkink (2010), RC are modeled by means of the matrix exponential and its inverse function, the matrix logarithm. The matrix exponential is a function on a square matrix A and given by the series representation

$$H = \operatorname{expm}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \,. \tag{16}$$

As a most important property of (16), if A is a real, symmetric, and positive-definite matrix, so is $H = \operatorname{expm}(A)$. With the converse being true as well, the inverse function of the matrix exponential, the matrix logarithm,

$$A = \operatorname{logm}(H) \,, \tag{17}$$

is a useful device for guaranteeing predicted covariance matrices to be positive-definite.

Given a time series of RC matrices H_t, $t = 1, \ldots, T$, of size $d \times d$, Bauer and Vorkink (2010) suggest to apply the matrix logarithm, $A_t = \operatorname{logm}(H_t)$. Now, A_t, $t = 1, \ldots, T$, forms a time series of symmetric $d \times d$ matrices. As a next step, the vech-operator is applied

$$a_t = \operatorname{vech}(A_t) \,, \tag{18}$$

which stacks the upper triangular of A_t columnwise into a $\frac{1}{2}d(d+1) \times 1$ vector. The vector time series a_t is now modeled along the lines of the univariate HAR model, i.e. by forming elementwise weekly and monthly aggregates of daily components. The resulting forecasting rules for these (averaged) aggregates take exactly the same form as presented in (9). By first applying the reverse vech-operator and then the matrix exponential to the predictions derived from this model, the respective predicted covariance matrix is obtained as $\widehat{H}_{t+1|t} = \operatorname{expm}(\widehat{A}_{t+1|t})$, which is positive-definite as long as the elements in $\widehat{a}_{t+1|t}$ are real.

A similar approach is followed by Chiriac and Voev (2011), but the series of covariance matrices is decomposed into a series of Cholesky factors, i.e. now (17) is replaced by

$$AA^\top = H \,, \tag{19}$$

where A is a real upper triangular $d \times d$ matrix with positive diagonal elements. As before, applying first (19) and subsequently the vech-operator to a time series H_t gives rise to the vector-valued time series a_t. Then weekly and monthly aggregates are derived and modeled along the forecasting rules in (9). Predictions $\widehat{a}_{t+1|t}$ are converted to positive-definite predicted covariance matrices by applying the reverse vech-operator, which yields an upper triangular matrix $\widehat{A}_{t+1|t}$, and by computing the matrix product $\widehat{H}_{t+1|t} = \widehat{A}_{t+1|t}\widehat{A}^{\top}_{t+1|t}$.

4 Empirical part

4.1 Data description, data filtering, and realized variance estimation

The empirical part of this work is based on stock price data obtained from NYSE's Trades and Quotes (TAQ) database, for the period from 2 January, 2009, to 31 December, 2010. It covers a total of 470 days and contains the daily transaction data observed between 9:30 till 16:00 local time. The stocks we consider are IBM, Google, Oracle, Pfizer (PFE), Exxon (XOM), which are among the most heavily traded names at NYSE.

High frequency data is known to be noisy such that the accuracy of the RV and RC estimates can be seriously impaired. We therefore subject the data to the filtering procedure established in by Barndorff-Nielsen et al. (2009) for TAQ data comprising the following steps:

1. Delete entries outside 9:30-16:00 and with zero transaction price.
2. Delete entries with corrected trades or abnormal sale condition.
3. Replace multiple trades for the same time stamp by the median price.
4. Delete entries with prices above ask plus bid-ask spread or below bid minus bid-ask spread.

After applying this cleaning procedure we estimate RC matrices by the realized kernel estimator due to Barndorff-Nielsen et al. (2011), see Appendix A for all relevant details on the procedure. The realized kernel estimator warrants a positive-definite estimate of RC and is robust to market micro structure noise, such as non-synchronous trading and the bid-ask-bounce. Descriptive statistics on estimated RC are displayed in the upper panels of Tables 3 and 4.[5] They are well in line with those reported on stock market data in general (Andersen, Bollerslev, Diebold and Ebens; 2001) or for realized kernel estimators specifically (Barndorff-Nielsen et al.; 2009). In Figure 2 we display the series of realized correlations of the two portfolios. As is visible for the first portfolio containing Google-IBM-Oracle, all realized correlations track each other very closely. This is natural given all stocks come from the information and communication technologies sector. The second portfolio, IBM-PFE-XOM, which is a mixed sector portfolio, comprises one pair of stocks (IBM-XOM) that in the first part of the sample period is slightly stronger correlated that the other two pairs.

[5]Note that entries for RV of IBM are differing between the two tables. This is due to refresh time sampling of the realized kernel estimator. Since both covariance matrices are estimated separately, refresh time sampling for both RV series differs for the two portfolios implying slightly differing estimates. An alternative would be to increase dimension and to directly estimate the six-dimensional RC. However, as a consequence of refresh time sampling, fewer data observations would be used in the resulting estimator. We therefore prefer to compute the smaller dimensional estimates. An estimator for RC overcoming this issue is suggested by Corsi, Peluso and Audrino (2012).

Aside from high-frequency intraday data, we also employ a sample of daily closing prices from 9 July 2007 to 31 December 2010, see Table 2 for the descriptive statistics. These data will be used for backtesting the out-of-sample VaR computations. The history is larger than the one for intraday data, since the competitor models based on daily data (rolling window and the LCP method) require a data history prior to the one which is under scrutiny in the VaR investigation.

4.2 Empirical results

Before looking at the out-of-sample VaR results it is instructive to study the in-sample estimates of the copula parameters and of the forecasting rules as outlined in Section 2.2. The discussion of the out-of-sample backtesting results follows.

4.2.1 In-sample results

For our empirical application we consider an Archimedean copula, the Clayton copula, and the rotated Gumbel (rGumbel) copula, which is not Archimedean. Both copulae exhibit lower tail-dependence, which is likely to be crucial for modeling risk measures for stock portfolios. In-sample results for the estimated copula parameters are displayed in the lower panels of Tables 3 and 4.

In the first portfolio, the estimated parameters of the rotated Gumbel fluctuates between one (independence case) and quite substantial dependence of around two (i.e. Kendall's tau of around 0.5) with the mean estimate being $\theta_{\text{rGum}}^{\text{MM}} \approx 1.4$. Similar findings apply to the Clayton copula whose parameter estimates are between zero (independence case) and two. Necessarily, the estimates for both copulae agree on the implied dependence expressed by Kendall's tau. For the second portfolio estimated copula shape parameters are somewhat lower, as is to be expected comparing the upper and the lower panel in Figure 2. Here estimates are on average around 1.3 in the rotated Gumbel and 0.58 in the Clayton case. As suggested by the findings of the simulation study in Section 2.3, for both portfolios, due to dependence being moderate overall, the ad hoc estimator provides estimates which are quite close to those in the exact case.

