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Reference Dependent Preferences and the EPK Puzzle

Maria Grith*
Wolfgang Karl Härdle*
Volker Krätschmer**



* Humboldt-Universität zu Berlin, Germany

** University of Duisburg-Essen, Germany

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Reference Dependent Preferences and the EPK Puzzle ^{*}

Maria Grith,[†] Wolfgang K. Härdle,[‡] Volker Krätschmer,[§]

Abstract

Supported by several recent investigations, the empirical pricing kernel (EPK) puzzle might be considered a stylized fact. Based on an economic model with state dependent preferences for the financial investors, we want to emphasize a microeconomic view that succeeds in explaining the puzzle. We retain the expected utility framework in a one period model and illustrate the case when the state is defined with respect to a reference point. We further investigate how the model relates the shape of the EPK to the economic conditions.

KEYWORDS: Pricing kernel, aggregate agent, empirical pricing kernel, EPK puzzle, state dependent utilities, reference dependent utilities, reference points. JEL CLASSIFICATION: D04, D53, C02, G13
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[†]Ladislav von Bortkiewicz Chair of Statistics, School of Business and Economics, Humboldt-Universität zu Berlin, Spandauer Str. 1, D-10178 Berlin; e-mail: gritmari@wiwi.hu-berlin.de.

[‡]Center for Applied Statistics and Economics, Ladislav von Bortkiewicz Chair of Statistics, School of Business and Economics, Humboldt-Universität zu Berlin, Spandauer Str. 1, D-10178 Berlin; e-mail: haerdle@wiwi.hu-berlin.de.

[§]Center for Applied Stochastics, Faculty of Mathematics, University of Duisburg-Essen, Forsthausweg 2, D-47057 Duisburg; e-mail: Volker.kraetschmer@uni-due.de.

1 Introduction

The empirical pricing kernel puzzle emerged as an empirical phenomenon in the financial markets, particularly with respect to the prices of European options written on the underlying stock index. Several authors have investigated if such patterns of the EPK can be justified in a general equilibrium setting and if the observed prices can be the outcome of investors' optimal behavior. The starting point for many of the investigations is settled within similar economic models that assume a representative agent in financial markets whose preferences have classical expected utility representation. Additionally, the risk neutral valuation principle is supposed to be valid for the financial markets by means of pricing kernels. If the pricing kernels represent state contingent equilibrium prices they might be identified with the v. Neumann-Morgenstern marginal utility indices of the representative agent.

Starting with Ait-Sahalia and Lo (2000), Jackwerth (2000), Engle and Rosenberg (2002), different econometric methods have been applied to estimate pricing kernels with varying underlying models for the financial markets. It turned out as a common result, that the estimates, the so called empirical pricing kernels (EPK), have non-monotonic shape regardless of the used data sets. Typically, we find either a U-shaped pricing kernel or a hump-shaped pricing kernel. In either cases the empirical kernels fail to be monotone, contrasting the standard theory of expected utility. This is what we shall call the *EPK puzzle*. Based on conditional estimates of the risk neutral and physical densities, it appears that periods of unusual low and stable realized and risk neutral volatility feature a hump shaped EPK, whereas during periods of high volatility the estimates look U-shaped. Several studies report the shape of the pricing kernel as being hump-shaped for most months between 2004 and 2007. This holds for both the German DAX 30 index Giacomini and Härdle (2008); Grith et al. (2012) and the American S&P 500 index Barone-Adesi et al. (2013); Beare and Schmidt (2012); Polkovnichenko and Zhao (2012).

Monotonicity tests for the EPK have been proposed by Golubev et al. (2008) who construct test for the local concavity of the utility function and Härdle et al. (2012) who build uniform confidence bands for the empirical pricing kernel; they apply the test to DAX 30 index EPK. Beare and Schmidt (2012) test the concavity of the ordinal dominance curve associated with the risk neutral and physical distributions associated with S&P 500 index. Typically, the null hypothesis of nonincreasing EPK was rejected.

Recent econometric models point at volatility as a state variable, that help explain the observed non-monotonocities in the pricing kernel. Chabi-Yo (2012); Song and Xiu (2012) find that, consistent with economic theory, the pricing kernel decreases in the market index return, conditional on the market volatility. As such, unconditional estimates of the PK may appear U-shaped. Christoffersen et al. (2012), propose an augmented Heston and Nandi (2000) model that allows for U-shaped pricing kernel in a one period model by introducing a variance preference parameter.

There is a large body of literature that investigates the mechanisms through which a locally increasing region in the pricing kernel can occur. Hens and Reichlin (2012) conduct a systematic analysis of the EPK puzzle by relaxing in turn the assumptions embedded in the standard expected utility models: complete markets, risk-averse investors and correct beliefs. They calibrate a hump-shaped pricing kernel and find that incomplete markets can alone explain the puzzle. The authors rule out local risk-proclivity, that works only as a 'pathological example with a few states'. With homogeneous agents, misestimation of objective probability in isolations misses some essential features of the data. This finding is in line with Ziegler (2007).

Closely related to the latter interpretation, heterogeneity in beliefs about the future realizations of the returns occurs in several papers as a possible interpretation for the EPK puzzle. Bakshi and Madan (2008); Bakshi et al. (2010) consider an equilibrium model with short and long equity investors that is able to explain U-shaped pricing kernel; in particular, the positively sloped regions in the pricing kernel occur when some investors are shorting equities. This model is able to explain some features of the option data: decreasing negative returns in strikes of the OTM calls and the even pronounced negative returns of put options, increasing in strike prices. However, it cannot capture the positive returns of call options for high strikes as reported in Bondarenko (2003). Ziegler (2007) considers three groups of heterogeneous agents with biased beliefs about the physical density but concludes that the estimates of the mean are not realistic for the pessimistic groups. Optimism and pessimism reflect biases in the first moment of the objective probabilities; Shefrin (2008) points out that one should consider higher order biases in order to explain the empirical findings and emphasizes the bias in the second moments that leads to risk neutral and physical distribution having different variance.

Some studies argue that modifications of standard preferences are needed to explain the data. Departing from the expected utility framework, Polkovnichenko and Zhao (2012) propose a rank dependent utility model and estimate probability weighting function nonparametrically. For most of the years the estimates are inverse S-shaped, consistent with a U-shaped PK but they become S-shaped in the years 2004-2007, suggesting a hump-shaped EPK. In line with experimental findings, inverse S-shaped weighting function imply that investors tend to overweight low-probability events while underweighting the likelihood of high-probability ones. The converse holds for the S-shaped probability weighting function but the authors do not make further investigations about the differences in these treatments. Hens and Reichlin (2012) show that a combination of reasonable pessimism and inverse S-shaped weighting function can explain the hump shaped EPK.

Shefrin (2008) rationalizes the EPK puzzle in a model with mixed expected utility maximizers and agents endowed with SP/A preferences - security, potential and aspiration theory, proposed by Lopes (1987) and developed in Lopes and Oden (1999). The idea that investors are endowed with utilities that mirror their concerns for portfolio maximization also pervades our paper.

Another stream of literature that tries to rationalize the EPK puzzle considers state dependence. State dependence has been traditionally used to explain the asset pricing puzzles in equilibrium models mainly based on two utility classes: habit formation, see Constantinides (1990), Campbell and Cochrane (1999), or recursive utilities, see Epstein and Zin (2001). In these papers, one typically assumes a Markov switching process for the evolution of states and derive asset related characteristics in a consumption based model. Garcia et al. (2003) investigate recursive utility functions with state dependency in the fundamentals. Melino and Yang (2003) disentangle the roles played by state dependent intertemporal substitution and time preference in explaining the risk aversion puzzle in a model with state dependent recursive preferences. Veronesi (2004) extends the state dependent utility by assuming that the agents possess a probability distribution over their state and introduces the concept of 'belief-dependent preferences'. A first explanation for the empirical pricing kernel puzzle via state dependence has been offered by Chabi-Yo et al. (2008), who generalize the setup of Melino and Yang (2003). The crucial idea of the authors is to suppose that regime switches are inherent of the price

process of the stock market. More specifically, within a discrete time period $\{0, 1, \dots, T\}$, there are two types of price processes $(S_t^0)_{t \in \{0, \dots, T\}}$, $(S_t^1)_{t \in \{0, \dots, T\}}$ for the risky asset which have joint continuous distributions, and constitute separately together with the riskless bond arbitrage free financial markets in the sense of section 2. Furthermore, they assume a latent regime switching variables in terms of an unobservable Markov-chain $(U_t)_{t \in \{0, 1, \dots, T\}}$ of Bernoulli-distributed random variables. The observable price process $(S_t)_{t \in \{0, 1, \dots, T\}}$ is then modeled by $S_t = U_t S_t^1 + (1 - U_t) S_t^0$ for $t \in \{0, \dots, T\}$. Assuming the risk neutral valuation principle for the latent two basic financial markets and for the observable one, the authors drew a comparison of the associated pricing kernels via a simulation study. Indeed it turned out that the empirical pricing kernels in the separated financial market were nonincreasing whereas the empirical pricing kernel in the integrated financial market failed to have the property of monotonicity. Therefore the empirical pricing kernel might be explained by a switch of the price processes of the underlying in the financial market. The authors also investigate what type of conditioning - in preferences, economic fundamentals or beliefs - are more likely to explain the EPK puzzle over time. The time variant shape of the EPK is explained in Barone-Adesi et al. (2013) through optimism/pessimism and overconfidence/underconfidence defined as the difference in the first and second moments of the physical and risk neutral distribution. In this sense the authors find that the hump-shaped pricing kernel stems from a mix of optimistic overconfident and pessimistic underconfident agents. Grith et al. (2012) use the shape invariant model, a semi-parametric approach for multiple curves with shape-related nonlinear variation, to model the dynamics of the empirical pricing kernel (EPK) based on the hump feature. The approach allows to summarize the nonlinear variability with a few interpretable parameters that can be used to conduct a further analysis that links the shape of the pricing kernel to the business condition. They find that over periods of concerted negative evolution of the economic indicators, the EPK hump will move to the right in the returns space, increase its spread and shrink in vertical direction. Based on the initializing thought that regime switching is caused by changes of the investors' preferences our aim is to make the influence of these changes on the shape of the pricing kernels more explicit. We conjecture that the existing models with variance dependent component can be improved

by exploiting the time varying and possible nonmonotone relationship between returns and volatility. We apply the concept of reference points in a different context that it has been previously used in prospect theory, underlying another type of behavior that is not focused on loss aversion but performance comparative to a benchmark.

We propose a model that can accommodate both shapes of the EPK observed in the empirical literature while retaining the expected utility framework in a one period model and endow the financial investors with preferences that might be state sensitive. More technically, investors switch between two utility indexes - over terminal wealth sets - at a point that projected on the market index space we call 'reference point'. As a consequence, while the individual utility indices are concave, the market utility may have jumps in the aggregate wealth space. In equilibrium, this renders pricing kernel non-monotonic. Agents' heterogeneity with respect to their 'reference point' is summarized in the model by a distribution of the reference points. This, together with preference parameters will characterize the shape of PK.

2 Financial Market and Preferences

We consider a simple one period two-dates exchange economy model. Let $[0, T]$ be the time interval of investment in the financial market, where $t = 0$ denotes the present time and $t = T \in]0, \infty[$ the time of maturity. It is assumed that a riskless bond and a risky asset are traded in the financial market as basic securities. The price process of the riskless bond $(B_t)_{t \in [0, T]}$ is defined by $B_t = \exp(-\int_0^t r_x dx)$ via a deterministic Riemannian-integrable interest process $(r_t)_{t \in [0, T]}$. The price process of the risky asset $(S_t)_{t \in [0, T]}$ is taken to be a nonnegative semimartingale with continuously distributed marginals S_t . Discrete time models may be also subsumed to this setting. Let us further suppose that the financial market is arbitrage free in the sense that there exists an equivalent martingale measure. We further assume that the risk neutral valuation principle is valid for nonnegative payoffs $\psi(S_T)$. Hence there is an unknown Radon-Nikodym density π of a martingale measure such that the price of any random payoffs $\psi(S_T)$ is characterized by

$$\mathbb{E} [B_T^{-1} \psi(S_T) \pi]. \quad (1)$$

By factorization with some Borel-measurable \mathcal{K}_π , that we call \mathcal{K}_π pricing kernel (w.r.t. π) with $\mathbb{E}[\pi|S_T] = \mathcal{K}_\pi(S_T)$ we obtain

$$\int_0^\infty B_T^{-1} \psi(x) \mathcal{K}_\pi(x) p_{S_T}(x) dx, \quad (2)$$

where p_{S_T} denotes a density function of the distribution of S_T .

We will consider a portfolio choice problem that links risk attitudes of investors to the pricing rule of the financial markets. Within the classical framework, that assumes a representative agent, investor preferences may be represented by expected utilities $\mathbb{E}[u\{\bar{w}(S_T)\}]$ depending on the aggregate final wealth $\bar{w}(S_T)$, with v. Neumann-Morgenstern utility index u . Under some further technical conditions one can show that there is some positive β such that

$$\left. \frac{du}{dx} \right|_{x=\bar{w}(s_T)} = \beta \mathcal{K}_\pi(s_T)$$

for every realization s_T of S_T . Within this framework the pricing kernel has to be nonincreasing due to concavity of the utility index u . We shall provide a simple economic model where the pricing kernel need not to be nonincreasing. The key idea is to consider the investors preferences representable by state dependent utilities. An axiomatic justification for this concept of state dependent preferences is provided by Karni et al. (1983).

3 A Microeconomic View on the EPK puzzle

3.1 State Dependent Preferences

Let us assume that we have m investors who have exogenous initial wealth $w_{10}, \dots, w_{m0} > 0$ and stochastic financial wealth in form of nonnegative random variables $e_1(S_T), \dots, e_m(S_T)$. Without loss of generality we assume that the numeraire bond equals one. This means that all the prices are discounted. The terminal wealth $w_i(S_T)$ fulfills the individual budget constraint:

$$\int_0^\infty w_i(x) \mathcal{K}_\pi(x) p_{S_T}(x) dx \leq w_{i0} + \int_0^\infty e_i(x) \mathcal{K}_\pi(x) p_{S_T}(x) dx, \quad i = 1, \dots, m. \quad (3)$$

Financial wealth $e_i(S_T)$ at $t = T$ depends on the initial holdings of securities and the investment choice at $t = 0$. If we denote by δ_i the fraction of the portfolio invested in the risky asset, $e_i(S_T) = \delta_i(S_T - 1) + 1$ and δ_i expresses the risk exposure given initial wealth w_{i0} .

The consumers are assumed to have state dependent utilities in terms of extended expected utility preferences within the terminology of Mas-Colell et al. (1995). In particular, this means that consumer i has numerical representation of her preferences as:

$$u^i\{S_T, w(S_T)\}$$

where $u^i : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$ denotes a state dependent v. Neumann-Morgenstern utility index satisfying:

$$u^i(x, y) \in \mathbb{R} \text{ for } x \geq 0, \quad y > 0, \quad (4)$$

$$u^i(x, \cdot) \text{ is strictly increasing and strictly concave for any } x \geq 0, \quad (5)$$

$$u^i(\cdot, y) \text{ is Borel-measurable for every } y \geq 0. \quad (6)$$

If $u^i(x, \cdot)$ is continuously differentiable the usual *Inada conditions* are assumed to hold for $i = 1, \dots, m$

$$\lim_{y \rightarrow 0} \frac{du^i(x, \cdot)}{dy} \Big|_y = \infty, \quad \lim_{y \rightarrow \infty} \frac{du^i(x, \cdot)}{dy} \Big|_y = 0. \quad (7)$$

Investors choose their optimal wealth $(\bar{w}_1(S_T), \dots, \bar{w}_m(S_T))$ such that the following properties are fulfilled.

(ii) *individual optimization*: For each consumer i , $\bar{w}_i(S_T)$ solves

$$\max_{w_i(S_T)} E \left[u^i\{S_T, w_i(S_T)\} \right] \quad (8)$$

s.t. $w_i(S_T)$ satisfies individual budget constraint (3).

(i) *market clearing*:

$$\sum_{i=1}^m \bar{w}_i(S_T) = \bar{w}(S_T). \quad (9)$$

The conditions (8) and (9) describe a weak version of a *contingent Arrow Debreu equilibrium* (Dana and Jeanblanc (2003), sect. 7.1). As a by product $\bar{w}_1(S_T), \dots, \bar{w}_m(S_T)$ are Pareto optimum too, i.e. there are no $w_1(S_T), \dots, w_m(S_T)$ with $U^i\{w_i(S_T)\} \geq U^i\{\bar{w}_i(S_T)\}$ for every i and such that $U^i\{w_i(S_T)\} > U^i\{\bar{w}_i(S_T)\}$ for at least one i . By *Negeishi method* cf. Dana and Jeanblanc (2003) we may find nonnegative weight vector α s.t. the aggregate preferences have extended expected utility representation

$$E[u_\alpha\{S_T, \bar{w}(S_T)\}],$$

for the aggregate state dependent utility $u_\alpha : \mathbb{R}_+^2 \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$u_\alpha(x, y) \stackrel{\text{def}}{=} \sup_{\{y_i\}_{i=1}^m} \left\{ \sum_{i=1}^m \alpha_i u^i(x, y_i) \mid y_1, \dots, y_m \geq 0, \sum_{i=1}^m y_i \leq y \right\}.$$

These can be concluded from Lemma B.1, B.2 (cf. Appendix B). We impose a further condition on the asymptotic elasticity of the utilities that represents a minimal requirement to describe the optimal investment in terms of the marginal utilities and a pricing kernel.

$$\limsup_{y \rightarrow \infty} \frac{du^i(x, \cdot)}{dy} \Big|_y < 1 \quad \text{for any } x \geq 0 \text{ and every } i \in \{1, \dots, m\}. \quad (10)$$

The condition follows the guidelines of Kramkov and Schachermayer (1999); a similar condition appears in Dana and Jeanblanc (2003), Duffie (1996), Karatzas and Shreve (1998). We find this formulation more convenient to establish the following theorem.