In the top panels of Figures 3 and 4 we plot the time series of the RCop shape parameters based on Hoeffding's lemma (red line, rotated Gumbel), which is estimated from high-frequency intraday data, against the time series of the copula parameters of the naïve rolling window (black line, window size 250 days) and the adaptive LCP method (blue line), both obtained for daily data. As is visible, the RCop structure differs markedly from the one recovered for the latter two approaches. First the copula parameters obtained for the daily data appear to be higher on average than is suggested from intraday data. Second, they are less noisy, but their reaction to fundamental changes in the economy is more inert than for RCop, as can be well discerned in the second half of the sample period (May to Sep. 2010). The reason becomes apparent in the lower panels of Figures 3 and 4 where we plot the estimated interval lengths of the rolling window and the LCP method. For both portfolios, also the LCP method tends to identify rather long intervals of homogeneity, which is why both approaches deliver very close estimates for these periods. It is first in September that LCP identifies much smaller intervals of homogeneity. In consequence the estimated copula parameter jumps up. In contrast RCop reacts already between May and July to higher levels of dependence and by September has already returned to usual levels.

In the lower panels of Tables 5 and 6 we provide in-sample estimates for the RCop forecasting rules. Estimation is accomplished by ordinary least squares.[6] Size of estimates for the log-RV models are as reported elsewhere in the literature (Corsi; 2009): dynamics of RV is mainly driven by yesterdays realization. The influence of the weekly component is only half as big, followed by the monthly RV aggregate being smallest in magnitude, but significant. Interestingly, this contrasts sharply with the covariance dynamics as implied by the copula parameters. The shape parameter of RCop appears to be mainly driven by the daily and the weekly aggregate, with the latter even outweighing the first. The monthly component is not significant at all. These findings suggest that the dynamics of spillovers and (lower) tail-dependence as reflected by the time-varying copula parameter are more sluggish than those of RV: after some initial shocks cross-sectional dependence tends to subside more slowly than RV, allowing the system to still maintain high coefficients of tail-dependence and thus a high probability to incur simultaneous price deteriorations in all stocks, even when individual variances might have calmed down already.

4.2.2 Out-of-sample VaR backtesting results

Since it appears difficult to subject a given copula assumption to a specification test, an out-of-sample study, for instance by backtesting VaR, is a vital means of model validation.

The backtesting proceeds as follows. We shrink the relevant time frame for backtesting to 19 October 2009 to 31 December 2010 taking the initial sample of 200 days to estimate the HAR-type prediction rules for all RV-based models: RCop (Hoeffding's lemma and ad hoc estimator), the matrix log transformation, and the Cholesky factorization. Given the linear prediction rules, a forecast is made for RC, RV and the RCop parameter. To achieve high accuracy of the relevant quantiles of the future P&L distribution, we simulate it with 100 000 random draws. We then check whether the following day's P&L realization is an exceedance or not. For the next VaR computation, the initial learning sample is shifted to include the new day with the initial day from the previous learning sample being dropped. We thus iterate through the entire sample. As described in Section 2.4 the portfolio weights are always adjusted to preserve the same relative weights within the portfolio.

As was explained in Section 3.1, the rolling window and the LCP method work with a degenerated forecasting rule: the current copula parameter, which is estimated on the current interval of homogeneity (either fixed at 250 days or locally adaptive in the LCP method), is extrapolated as a constant to the next following day. To initialize the LCP, we start at 2 January 2009 and go into the past until the smallest interval of a constant parameter is found by rejecting the homogeneity test. The relevant variances of the rolling window and the LCP method are computed from the daily returns on the respective intervals of homogeneity. We then iterate through the backtesting sample as described above.

Tables 7 and 8 summarize the results for 1-day ahead quantiles of 1%, 5% and 10%. For the first portfolio, which is reported in Table 7, the RCop approaches are best performing, with rotated Gumbel and Clayton being hardly distinguishable from each other. In particular the smallest quantiles are very well captured. Unconditional coverage testing based on the Kupiec test confirms this observation. In the second portfolio (Table 8) rolling window and LCP perform slightly better than RCop at the 1% quantile, but at the 5% and the 10% quantiles it is RCop that is superior.

[6] The estimation problem could also be treated in a seemingly unrelated regressions framework.

As before rotated Gumbel and Clayton are very similar.

As a result, the RV approaches based on the matrix log transformation and the Cholesky factorization, which work with a Gaussian structure, appear to be dominated by the methods allowing for a non-Gaussian multivariate distribution. This is particularly evident for the small quantiles. It is important to note that at the margins *all* methods assume normality. The strikingly better performance of the copula-based methods need therefore be attributed to non-trivial forms of tail-dependence relevant for VaR-computations.

In Figures 5 and 6 we present the exceedances plots for the 1%-VaR risk for both portfolios. Both figures elucidate the findings of the previous tables. As is visible in the top panels of both figures, the rolling window and the LCP method exhibit a much smoother quantile history than the RV-based approaches. In contrast, RCop (middle panel) responds very quickly to shocks in the economy and quantiles widen accordingly. In Figure 5, this is nicely visible in the mid of the sample (June 2010), where many exceedances occur. While rolling window and LCP do not detect these outbursts, RCop does and only few exceedances are recorded. Like RCop, also the two other Gaussian RV approaches are very sensitive to these events, but having zero tail-dependence their quantiles are not sufficiently fat-tailed, which leads to a number of exceedances. The same deficiency inherent to Gaussian RV approaches is observed in Figure 6, where many exceedances occur during the first days of the backtesting period.

Finally, as is also apparent from Tables 7 and 8 and Figures 5 and 6, for the moderate dependence in our sample, RCop based on the ad hoc estimator essentially delivers the same results as the accurate estimator using Hoeffding's lemma. In many circumstances, we therefore expect the ad hoc estimator to be a suitable practical replacement for the exact estimator, making computations even more straight forward.

5 Conclusion

Based on assumptions of the marginal distributions of daily stock returns and a copula family, we introduce realized copula as the copula structure materialized in realized covariance estimated from within-day high-frequency data. We estimate the copula parameters in a method-of-moments type of fashion using Hoeffding's lemma. The resulting time series of copula parameters is captured using a heterogeneous autoregressive model which is well established in the realized variance literature.