Theorem 3.1 *In addition to (4) – (10) let $u^1(x, \cdot), \dots, u^m(x, \cdot)$ be twice continuously differentiable for $x \geq 0$. Then $u_\alpha(x, \cdot)$ is continuously differentiable for every realization s_T of S_T . Furthermore for any $\alpha_i > 0$ there exists some $\beta_i > 0$ such that*

$$\frac{du_\alpha(s_T, \cdot)}{dy} \Big|_{y=\bar{w}(s_T)} = \alpha_i \frac{du^i(s_T, \cdot)}{dy} \Big|_{y=\bar{w}_i(s_T)} = \alpha_i \beta_i \mathcal{K}_\pi(s_T) = \beta \mathcal{K}_\pi(s_T)$$

for every realization s_T .

The proof of Theorem 3.1 is delegated to the end of Appendix A.

Theorem 3.1 is the corner stone for linking aggregated individual preferences to the market pricing kernel with its potential nonmonotonicities. If we assume that the initial aggregate wealth sums up to zero it is reasonable to conclude that market final wealth specializes to $\bar{w}(S_T) = S_T$ if the bond is in zero net supply. Let $R_T = \frac{S_T}{S_0}$ be the return at maturity. Theorem 3.1 reads as follows in terms of relative price.

Corollary 3.2 *Let $\bar{w}(R_T) = R_T$ and let $u^1(x, \cdot), \dots, u^m(x, \cdot)$ be twice continuously differentiable for $x \geq 0$. Then under (4) – (10), $u_\alpha(x, \cdot)$ is continuously differentiable for every realization r_T , of R_T and for any $\alpha_i > 0$ there exists some $\beta_i > 0$ such that*

$$\frac{du_\alpha(r_T, \cdot)}{dy} \Big|_{y=r_T} = \alpha_i \frac{du^i(r_T, \cdot)}{dy} \Big|_{y=\bar{w}_i(r_T)} = \beta \mathcal{K}_\pi(r_T) \stackrel{\text{def}}{=} \tilde{\mathcal{K}}_\pi(r_T),$$

for $\bar{w}(R_T) = R_T$. Without loss of generality we can assume that $\beta = 1$.

3.2 Reference Dependent Preferences

The framework of state dependent utilities of the investors allows us to describe a switching behavior of them when facing a threshold or a reference. We will consider a simple case when the reference is with respect to the future realization of the market return R_T . In more detail, let us assume that each investor i is disposed of two basic continuous, strictly increasing and strictly concave utility indices $u_i^0, u_i^1 : [0, \infty[\rightarrow \mathbb{R} \cup \{-\infty\}$ with $u_i^0(y), u_i^1(y) \in \mathbb{R}$ for $y > 0$. She is changing between these indices dependent on a threshold $x_i > 0$ in the space of future returns i.e.

$$u^i \{r_T, w_i(r_T)\} = u_i^0 \{w_i(r_T)\} \mathbf{I}\{r_T \in [0, x_i]\} + u_i^1 \{w_i(r_T)\} \mathbf{I}\{r_T \in (x_i, \infty)\} \quad (11)$$

for every realization r_T of R_T . The reader may think of u_i^0, u_i^1 as utility indices representing bearish and bullish risk attitudes of investor i , and that her revealed attitudes are adapted to the prices of the financial market.

In order to simplify notations, let us assume that the thresholds are ordered by $x_1 \leq \dots \leq x_m$. There exist different competing potential representative agent groups in the market with representations of aggregate utility indices defined by

$$u_\alpha^j\{\bar{w}(R_T)\} = \sum_{k=1}^m \alpha_k u_k^0\{\bar{w}_k(R_T)\} \mathbf{I}\{k \geq j\} + \sum_{k=1}^m \alpha_k u_k^1\{\bar{w}_k(R_T)\} \mathbf{I}\{k < j\} \quad (12)$$

In view of Lemma B.1, B.2 in Appendix B they have expected utility representations

$$\mathbb{E} \left[u_\alpha^j\{\bar{w}(R_T)\} \right],$$

$j = 1, \dots, m+1$. It is now a routine exercise to verify that

$$u_\alpha(x, y) = u_\alpha^1(y) \mathbf{I}\{x \in [0, x_1]\} + \sum_{i=1}^{m-1} u_\alpha^{i+1}(y) \mathbf{I}\{x \in (x_i, x_{i+1}]\} + u_\alpha^{m+1}(y) \mathbf{I}\{x \in (x_m, \infty)\} \text{ for } x, y \geq 0.$$

As a consequence the aggregate utility index might be interpreted as expressing the hegemony of different potential representative agents. Moreover, via Corollary 3.2 we obtain for some $\beta > 0$ and any realisation r_T of R_T the expression for $\tilde{\mathcal{K}}_\pi(r_T)$ is

$$\frac{du_\alpha^1(y)}{dy} \Big|_{y=r_T} \mathbf{I}\{r_T \in [0, x_1]\} + \sum_{i=1}^{m-1} \frac{du_\alpha^{i+1}(y)}{dy} \Big|_{y=r_T} \mathbf{I}\{r_T \in (x_i, x_{i+1}]\} + \frac{du_\alpha^{m+1}(y)}{dy} \Big|_{y=r_T} \mathbf{I}\{r_T \in (x_m, \infty)\}$$

From this observation it becomes clear that the pricing kernel is nonincreasing separately on the intervals $[0, x_1[,]x_1, x_2[, \dots,]x_m, \infty[$, but it might fail to be monotone just at the switching points x_1, \dots, x_m .

3.3 Reference Points and Pricing Kernel

To illustrate this point let us assume that the distribution of R_T has $[0, \infty[$ as support, and that the investors have an identical switching point say x_1 ; the market pricing kernel has the following representation

$$\frac{du_\alpha^1(y)}{dy} \Big|_{y=r_T} \mathbf{I}\{r_T \in [0, x_1]\} + \frac{du_\alpha^{m+1}(y)}{dy} \Big|_{y=r_T} \mathbf{I}\{r_T \in (x_1, \infty)\} = \tilde{\mathcal{K}}_\pi(r_T) \quad (13)$$

for every realization r_T of R_T .

From (12) one can show that u_α^j inherits the properties of utility indices u_i^0 and u_i^1 : it is continuous, strictly increasing and strictly concave and fulfills the Inada conditions. Its first derivative has an inverse F_α^j that is continuously differentiable and strictly decreasing. The application of Lemma B.1 and Proposition B.3 in Appendix B yields

$$r_T = F^1 \left(\frac{du_\alpha^1(y)}{dy} \Big|_{y=r_T} \right) = F^{m+1} \left(\frac{du_\alpha^{m+1}(y)}{dy} \Big|_{y=r_T} \right)$$

for any positive realization r_T ,

For example, let us suppose that each investor i switches between CRRA utilities $u_i^j(y) = \frac{y^{1-\gamma_i^j}}{1-\gamma_i^j}$ with $y > 0$ and Arrow-Pratt coefficients of relative risk aversion γ_i^j ($j = 0, 1; 1 > \gamma_i^0 > \gamma_i^1 > 0$). It follows that u_1^0, \dots, u_m^0 represent more risk averse attitudes than u_1^1, \dots, u_m^1 . In particular for stock returns lower or equal x_1 we have a bullish market, whereas we obtain a bearish market when stock returns exceed x_1 . For this parametrization of the utility indices, the mappings $F^j : [0, \infty) \rightarrow [0, \infty)$ are defined

$$F^j(z) = \sum_{\substack{i=1 \\ \alpha_i > 0}}^{m+1} \left(\frac{z}{\alpha_i} \right)^{\gamma_i^j} \frac{1}{\gamma_i^j} \quad (j = 0, 1)$$

If x_1 is larger than the intersection of F^1 and F^{m+1} then

$$F^1 \left(\frac{du_\alpha^1(y)}{dy} \Big|_{y=x_1} \right) = x_1 = F^{m+1} \left(\frac{du_\alpha^{m+1}(y)}{dy} \Big|_{y=x_1} \right) > F^1 \left(\frac{du_\alpha^{m+1}(y)}{dy} \Big|_{y=x_1} \right).$$

for any realization $r_T \geq x_1$. Therefore

$$\frac{du_\alpha^{m+1}(y)}{dy} \Big|_{y=x_1} > \frac{du_\alpha^1(y)}{dy} \Big|_{y=x_1}$$

That means that $\widetilde{\mathcal{K}}_\pi$ is not monotone at x_1 .

We illustrate the case of a single reference point for the following cases.

Example 1. Market utility indexes have u_α^1 and u_α^{m+1} have power representation with different aggregate constant coefficients of relative risk aversion γ_α^0 and γ_α^1 .

$$r_T^{-\gamma_\alpha^0} \mathbf{I}\{r_T \in [0, x_1]\} + r_T^{-\gamma_\alpha^1} \mathbf{I}\{r_T \in (x_1, \infty)\} = \widetilde{\mathcal{K}}_\pi(r_T)$$

Example 2. Market utility indexes u_α^1 and u_α^{m+1} have power representation with equal aggregate constant coefficients of relative risk aversion γ_α but differ by a multiplicative constant $b > 1$.

$$r_T^{-\gamma_\alpha} \mathbf{I}\{r_T \in [0, x_1]\} + b r_T^{-\gamma_\alpha} \mathbf{I}\{r_T \in (x_1, \infty)\} = \widetilde{\mathcal{K}}_\pi(r_T)$$

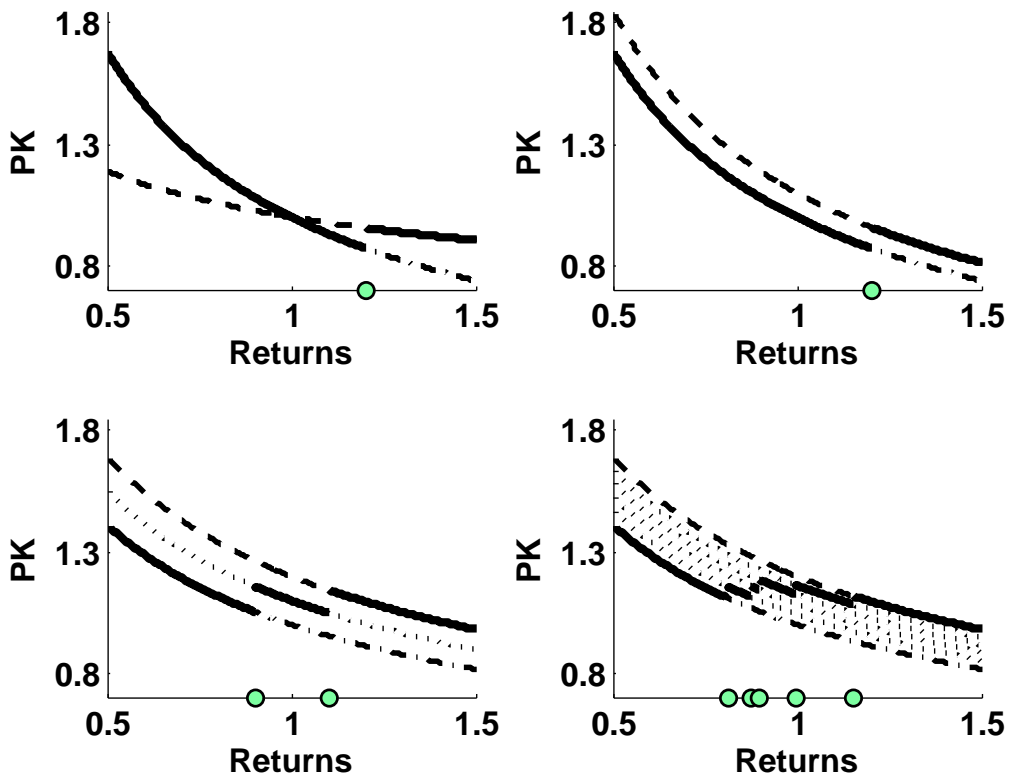


Figure 1: $\frac{du_\alpha(r_T)}{dr_T}$ (solid), $\frac{du_\alpha^j(r_T)}{dr_T}$ (dotted), $\frac{du_\alpha^l(r_T)}{dr_T}$ (dashed-dotted) and $\frac{du_\alpha^{m+1}(r_T)}{dr_T}$ (dashed)

A graphical illustration for these example is in figure 1: left panel top for $\gamma_\alpha^0 = 0.75$ and $\gamma_\alpha^1 = 0.25$ and $x_1 = 1.2$; a jump of similar size is depicted in the right upper panel of the same figure for the case when utilities differ just by a constant $u_i^1 = bu_i^0$ with $b = 1.2$ and $\gamma_\alpha = 0.75$.

Next, we exemplify the case of investors with heterogeneous reference points x_i . For exposition purposes we will assume that the investors are equally important, that is $\alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha$. In a simple case, we assume that all agents switch between the same two utility indices $u_i^j(y) = u^j(y)$, ($j = 0, 1$) for all $i = 1, \dots, m$. Let us denote

$$F(r_T) = \frac{1}{m} \sum_{i=1}^m \mathbf{I}\{r_T \in (0, x_i]\}$$

the cumulative distribution function of the reference points; F is basically the share of agents that have preferences described by u^1 at the realization r_T . The interpretation of the ordered reference points is the following: for $x_1 < x_2$ we will say the investor 1 is more optimistic than the agent 2. The degree of heterogeneity of the agents with respect to their reference points is an indicator for market uncertainty. This point will be extended upon in section 6.

Example 3. We exemplify with the individual utility functions u^j , $j = 0, 1$.

$$u^j(y) = \begin{cases} b_j \frac{y^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\ b_j \log(y) & \text{if } \gamma = 1 \end{cases}$$

The positive constants $b_0 < b_1$ retain the relationship between u^0 and u^1 in the previous example; in that sense b_1 represent bullish attitudes. Given our parametric specifications for the utility indices and F we can rewrite the formulas for $\tilde{\mathcal{K}}_\pi(r_T)$ developed in section 3.2 as

$$\tilde{\mathcal{K}}_\pi(r_T) = \left[\frac{r_T}{\{1 - F(r_T)\} b_0^{\frac{1}{\gamma}} + F(r_T) b_1^{\frac{1}{\gamma}}} \right]^{-\gamma}, \quad (14)$$

for every possible realization r_T of R_T . We illustrate the results in Figure 1 for $\gamma^0 = \gamma^1 = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 2$ (lower panel left) and $m = 5$ respectively (lower panel right).

Example 4. If agents have homogeneous, state dependent CRRA preferences

$$u^j(y) = \begin{cases} \frac{y^{1-\gamma^j}}{1-\gamma^j} & \text{if } \gamma^j > 0 \text{ and } \gamma^j \neq 1 \\ \log(y) & \text{if } \gamma^j = 1 \end{cases}$$

the market pricing kernel can be written as a power function

$$\tilde{\mathcal{K}}_\pi(r_T) = b(r_T) r_T^{-\gamma_\alpha(r_T)} \quad (15)$$

with non-constant coefficient of relative risk aversion $\gamma_\alpha(r_T)$

$$\gamma_\alpha(r_T) = r_T \left[\{1 - F(r_T)\} \frac{\bar{w}^0}{\gamma^1} + F(r_T) \frac{\bar{w}^1}{\gamma^0} \right]^{-1}$$

and

$$b(r_T) = \left[\left\{ 1 - F(r_T) b_0^{\frac{1}{\gamma_0}} \right\} \frac{\bar{w}^0}{r_T} + F(r_T) b_1^{\frac{1}{\gamma_1}} \frac{\bar{w}^1}{r_T} \right]^{\gamma_\alpha(r_T)}$$

for \bar{w}^j the optimal wealth path in state j , $j = 0, 1$.

Example 5. Introducing state dependence in both b and γ results in a pricing kernel of the form (15) with $b(r_T) = \left[\{1 - F(r_T)\} \frac{\bar{w}^0}{r_T} + F(r_T) \frac{\bar{w}^1}{r_T} \right]^{\gamma_\alpha(r_T)}$.

A further generalization of the previous examples is possible if we consider heterogeneity of agents in CRRA, γ_i^j and/or constants b_i^j . However, then the link to F is lost. We will use notations $\mathcal{K}_{\theta, F} = \tilde{\mathcal{K}}_\pi(r_T)$ for the models described in Examples 3 through 5, for $\theta = (b, \gamma) \top$ a parameters vector describing preferences.