Realized copula allows to move beyond the usual Gaussian structure which realized variance models typically adopt. In an out-of-sample VaR backtesting analysis, we demonstrate the relevance of this feature. Comparing our approach with a rolling window and an adaptive change point algorithm (both estimated for daily data) and two classical multivariate realized variance based benchmark models (matrix log transformation, Cholesky factorization), we find that models adopting a multivariate Gaussian structure are dominated by copula models. On the other hand, models that are only based on daily data appear to be too sluggish to respond to structural shifts in the economy. Realized copula unites advantages of both modeling approaches in being highly responsive to shocks in the economic system, but at the same time allowing for non-trivial forms of tail-dependence. Both features are most crucial for accurate risk-management and portfolio optimization.

Our empirical results demonstrate that judicious combinations of low and high frequency information, as pioneered by Engle and Gallo (2006) and Ghysels et al. (2006), can generate substantial improvements in the out-of-sample forecasting accuracy, see also Hautsch et al. (2011) for a recent account in portfolio allocation. It would therefore be desirable to carry the approach to larger dimensions than the two- and three-dimensional cases considered. While technically possible, such a model would still have a single copula parameter and thus come at the cost of a very strong homogeneity assumption, which presumably one does not want to maintain in a high-dimensional setting. This issue could be addressed by using richer, yet still parsimoniously parametrized copulae, such as hierarchical Archimedean copulae (Whelan; 2004; Härdle et al.; 2010; Savu and Trede; 2010) or vine copulae (Joe; 1996; Bedford and Cooke; 2002). As a critical challenge of such a realized copula framework, one would not only need to estimate the copula shape parameters, but also has to simultaneously identify the embedded copula structure. We therefore suggest this topic for future research.

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A Estimation of realized variance

On the intra-day level, we adopt the model framework used in the realized variance literature described e.g. in Barndorff-Nielsen et al. (2011). We suppose that during the trading day $[t-1;t]$ the log-price process follows a Brownian semimartingale (BSM) which is superimposed by market micro structure noise. More precisely, denote the d-dimensional efficient equilibrium price process by

$$Y_t = Y_{t-1} + \int_{t-1}^{t} \sigma_u \, dW_u$$

where σ_t càdlàg is a volatility matrix process and W_t is a d-dimensional vector of independent Brownian motions. At observation times $t-1 = \tau_0 < \tau_1 < \ldots < \tau_N = t$, we record the efficient price plus a covariance stationary additive component

$$P_{\tau_i} \;=\; Y_{\tau_i} + U_{\tau_i}\;,$$

where $\mathsf{E}(U_{\tau_i}) = 0$ and $\sum_h |h\Omega_h| < \infty$ with $\Omega_h = \mathsf{Cov}(U_{\tau_i}, U_{\tau_{i-h}})$ for $h > 0$. The purpose is to estimate the quadratic variation of Y, i.e. $[Y]_{t,t-1} = \int_{t-1}^{t} \Sigma_u du$ with $\Sigma = \sigma\sigma^\top$. An ex-post estimate of integrated covariance $[Y]_{t,t-1} = \int_{t-1}^{t} \Sigma_u du$ will be called *realized covariance* and was denoted by H_t in Section 2.1. For the estimation of $[Y]$ from discrete, non synchronous, and noisy price observations, we use the realized kernel estimator, which is described in the following.

The multivariate realized kernel method introduced by Barndorff-Nielsen et al. (2011) yields consistent, positive semi-definite estimates of the covariation of equity prices in the presence of noise and non-synchronous trading. Data synchronization is accomplished via refresh time sampling (RTS), see also Harris et al. (1995) and Hautsch et al. (2009). The refresh times are defined as the times needed for all the assets in the portfolio to trade or to refresh posted prices. After all assets being traded, the most recent price is used to form the RTS time scale. Formally, the first refresh time is defined as $\tau_1^* = \max\{\tau_{1,1}, \ldots, \tau_{d,1}\}$. All subsequent refresh times are defined as $\tau_{i+1}^* = \arg\min\{\tau_{j,k_j} | \tau_{j,k_j} > \tau_i^*,\ \forall k_j = 1, \ldots, N_j;\ j \in 1 \ldots d\}$, where N_j denotes the number of price observations made for asset j. This synchronization leads to a new high-frequency vector of returns $p_i = P_{\tau_i^*} - P_{\tau_{i-1}^*}$, where $i = 1, \ldots, n$, and n is the number of RTS observations. With RTS, the sample size n of retained data depends on the degree of non-synchronicity between assets.

The multivariate realized kernel estimator is defined as

$$K(P) = \sum_{h=-H}^{H} k\left(\frac{|h|}{H+1}\right) \Gamma_h,$$

with Γ_h being a matrix of autocovariances given by

$$\Gamma_h = \begin{cases} \sum_{j=|h|+1}^{n} p_j p_{j-h}^\top, & h \geq 0 \\ \sum_{j=|h|+1}^{n} p_{j-h} p_j^\top, & h < 0 \end{cases},$$

and $k(x)$ being the Parzen kernel

$$k(x) = \begin{cases} 1 - 6x^2 + 6x^3 & 0 \leq x \leq 1/2 \\ 2(1-x)^3 & 1/2 \leq x \leq 1 \\ 0 & x > 1 \end{cases}.$$

In the estimation of the multivariate bandwidth parameter H we strictly follow the suggestions outlined in Barndorff-Nielsen et al. (2009).

B Tables and figures

family	ϕ_θ	${\phi_\theta}^{-1}$	domain	f_τ
Gumbel	$\exp\{-x^{1/\theta}\}$	$(-\log x)^\theta$	$\theta \in [1, \infty)$	$1 - 1/\theta$
Clayton	$(\theta x + 1)^{-1/\theta}$	$(x^{-\theta} - 1)/\theta$	$\theta \in (0, \infty)$	$\theta/(2 + \theta)$
Nelsen (2006) (4.2.2)	$1 - x^{1/\theta}$	$(1 - x)^\theta$	$\theta \in [1, \infty)$	$(\theta - 2)/\theta$
Nelsen (2006) (4.2.3)	$(1 - \theta)/(e^x - \theta)$	$\log \frac{1-\theta(1-x)}{x}$	$\theta \in [0, 1)$	$1 - 2\frac{\theta+(\theta-1)^2 \log(1-\theta)}{3\theta^2}$
Frank	$-\frac{1}{\theta} \log\{e^{-x}(e^{-\theta} - 1) + 1\}$	$-\log \frac{e^{-\theta x}-1}{e^{-\theta}-1}$	$\theta \in (0, \infty)$	$1 + 4\{D_1(\theta) - 1\}/\theta$