4 Investors' Portfolio Choice

From Corrolary 3.2 and Appendix A we can establish the relationship between the optimal terminal wealth of investor i and the market pricing kernel

$$\bar{w}_i(r_T) = I_i\left\{r_T, \frac{1}{\alpha_i} \widetilde{\mathcal{K}}_\pi(r_T)\right\} \text{ for } i = 1, \dots, m \quad (16)$$

More explicitly, given the reference dependent utility specification in equation (11)

$$I_i\left\{r_T, \frac{1}{\alpha_i} \widetilde{\mathcal{K}}_\pi(r_T)\right\} = \bar{w}_i^0(r_T) \mathbf{I}\{r_T \in [0, x_i]\} + \bar{w}_i^1(r_T) \mathbf{I}\{r_T \in (x_i, \infty)\} \quad (17)$$

where $\bar{w}_i^j(r_T) = I_i^j\left\{\frac{1}{\alpha_i} \widetilde{\mathcal{K}}_\pi(r_T)\right\}$, for $I_i^j(\cdot)$ continuously differentiable, strictly decreasing on $]0, \infty[$, the inverse functions of $\frac{du_i^j(y)}{dy}$, $j = 0, 1$.

At the same time, the optimal wealth $\bar{w}_i(r_T)$ also satisfies

$$\bar{w}_i(r_T) = w_{i0} + \delta_i(r_T - 1) + 1. \quad (18)$$

for every realization r_T of R_T . Equating the right hand side of equations (16) and (18), and taking expectations we can derive the optimal weight invested in the risky asset

$$\delta_i^* = \frac{\mathbb{E}\left[\bar{w}_i^0(r_T) \mathbf{I}\{r_T \in [0, x_i]\}\right] + \mathbb{E}\left[\bar{w}_i^1(r_T) \mathbf{I}\{r_T \in (x_i, \infty)\}\right] - w_{i0} - 1}{\mathbb{E}(r_T) - 1} \quad (19)$$

For u_i^0 denoting bearish and u_i^1 bullish attitudes, in the sense that there exists a threshold x so that for

$$\frac{du_i^1(y)}{dy} > \frac{du_i^0(y)}{dy} \text{ for } y \geq x,$$

the investors invest a higher fraction of wealth in the risky assets when $x_i \geq x$ is small.

This is because $\bar{w}_i^1(r_T) > \bar{w}_i^0(r_T)$ for $r_T \geq x$. The risk attitudes induced by a relatively smaller reference point x_i we will call 'optimism'. Obviously, the higher δ_i^* is the higher is investors's expected wealth $\mathbb{E}[\bar{w}_i(r_T)]$. These are typically the agents that will take a long position in the risky assets, while short selling might occur for agents that have their reference points further to the right. Bakshi et al. (2010)

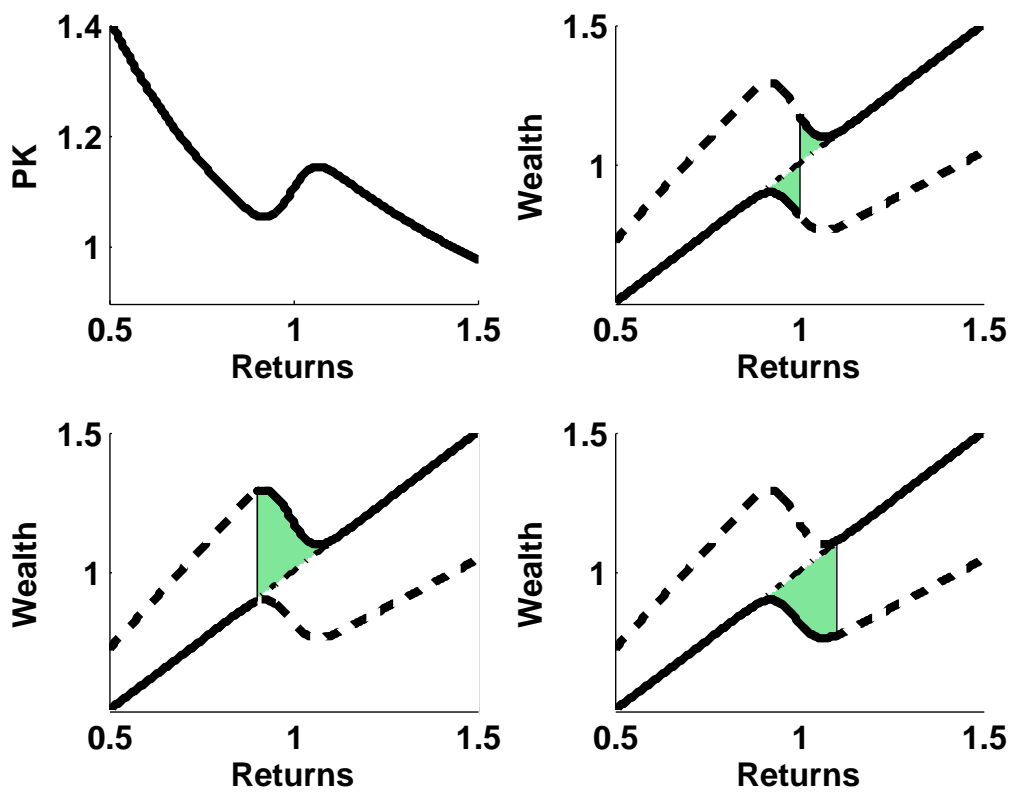


Figure 2: Market pricing kernel and (scaled) final wealth of three type of agents: mixed agent (upper right); optimistic agent (lower left) and pessimistic agents (lower right); $m\bar{w}_i(r_T)$ (solid), $m\bar{w}_i^0(r_T)$ and $m\bar{w}_i^1(r_T)$ (dotted)

suggest that investors shorting equities possibly generate a positively sloped region in the pricing kernel.

Terminal wealth for three types of agents is illustrated in figure 2. The 45 degree line depicts the wealth of the aggregate agent contrasting to the optimal wealth allocated to the individual investors. The portfolio of an 'optimistic' investor 'beats' the market for realizations of r_T at the right of its reference point for the increasing region of the pricing kernel, whereas the portfolio of a pessimistic agents underperforms compared to the benchmark at the left of the reference points for the mixed and pessimistic type.

5 Simulation Study

5.1 Comparative Statics

According to section 2, the price of the risky asset at $t = 0$ is given by

$$S_0 = \int_0^\infty s_T \tilde{\mathcal{K}}_\pi(s_T) p_{s_T}(s_T) ds_T. \quad (20)$$

For a fixed probability density function p_{s_T} the pricing kernel \mathcal{K}_π has a direct effect on the price at $t = 0$ through the way it weights the possible realizations of s_T . For $\mathcal{K}_{\theta,F} = \tilde{\mathcal{K}}_\pi$ we analyze the effects that the model's F and θ have on the price S_0 . The baseline model given by equation (14) for $b = b_1/b_0$ and $b_0 = 1$ is marked with solid line in figure (3).

We parametrize F to be $N(1, 0.05)$ and we investigate the effect that the change in the mean and variance of the distribution has on the price in the upper panels of figure (3). A decrease in the mean results in higher weights associated with higher realizations for nonzero values of $dF(\cdot)$, while a decrease in the variance makes the hump more pronounced by simultaneously lowering the weights of lower realizations and increasing those of higher realizations around the nonzero values of $dF(\cdot)$. In the first case this is due to the prevalence of optimistic investors that tilt their portfolios towards the risky asset, triggering an increase in price S_0 ; in the second case, the heterogeneity of investors' reference points