Table 1: Generator function ϕ_θ, the inverse ${\phi_\theta}^{-1}$, and Kendall's tau f_τ for selected Archimedean copulae, see Nelsen (2006), Table 4.1. D_1 is the Debye function of order 1, $D_1(\theta) = \int_0^\infty t/\theta(e^t - 1)dt$.

	min.	median	mean	max.	std.	skewness	kurtosis
Google	-0.123	-0.143e-3	0.254e-3	0.182	0.022	0.331	7.192
Oracle	-0.103	0.000e-3	0.743e-3	0.122	0.021	0.393	4.729
IBM	-0.061	0.604e-3	0.475e-3	0.109	0.015	0.166	4.712
Pfizer	-0.112	-0.690e-3	-0.250e-3	0.096	0.017	-0.236	6.233
Exxon	-0.150	0.290e-3	0.182e-3	0.158	0.019	0.126	12.546

Table 2: Descriptive statistics of the daily log return data of the stocks under consideration. Sample period 9 July 2007 to 31 December 2010.

	min.	median	mean	max.	std.
RV(Google)	2.277e-05	1.714e-04	2.503e-04	0.003	0.269e-3
RV(IBM)	1.431e-05	1.048e-04	1.704e-04	0.001	0.180e-3
RV(Oracle)	5.220e-05	2.208e-04	3.082e-04	0.002	0.253e-3
RC(Google,IBM)	1.978e-06	5.758e-05	9.112e-05	0.001	0.110e-3
RC(Google,Oracle)	5.359e-06	7.628e-05	1.112e-04	0.001	0.128e-3
RC(IBM,Oracle)	2.106e-06	6.749e-05	1.015e-04	0.001	0.113e-3
$\hat{\theta}_t^{\mathrm{MM}}$ (rGumbel)	1.073e-00	1.353e-00	1.375e-00	1.935	0.146
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (rGumbel)	1.078e-00	1.361e-00	1.382e-00	1.937	0.144
$\hat{\theta}_t^{\mathrm{MM}}$ (Clayton)	1.475e-01	7.028e-01	7.538e-01	1.986	0.304
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (Clayton)	1.554e-01	7.225e-01	7.645e-01	1.875	0.288

Table 3: Descriptive statistics of the realized kernels (variances and covariances) and realized copulae (method of moments and ad hoc for Clayton and rotated Gumbel copulae) of the Google-IBM-Oracle portfolio.

	min.	median	mean	max.	std.
RV(IBM)	1.474e-05	1.014e-04	1.704e-04	0.194e-04	1.820e-04
RV(Pfizer)	2.819e-05	2.067e-04	2.837e-04	0.311e-04	2.467e-04
RV(Exxon)	2.455e-05	1.281e-04	1.810e-04	0.229e-04	1.786e-04
RC(IBM,Pfizer)	-1.550e-06	4.069e-05	6.553e-05	0.161e-04	9.599e-05
RC(IBM,Exxon)	4.231e-08	5.198e-05	8.442e-05	0.111e-04	1.010e-04
RC(Pfizer,Exxon)	-3.858e-06	4.691e-05	7.187e-05	0.112e-04	8.744e-05
$\hat{\theta}_t^{\mathrm{MM}}$ (rGumbel)	1.011e-00	1.264e-00	1.286e-00	1.788e-00	1.275e-01
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (rGumbel)	1.012e-00	1.284e-00	1.303e-00	1.793e-00	1.302e-01
$\hat{\theta}_t^{\mathrm{MM}}$ (Clayton)	2.271e-02	5.247e-01	5.724e-01	1.644e-00	2.599e-01
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (Clayton)	2.473e-02	5.672e-01	6.057e-01	1.586e-00	2.604e-01

Table 4: Descriptive statistics of the realized kernels (variances and covariances) and realized copulae (method of moments and ad hoc for Clayton and rotated Gumbel copulae) of the IBM-PFE-XOM portfolio.

	β_0	β_{D}	β_{W}	β_{M}
Google	-0.597 (0.234)	0.559 (0.054)	0.254 (0.074)	0.121 (0.054)
IBM	-0.586 (0.232)	0.509 (0.055)	0.254 (0.078)	0.175 (0.059)
Oracle	-0.705 (0.260)	0.505 (0.054)	0.155 (0.079)	0.259 (0.066)
$\hat{\theta}_t^{\mathrm{MM}}$ (rGumbel)	0.047 (0.015)	0.354 (0.056)	0.449 (0.083)	0.039 (0.073)
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (rGumbel)	0.048 (0.016)	0.354 (0.056)	0.454 (0.083)	0.036 (0.073)
$\hat{\theta}_t^{\mathrm{MM}}$ (Clayton)	-0.068 (0.022)	0.364 (0.056)	0.444 (0.084)	0.047 (0.077)
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (Clayton)	-0.063 (0.020)	0.363 (0.056)	0.450 (0.084)	0.043 (0.077)

Table 5: Parameters of the in-sample HAR models for the Google-IBM-Oracle portfolio. Standard errors in brackets.

	β_0	β_{D}	β_{W}	β_{M}
IBM	-0.599 (0.236)	0.492 (0.056)	0.255 (0.079)	0.190 (0.061)
Pfizer	-0.523 (0.241)	0.488 (0.055)	0.238 (0.080)	0.216 (0.065)
Exxon	-0.734 (0.258)	0.418 (0.056)	0.411 (0.078)	0.092 (0.059)
$\hat{\theta}_t^{\mathrm{MM}}$ (rGumbel)	0.048 (0.016)	0.270 (0.056)	0.441 (0.090)	0.085 (0.088)
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (rGumbel)	0.049 (0.016)	0.280 (0.056)	0.437 (0.090)	0.084 (0.088)
$\hat{\theta}_t^{\mathrm{MM}}$ (Clayton)	-0.139 (0.047)	0.255 (0.056)	0.466 (0.099)	0.126 (0.102)
$\hat{\theta}_t^{\mathrm{ad\ hoc}}$ (Clayton)	-0.124 (0.043)	0.262 (0.056)	0.468 (0.100)	0.121 (0.103)

Table 6: Parameters of the in-sample HAR models for the IBM-Pfizer-Exxon portfolio. Standard errors in brackets.

model \ α	0.01	0.05	0.1
LCP $m_0 = 40$ (rGumbel)	0.0258 (0.028)	0.0369 (0.3003)	0.0775 (0.2002)
ROL $w = 250$ (rGumbel)	0.0221 (0.083)	0.0332 (0.1779)	0.0664 (0.0511)
MM (rGumbel)	**0.0148** (0.462)	0.0590 (0.5062)	**0.0996** (0.9838)
ad hoc (rGumbel)	**0.0148** (0.462)	0.0590 (0.5062)	**0.0996** (0.9838)
LCP $m_0 = 40$ (Clayton)	0.0258 (0.028)	**0.0517** (0.9007)	0.0849 (0.3953)
ROL $w = 250$ (Clayton)	0.0221 (0.083)	0.0443 (0.6598)	0.0738 (0.1334)
MM (Clayton)	**0.0148** (0.462)	0.0554 (0.6909)	0.0959 (0.8227)
ad hoc (Clayton)	**0.0148** (0.462)	0.0554 (0.6909)	0.0886 (0.5229)
Gauss (Bauer and Vorkink; 2010)	0.0406 (1e-04)	0.0738 (0.0921)	0.1218 (0.2463)
Gauss (Chiriac and Voev; 2011)	0.0369 (6e-04)	0.0812 (0.0301)	0.1255 (0.1772)

Table 7: VaR performance ($\hat{\alpha}$) for the Google-IBM-Oracle portfolio. p-values of the Kupiec test in brackets. m_0 denotes the smallest possible interval of homogeneity for the LCP method, w is the window width for the rolling window approach.

model \ α	0.01	0.05	0.1
LCP $m_0 = 40$ (rGumbel)	**0.0111** (0.8618)	0.0443 (0.6598)	0.0701 (0.0847)
ROL $w = 250$ (rGumbel)	**0.0111** (0.8618)	0.0332 (0.1779)	0.0517 (0.0038)
MM (rGumbel)	0.0074 (0.6494)	0.0554 (0.6909)	**0.1033** (0.8561)
ad hoc (rGumbel)	0.0074 (0.6494)	**0.0517** (0.9007)	**0.1033** (0.8561)
LCP $m_0 = 40$ (Clayton)	0.0185 (0.2110)	0.0554 (0.6909)	0.0923 (0.6670)
ROL $w = 250$ (Clayton)	**0.0111** (0.8618)	0.0369 (0.3003)	0.0590 (0.0157)
MM (Clayton)	0.0074 (0.6494)	0.0554 (0.6909)	**0.1033** (0.8561)
ad hoc (Clayton)	0.0074 (0.6494)	0.0554 (0.6909)	**0.1033** (0.8561)
Gauss (Bauer and Vorkink; 2010)	0.0369 (0.0006)	0.0738 (0.0921)	0.1107 (0.5631)
Gauss (Chiriac and Voev; 2011)	0.0406 (0.0001)	0.0738 (0.0921)	0.1144 (0.4390)

Table 8: VaR performance ($\hat{\alpha}$) for the IBM-Pfizer-Exxon portfolio. p-values of the Kupiec test in brackets. m_0 denotes the smallest possible interval of homogeneity for the LCP method, w is the window width for the rolling window approach.

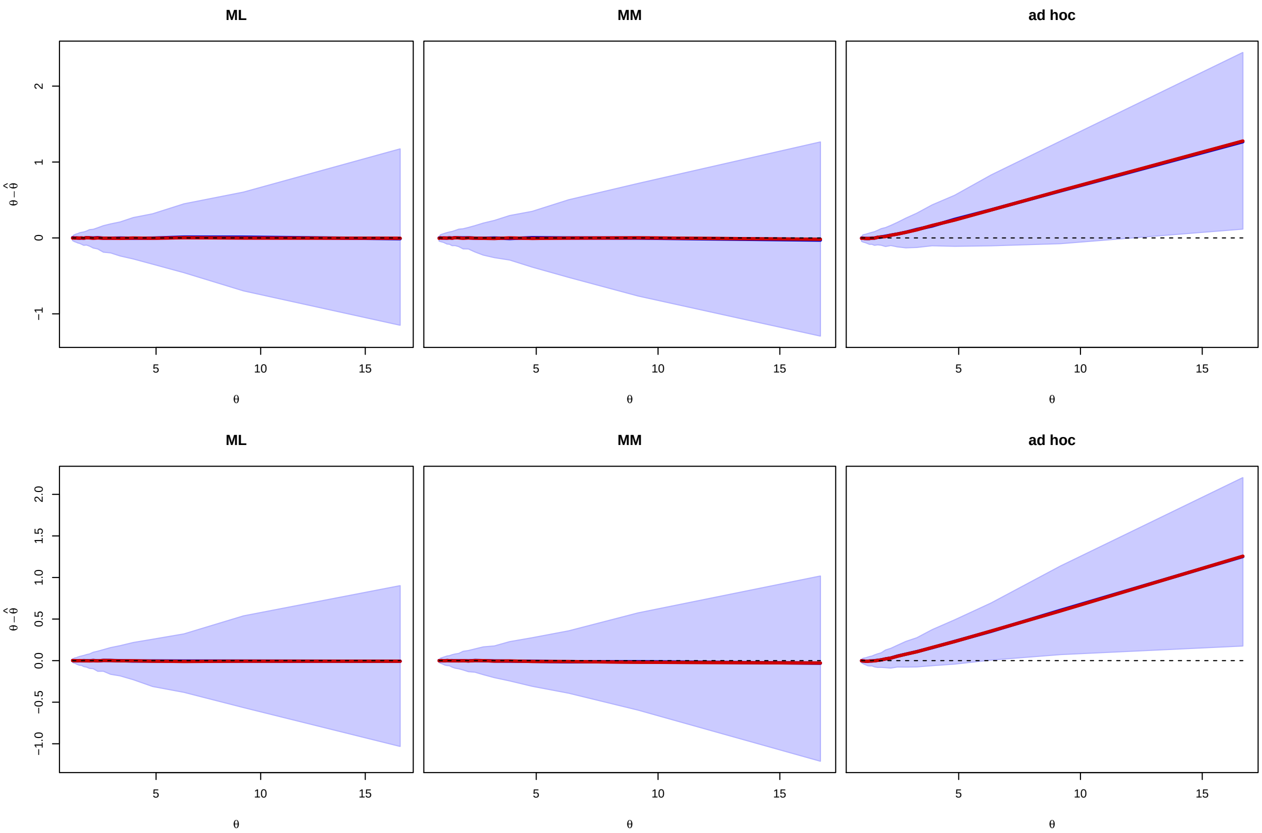

Figure 1: Monte Carlo simulation. Top panel: two-dimensional case with Gumbel copula. Lower panel: three-dimensional case with Gumbel copula. Left plot: ML-IFM estimator. Middle plot: moment estimator based on Hoeffding's lemma. Right plot: ad-hoc estimator based on linear correlation. Numbers are computed based on 1000 estimates of the respective parameters. Each parameter estimate is based on a sample size of 1000 randomly drawn observations. Red line is the mean of the differences between estimate and true value, blue the median (hardly visible). Shaded area is the 95% interval based on the 1000 repetitions.