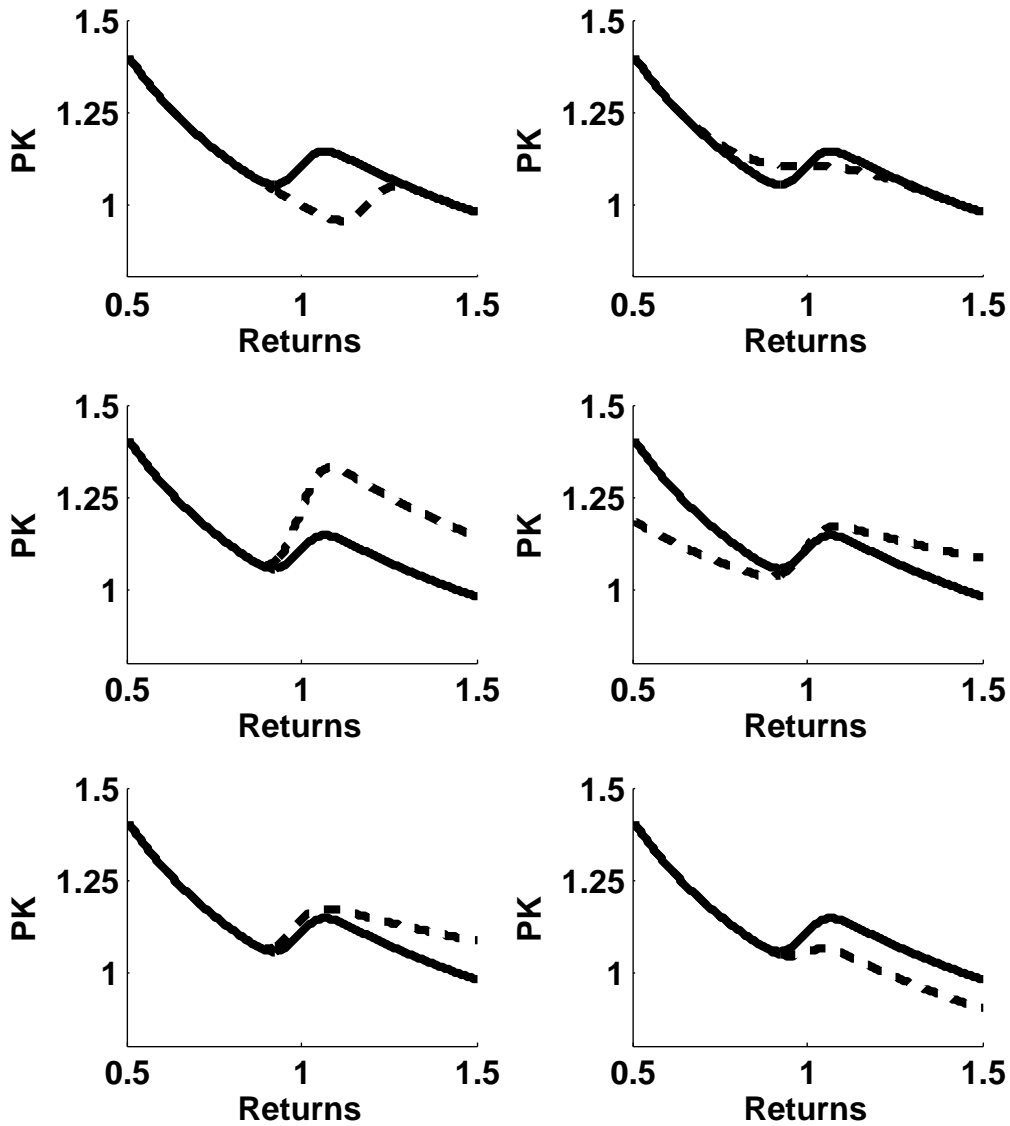


Figure 3: Impact of model parameters on the shape of PK: baseline model (solid): $\gamma = 0.5$, $b = 1.2$, $F = N(1, 0.05)$; comparative models (dashed) left panel up $F = N(1.2, 0.05)$; right panel up $F = N(1, 0.15)$; left panel middle $b = 1.4$; right panel middle $\gamma = 0.25$; left panel down $\gamma_1 = 0.25$; right panel down $F = 1/2N(1, 0.05)$

x_i -s is lower; this increases the slope of the upward region without significant effects on the price. We also observe that for small mean and large variance of F the humped feature disappears.

The next two panels depict the shape of the pricing kernel under various b -s and γ -s. We notice that for higher b the weights associated with higher returns are higher and hence large price S_0 . In this example, varying γ makes pricing kernel 'rotate' around the value of r_T corresponding to the mean of F . Lower CRRA results in higher weights for higher returns and lower rates for lower returns realizations, over all domain of r_T . The overall effect is an increase in the price in a similar fashion it produces in state independent preferences case, by reducing the price per unit of probability of bad states and conversely for the good states. If we let CRRA to vary between the two states and apply pricing kernel specification in equation (15) we can see how the divergence between γ_1 and γ_2 affect the shape of the pricing kernel and consequently the price S_0 .

Finally, in the lower panel right, we allow for a ratio of investors to have state independent preferences of type u^0 (as specified in the baseline model). These influence the price S_0 in a negative and this effect is more pronounced the higher the ratio of agents with preferences u^0 is. Obviously, the predictions for the change in S_0 will be in the opposite direction for state independent preferences of type u^1 .

5.2 Identifiability

In this subsection, we discuss some aspects related to the applicability of the model proposed in the previous section in practice, when we try to fit it to empirical pricing kernel $\widehat{\mathcal{K}}$. If we denote $\widehat{\mathcal{K}}(s_j) = y_j$ the estimates of the pricing kernel at observation points s_j , for $j = 1, \dots, n$ and assume that

$$y_j = \mathcal{K}_{\theta, F} + \varepsilon_j, \text{ with } \varepsilon_j \sim (0, \sigma^2) \quad (21)$$

the fitting problem involves finding θ^* and F^* that minimize

$$\sum_{j=1}^n \{y_j - \mathcal{K}_{\theta, F}(s_j)\}^2, \quad (22)$$

or a weighted version of it. We demonstrate the inverse problem in a simulation exercise, for $\mathcal{K}_{\theta, F}$

given by (14) and zero error term. The pricing kernel in figure 4 was generated for parameters $\gamma^0 = \gamma^1 = 0.5$, $b = 1.2$ and F a *edf* of 400 random reference points from a normal distribution $N(1, 1.2)$. The two panels on the left depict the pricing kernel and F ; the dashed line marks the regions where F takes values 0 or 1. These are the regions that allow us to identify parameters b and γ , and consequently F . However, if the probability density function associated with F doesn't have compact support on the observed domain, these components can not be identified without further restrictions. The right panel up in figure 4 zooms in the pricing kernel at its left side so that the dashed lines are no more visible. This allows us to illustrate the case of non-identifiability of the model; underneath this panel we plot different combinations for γ , b and F that give a perfect fit of the PK above. For instance, the top fascicle of dotted curves depicts F for $b = 1.2$ and $\gamma = (0.46, 0.47, 0.48, 0.49, 0.50, 0.52)$, and for the next two bundles of curves we vary b to 1.3 and 1.5 respectively. Obviously, these combinations of parameters will determine the shape of the pricing kernel in the tails, where they diverge from the true pricing kernel in various degrees.

This exercise is relevant in practice; in particular, observations in the tails are sparse and the pointwise confidence intervals (or confidence bands) for the EPK are wider in the tails regions. This means that when trying to fit the model to the real data there will be a set of possible solutions that minimize the objective function (22). The characterization of these solutions are beyond the scope of this paper and constitutes the object of future work.

6 Real Data Analysis

Due to the identification problems explained in section 5.2, a quantitative analysis in terms of θ and F over time is not feasible due to the multiplicity of solutions. The authors are investigating possible solutions under suitable constraints in a concurring study. However, the comparative statics analysis in subsection 5.1 allows us to make a qualitative evaluation of the model for dynamically estimated PK. Further on, we refer to the results of Grith et al. (2012), GHP as of now. Their EPK estimates relate to the European call and put options written on the German DAX 30 index, between June 2003 and May

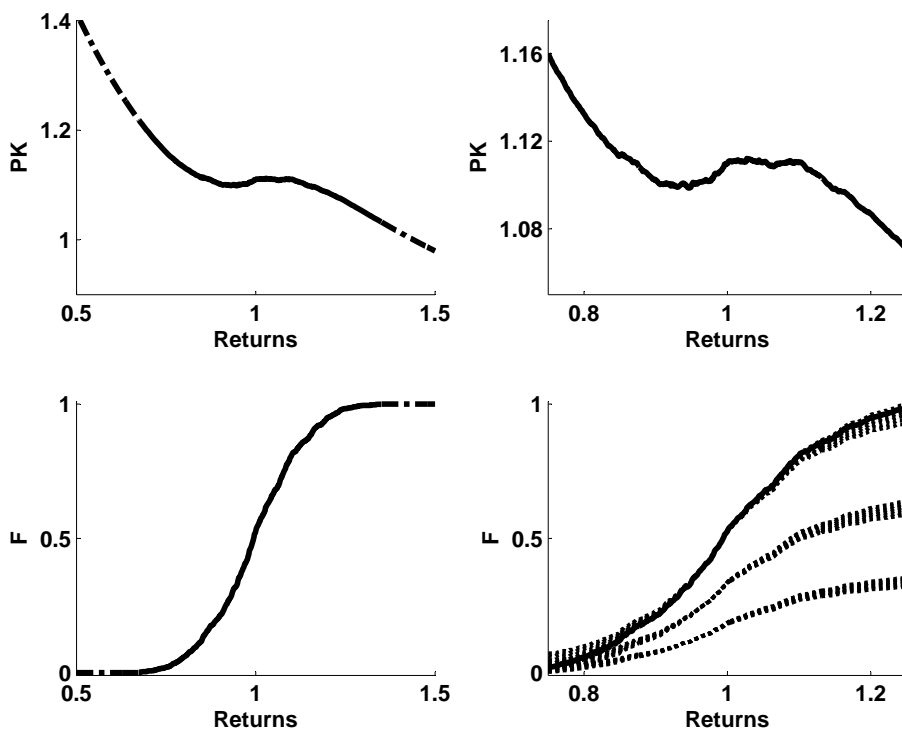


Figure 4: Parameters identifiably: compact support (left) and non-compact support (right) for the pdf of F on the observed domain

2006, at a monthly frequency. The authors assume that the conditional physical density is stationary, that is, p_{S_t} evolves slowly and most of the variation in the pricing kernel is due to q_{S_t} . If we extend the equation 20 to the contingent claims, we can explain the time variable patterns of the option prices through the changes in the pricing kernel. GHP relate the time variability of the pricing kernel hump to the economic conditions; in table 4 they report significant correlations between the changes in the shape of the EPK and the business indicators.

The changes in the height of the hump varies positively with the return on the index. The increase in the 'peak' in our model can be induced either via F or through a larger b (b_1) or lower γ (γ_1). The later causes an increase in the hump's spread, which is at odds with another finding of the GHP paper that suggests that the spread and the height of the peak are negatively related. It means, that in terms of our model, the mechanism that triggers an increase in the peak works through b and/or F . This suggestion is supported in the model proposed by Basak and Pavlova (2012), who add to the utility function of their institutional investors a state dependent component that is directly related to the performance of the index; while the retail investors have standard preferences. The fraction of institutional investors is a key parameter in their model and its increase exercises pressure on the stock index pushing it up; the same effect is present in our model by increasing the number of agents that have u^1 type of preferences (or have reference dependent preferences).

The height of the EPK hump might respond to the business conditions as well, as suggested by the correlations with the credit spread - the difference between the yield on the corporate bond, based on the German CORPTOP Bond maturing in 3-5 years, and the government bond maturing in 5 years. Its countercyclical relation to the economic conditions and the negative relation to the height of the peak imply that its decrease pushes up the level of the peak. It is not yet clear how co-movements between b and F happen in the dynamics but the evidence so far seems to suggest that b may be interpreted as a magnitude parameter, that is increasing in S_t over time, while the overall economic conditions impact F .

The scale and shape parameters that modify the PK in the horizontal direction respond to changes in the yield term slope. The slope, computed as the difference between the 30-year government bond

yield and three-month interbank rate, has been shown to be pro-cyclical in Estrella and Hardouvelis (1991). A smaller slope shifts the increasing region of the PK to the right and widens its spread. These effects become effective in our model through the positive changes in the first two moments of F , meaning an increase in the pessimism and diversion of investors' reference points on the domain of future returns.

The arguments above suggest that our model delivers sensible mechanisms of PK's dynamics. We observe that at least what the changes in the EPK shape are concerned, they do not necessarily involve γ . It is possible that through this parameter, models that mimic other features of the pricing kernels, that are not consistent with the PK puzzle - e.g. generalized disappointment aversion model in Routledge and Zin (2010) - be reproduced; such generalizations necessitate further efforts and constitute material for new studies. On the other hand, it is possible that the mechanism that we suggest only manifests in certain circumstances while agents have permanent structural biases; explanation of inverse S-shaped weighting function Polkovnichenko and Zhao (2012) may practically hold for all periods but cease to capture some features in the data during some economic conditions. We do not rule out the possibility that the asset prices depend on investors' subjective beliefs regarding future realizations of S_T and our model can incorporate such extensions. Based on our analysis, we find that the investors incorporate information from the other part of the economy when making investment decisions. Our explanation of reference dependent preferences seems a plausible explanation for the time varying shape of the EPK.

7 Conclusions

Based on our specification for the marginal investors' preferences, the v. Neumann-Morgenstern utility index of the aggregate agent might switch between different 'regimes', meaning possible jumps in the pricing kernel. We empirically investigate its switching behavior in a simulation study and interpret the time varying patterns of real data in connection to our model. The theoretical model encompasses a fixed investment horizon, since we are only taking a snapshot of the market and try to explain

the observed shape in the pricing kernel. The natural extension for building a dynamic equilibrium model, starting from the static approach is to endogenize the formation of reference points. 'Keeping up with the Joneses' or status concerns Hong et al. (2012), the history of previous gains and losses Barberis et al. (2001), learning Benzoni et al. (2011), performance relative to a benchmark Basak and Pavlova (2012); Tang and Xiong (2012) are further possible explanations and extensions that need to be investigated and that come close to our approach. The model can be extended to other markets: commodities, interest rate and credit derivatives, in order to investigate if similar behavior occurs.

A Appendix

The aim of this section is to provide a proof for Theorem 3.1. We continue with the model of section 3, retaking all assumptions and notations. Firstly, we characterize the optimal terminal wealth $\bar{w}_1(S_T), \dots, \bar{w}_m(S_T)$ of the individual investor.

The Inada conditions together with (5) imply that for any $i \in \{1, \dots, m\}$ and every $x \geq 0$ the mapping $\frac{du^i(x, \cdot)}{dy} \Big|_{]0, \infty[}$ is injective onto $]0, \infty[$ with continuously differentiable, strictly decreasing inverse say $I_i(x, \cdot)$. This enables us to apply the dominated convergence theorem to show

(A1) continuity of mappings

$$g_{s_T}^i :]0, \infty[\rightarrow \mathbb{R}, y \mapsto I_i\{s_T, y, \mathcal{K}_\pi(s_T)\} \cdot \mathcal{K}_\pi(s_T) \quad (s_T \geq 0, i \in \{1, \dots, m\}).$$

(A2) $\lim_{y \rightarrow 0} g_{s_T}^i(y) = \infty$ and $\lim_{y \rightarrow \infty} g_{s_T}^i(y) = 0$.

We are now ready to extend the classical characterization of the optimal terminal wealth to the case of extended expected utility preferences.

Theorem A.1 *Assuming (4) – (10), there exists $y_i > 0$ such that*

$$\bar{w}_i(S_T) = I_i\{S_T, y_i, \mathcal{K}_\pi(S_T)\} \text{ for every } i = 1, \dots, m$$

Proof:

Let us fix $i \in \{1, \dots, m\}$ and denote $z_i \stackrel{\text{def}}{=} w_0^i + E[e_i(S_T) \mathcal{K}_\pi(S_T)]$. Since $z_i > 0$ we may find in view of (A1), (A2) some $y_i > 0$ with $g(y_i) = x_i$.

Let $w(S_T)$ be a nonnegative random variable with $E[w(S_T) \mathcal{K}_\pi(S_T)] \leq z_i$. Then

$$\begin{aligned} E[u\{S_T, w(S_T)\}] + y_i \{z_i - E[w(S_T) \mathcal{K}_\pi(S_T)]\} &= y_i z_i + E[u\{S_T, w(S_T)\} - y_i w(S_T) \mathcal{K}_\pi(S_T)] \leq \\ y_i z_i + \sup_{x \geq 0} E[u(S_T, x) - y_i x \mathcal{K}_\pi(S_T)] &= \\ y_i z_i + E[u\{S_T, I_i\{S_T, y_i \mathcal{K}_\pi(S_T)\}\} - y_i I_i\{S_T, y_i \mathcal{K}_\pi(S_T)\} \mathcal{K}_\pi(S_T)] &= E[u\{S_T, I_i(S_T, y_i \mathcal{K}_\pi(S_T))\}]. \end{aligned}$$

Therefore $I_i(S_T, y_i \mathcal{K}_\pi(S_T))$ solves the optimization problem of investor i . Moreover, the numerical representation U_i of investor's i preferences is strictly concave in view of strict concavity of $u^i(x, \cdot)$ for every $x \geq 0$. In particular $I_i(S_T, y_i \mathcal{K}_\pi(S_T))$ is the unique solution, hence being identical with $\bar{w}_i(S_T)$. \square

Before starting with the proof of Theorem 3.1 let us consider for purposes of reference the classical case of the investor being expected utility maximizer. Indeed as an additional corollary of Theorem 3.1, we may retain the folk result concerning the risk neutral price valuation and the v. Neumann-Morgenstern utility index of the representative agent. More precisely, let us assume that there exist mappings u_1, \dots, u_r from \mathbb{R}_+ into $\mathbb{R} \cup \{-\infty\}$ satisfying $u^1(x, \cdot) = u_1, \dots, u^m(x, \cdot) = u_m$ for $x \geq 0$, and

$$(A3) \quad u_1(y), \dots, u_m(y) \in \mathbb{R} \text{ for } y > 0,$$

$$(A4) \quad u_1, \dots, u_m \text{ are continuous, strictly increasing as well as strictly concave.}$$

Then

$$u(y) \stackrel{\text{def}}{=} \sup \left\{ \sum_{i=1}^m \alpha_i u_i(y_i) \mid y_1, \dots, y_m \geq 0, \sum_{i=1}^m y_i \leq y \right\} = u_\alpha(x, y) \text{ for } x, y \geq 0.$$

We shall impose the so called *Inada conditions* on the state independent utility indices u_1, \dots, u_m , i.e.

$$(A5) \quad u_1|]0, \infty[, \dots, u_m|]0, \infty[\text{ are assumed to be continuously differentiable satisfying}$$

$$\lim_{e \rightarrow 0} \frac{du_i}{dy} \Big|_{y=e} = \infty, \quad \lim_{e \rightarrow \infty} \frac{du_i}{dy} \Big|_{y=e} = 0 \quad (i = 1, \dots, m).$$

(A6) $E[I_1(y\mathcal{K}_\pi(S_T))], \dots, E[I_m(y\mathcal{K}_\pi(S_T))] < \infty$ for any $y > 0$, where I_1, \dots, I_r denote the inverses of $\frac{du_1}{dy}, \dots, \frac{du_m}{dy}$ respectively.