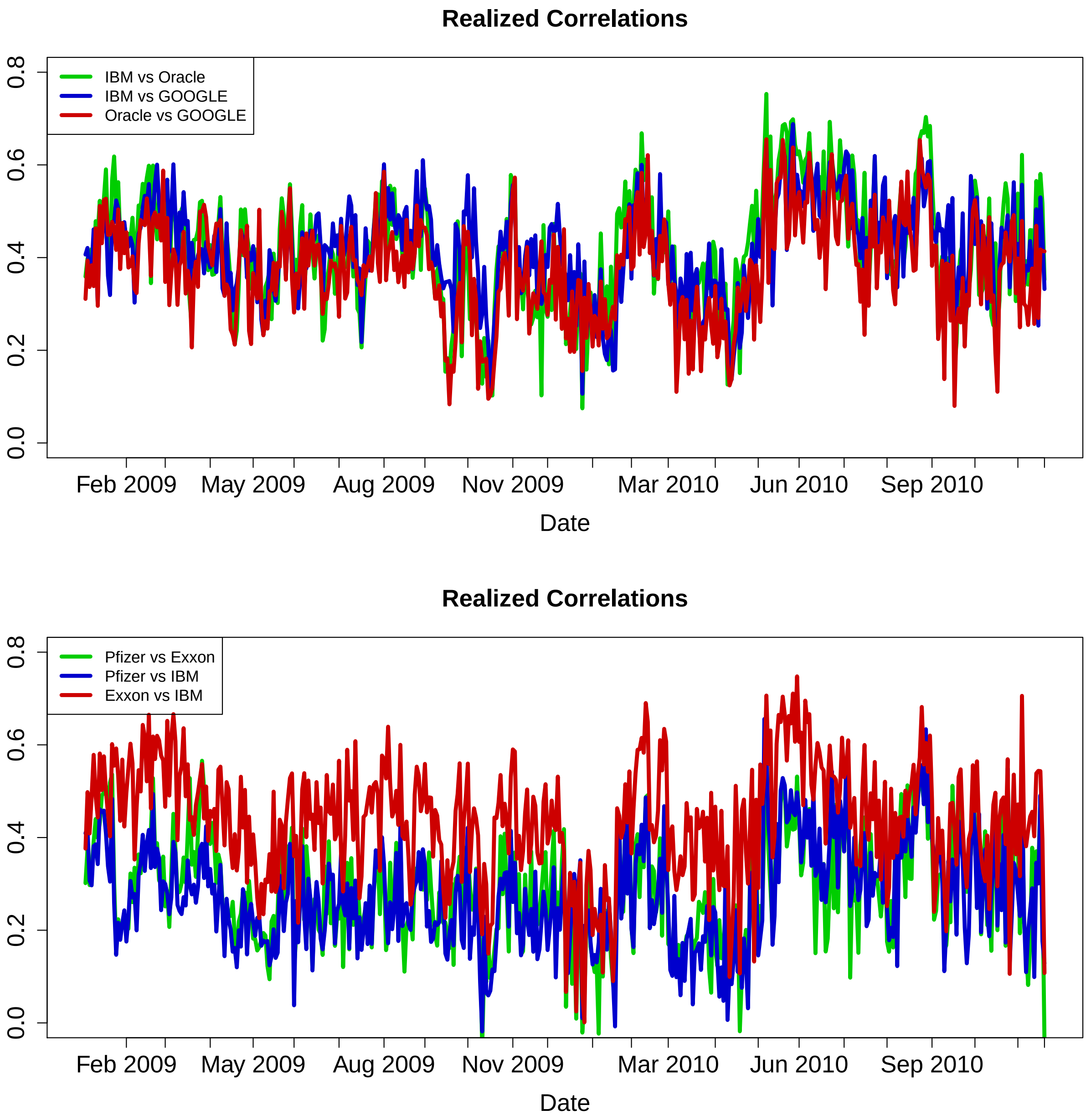

Figure 2: Realized correlation estimated by means of the realized kernel estimator between 2 Jan. 2009 to 31 Dec. 2010. Top panel: Google-IBM-Oracle portfolio. Lower panel: IBM-Pfizer(PFE)-Exxon(XOM) portfolio.

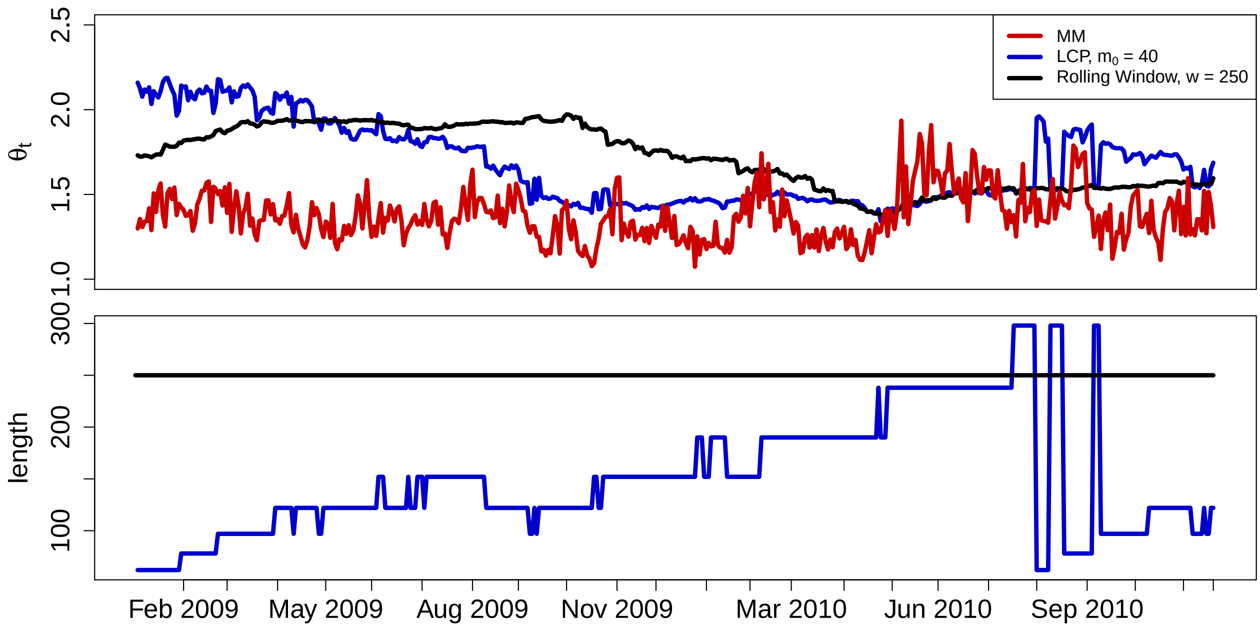

Figure 3: Estimated copula parameters for the case of the rotated Gumbel copula on the Google-IBM-Oracle portfolio. Top panel: the naïve rolling window estimates (black line), the LCP estimates (blue line), and realized copula estimates based on Hoeffding's lemma (red line). Lower panel: estimated interval length of the LCP procedure (blue line) and (constant) interval length of naïve rolling window approach (black line). LCP (blue) is run with $m_0 = 40$, which is the smallest possible interval of homogeneity.

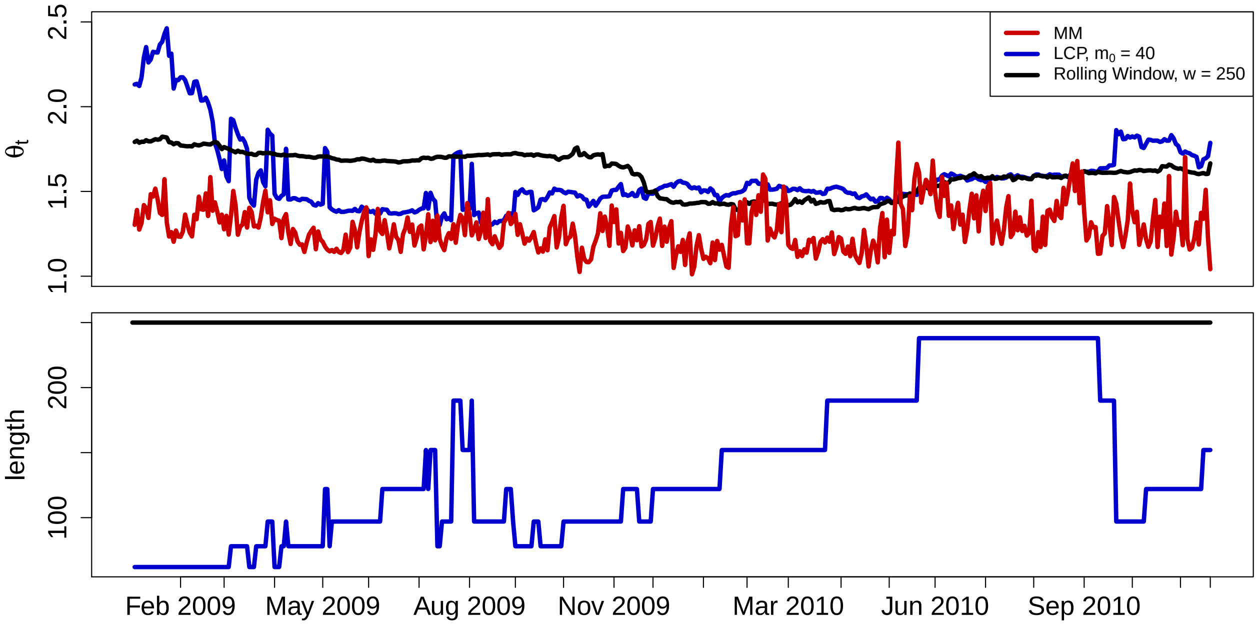

Figure 4: Estimated copula parameters for the case of the rotated Gumbel copula on the IBM-Pfizer-Exxon portfolio. Top panel: the naïve rolling window estimates (black line), the LCP estimates (blue line), and realized copula estimates based on Hoeffding's lemma (red line). Lower panel: estimated interval length of the LCP procedure (blue line) and (constant) interval length of naïve rolling window approach (black line). LCP (blue) is run with $m_0 = 40$, which is the smallest possible interval of homogeneity.

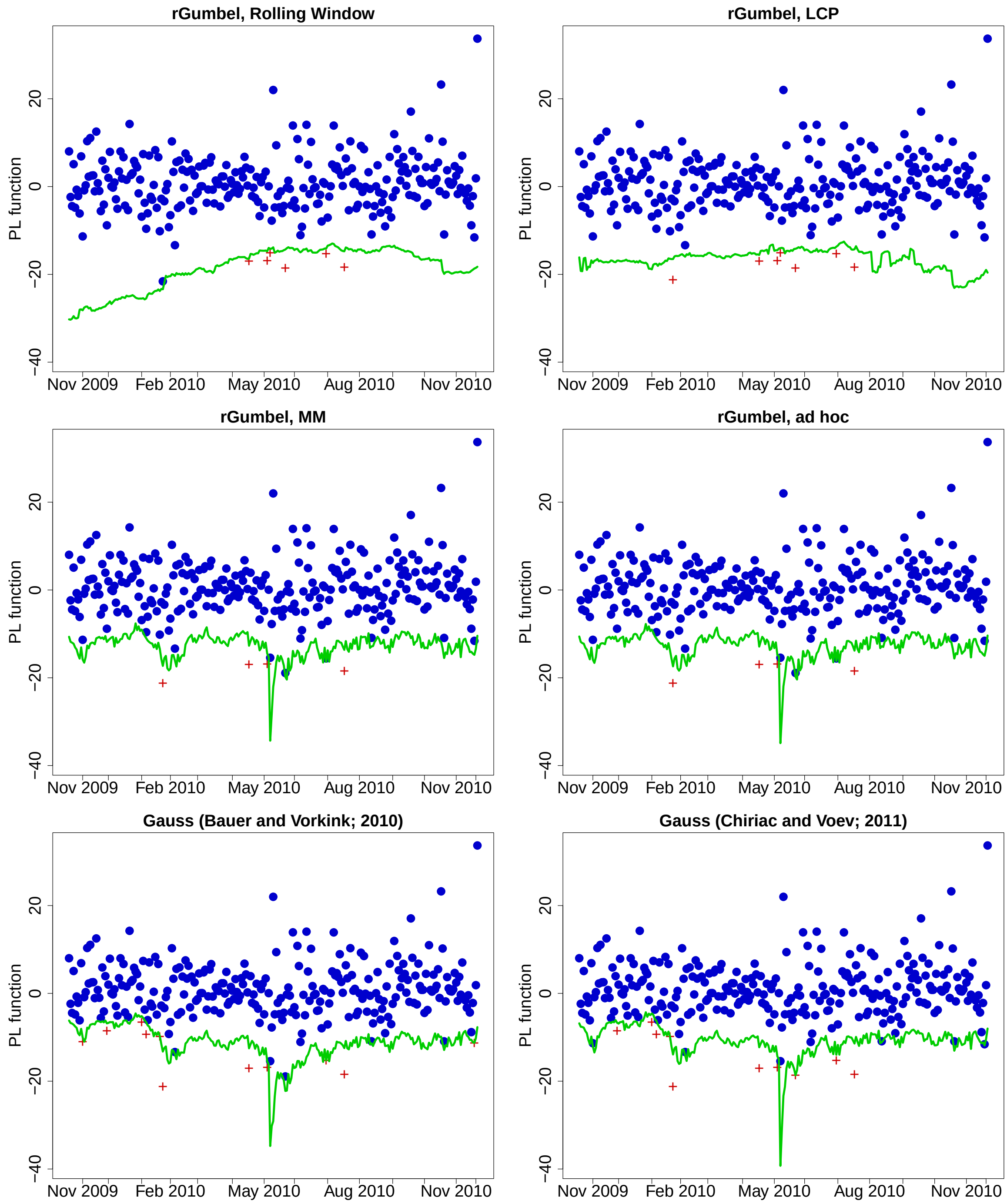

Figure 5: Exceedances plot for the VaR(0.01) for the rotated Gumbel copula on the Google-IBM-Oracle portfolio. From top left to lower right: Rolling window (window width $w = 250$), LCP method (smallest possible interval of homogeneity $m_0 = 40$), realized copula (Hoeffding's lemma), realized copula (ad hoc), Bauer and Vorkink (2010) and Chiriac and Voev (2011). Profit & loss (blue dots), the lower VaR(0.01) (green solid line), exceedances (red crosses).

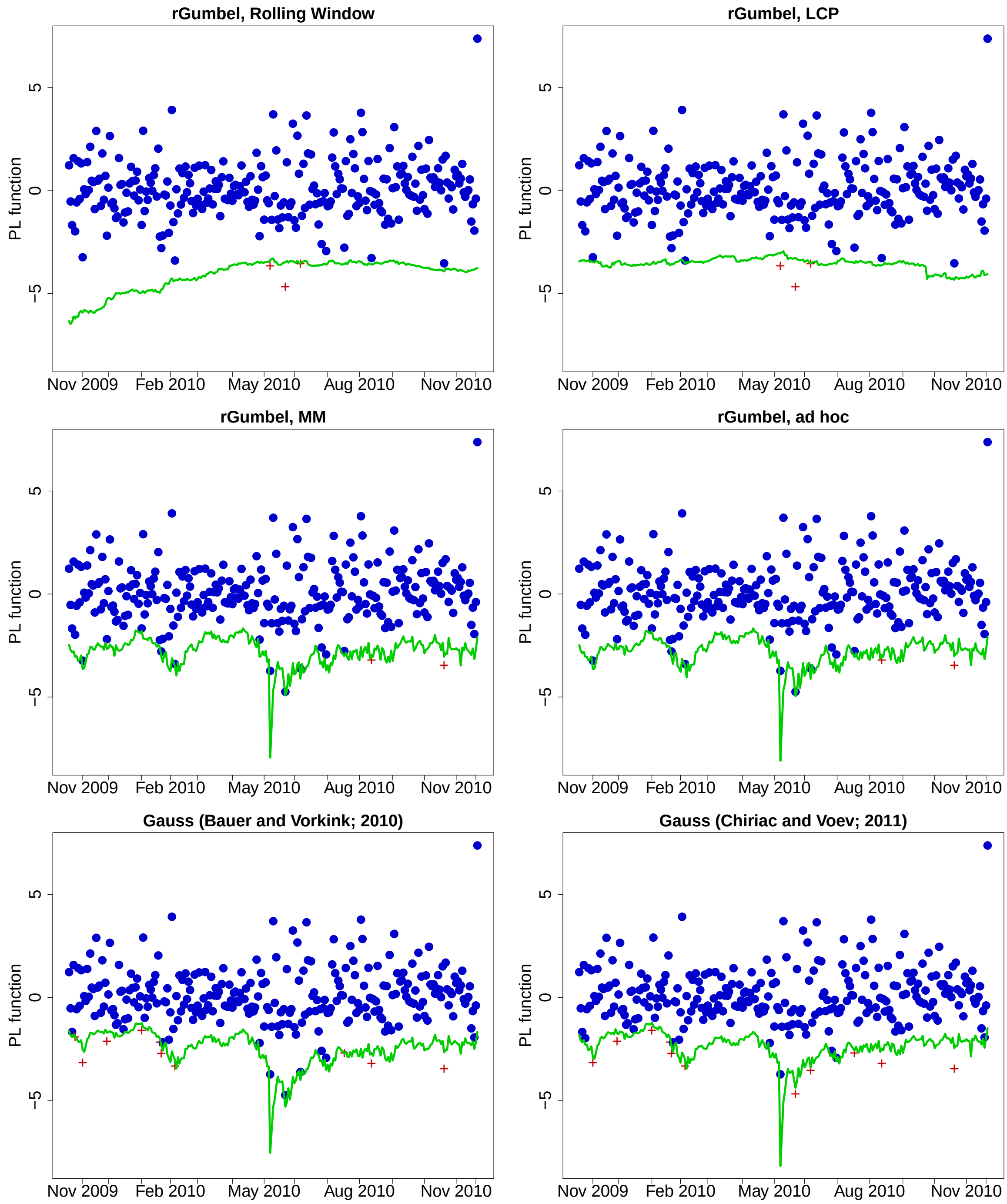

Figure 6: Exceedances plot for the VaR(0.01) for the rotated Gumbel copula on the IBM-Pfizer-Exxon portfolio. From top left to lower right: Rolling window (window width $w = 250$), LCP method (smallest possible interval of homogeneity $m_0 = 40$), realized copula (Hoeffding's lemma), realized copula (ad hoc), Bauer and Vorkink (2010) and Chiriac and Voev (2011). Profit & loss (blue dots), the lower VaR(0.01) (green solid line), exceedances (red crosses).

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".