We may conclude immediately from Theorem 3.1 the announced result.

Proof of Theorem 3.1:

Without loss of generality let us set $\{1, \dots, r\} \stackrel{\text{def}}{=} \{i \in \{1, \dots, m\} \mid \alpha_i > 0\}$. Then, defining $g_i \stackrel{\text{def}}{=} \alpha_i u_i$, we have $u_\alpha = \sum_{i=1}^r g_i$, and we may apply Lemma B.1, B.2 and Proposition B.3 (cf. Appendix B). Then, in view of Lemma B.1, B.2 and B.3, we obtain

$$u_\alpha \{s_T, \bar{w}(s_T)\} = \sum_{i=1}^r \alpha_i u^i \left(s_T, \bar{w}^i(s_T) \right)$$

for every realization s_T of S_T .

On one hand by Theorem A.1, there exist $y_1, \dots, y_m > 0$ such that

$$\bar{w}_i(s_T) = I_i(s_T, y_i \mathcal{K}_\pi(s_T)) > 0 \text{ for } i = 1, \dots, r.$$

On the other hand, due to Proposition B.3, $u_\alpha(s_T, \cdot) \mid]0, \infty[$ is differentiable for every realization s_T , satisfying

$$\alpha_i \frac{du^i(s_T, \cdot)}{dy} \Big|_{y=\bar{w}^i(s_T)} = \frac{du_\alpha(s_T, \cdot)}{dy} \Big|_{y=\bar{w}(s_T)}$$

for $i \in \{1, \dots, r\}$ and any realization s_T . Notice that by construction the random variable $\bar{w}(S_T)$ has strictly positive outcomes only. Now, the statement of Theorem 3.1 is clear.

B Appendix

Throughout this section let the mappings $g_1, \dots, g_r : \mathbb{R}_+^2 \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfy the following conditions:

(B0) $g_1(x, y), \dots, g_r(x, y) \in \mathbb{R}$ for $x \geq 0, y > 0$;

(B1) $g_1(x, \cdot), \dots, g_r(x, \cdot)$ are continuous, strictly increasing and strictly concave for $x \geq 0$;

(B2) $g_1(\cdot, y), \dots, g_r(\cdot, y)$ are Borel-measurable for $y \geq 0$.

Furthermore, let $g : \mathbb{R}_+^2 \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be defined by

$$g(x, y) = \sup \left\{ \sum_{i=1}^r g_i(x, y_i) \mid y_1, \dots, y_r \geq 0, \sum_{i=1}^r y_i \leq y \right\}.$$

Indeed $g(x, 0) = \sum_{i=1}^r g_i(x, 0) \in \mathbb{R} \cup \{-\infty\}$ for $x \geq 0$, and

$$-\infty < \sum_{i=1}^r g_i(x, \frac{y}{r}) \leq g(x, y) \leq \sum_{i=1}^r g_i(x, y) < \infty$$

for $x \geq 0, y > 0$ due to (B0), (B1).

Lemma B.1 For any $x, y \geq 0$ there is some unique $\phi(x, y) = (\phi_1(x, y), \dots, \phi_r(x, y)) \in \mathbb{R}_+^r$ such that $\sum_{i=1}^r \phi_i(x, y) \leq y$ and

$$\sum_{i=1}^r g_i \{x, \phi_i(x, y)\} = g(x, y).$$

Furthermore, $\sum_{i=1}^r \phi_i(x, y) = y$.

Proof:

Let $x, y \geq 0$. For $y = 0$ the statement of Lemma B.1 is obvious. So let $y > 0$, which means $g(x, y) \in \mathbb{R}$.

Due to (B1), the mapping

$$f : \left\{ (y_1, \dots, y_r) \in \mathbb{R}_+^r \mid \sum_{i=1}^r y_i \leq y, \sum_{i=1}^r g_i(x, y_i) \geq g(x, y) - 1 \right\} \rightarrow \mathbb{R}, (y_1, \dots, y_r) \mapsto \sum_{i=1}^r g_i(x, y_i)$$

is continuous, strictly concave, and defined on a nonvoid convex compact set. Therefore f attains its maximum at a unique $\phi(x, y)$. Obviously, $\sum_{i=1}^r \phi_i(x, y) = y$ because f is strictly increasing too by (B1).

The proof is complete.

Lemma B.1 defines a mapping $\phi = (\phi_1, \dots, \phi_r) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^r$. It is Borel-measurable as will be shown now.

Lemma B.2 ϕ is Borel-measurable.

Proof:

It suffices to show that $\phi^{-1} \left(\prod_{i=1}^r [0, a_i] \right)$ is a Borel-subset of \mathbb{R}_+^2 . For this purpose define for any (a_1, \dots, a_r) from \mathbb{R}_+^r the mapping $g_{a_1 \dots a_r} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$g_{a_1 \dots a_r}(x, y) = \sup \left\{ \sum_{i=1}^r g_i(x, y_i) \mid (y_1, \dots, y_r) \in \prod_{i=1}^r [0, a_i], \sum_{i=1}^r y_i \leq y \right\}.$$

Notice that $g_{a_1 \dots a_r}(x, y) \in \mathbb{R}$ for $x \geq 0, y > 0$, analogously to $g(x, y) \in \mathbb{R}$ for $x \geq 0, y > 0$. Furthermore $g_1(x, \cdot), \dots, g_r(x, \cdot)$ are continuous for any $x \geq 0$. Hence, setting $\mathcal{R}_{a_1 \dots a_r} = \prod_{i=1}^r [0, a_i] \times \mathbb{Q}^m$,

$$g_{a_1 \dots a_r}^{-1}(]z, \infty[) = \bigcup_{(y_1, \dots, y_r) \in \mathcal{R}_{a_1 \dots a_r}} \left(\sum_{i=1}^r \alpha_i g_i(\cdot, y_i) \right)^{-1} (]z, \infty[) \times \left[\sum_{i=1}^r y_i, \infty[\quad (z \in \mathbb{R}).$$

Thus $g_{a_1 \dots a_r}^{-1}(]z, \infty[)$ is a Borel-subset of \mathbb{R}_+^2 for every $z \in \mathbb{R}$ by assumption (B2). Then we may conclude that

$$\phi^{-1} \left(\prod_{i=1}^r [0, a_i] \right) = \left(\sup_{(b_1, \dots, b_r) \in \mathbb{Q}_+^r} g_{b_1 \dots b_r} - g_{a_1 \dots a_r} \right)^{-1} (\{0\})$$

is a Borel subset of \mathbb{R}_+^2 for any $(a_1, \dots, a_r) \in \mathbb{R}_+^r$, which completes the proof.

In order to characterize the mapping ϕ in terms of derivatives of the functions $g_1(x, \cdot), \dots, g_r(x, \cdot)$, it is customary to impose the *Inada conditions*, i.e.

(B3) for any $x \geq 0$ the mappings $g_1(x, \cdot)|]0, \infty[, \dots, g_r(x, \cdot)|]0, \infty[$ are assumed to be continuously differentiable satisfying

$$\lim_{\epsilon \rightarrow 0} \frac{\partial g^i(x, \cdot)}{\partial y} \Big|_{y=\epsilon} = \infty, \quad \lim_{\epsilon \rightarrow \infty} \frac{\partial g^i(x, \cdot)}{\partial y} \Big|_{y=\epsilon} = 0, \quad i = 1, \dots, r.$$

The Inada conditions together with condition (B1) imply that for any $i \in \{1, \dots, r\}$ and every $x \geq 0$ the mapping $\frac{\partial g^i(x, \cdot)}{\partial y} |]0, \infty[$ is injective onto $]0, \infty[$ with continuously differentiable, strictly decreasing inverse say $I_i(x, \cdot)$.

Proposition B.3 *Let the assumptions (B0) - (B3) be fulfilled, and let $g_1(x, \cdot)|]0, \infty[, \dots, g_r(x, \cdot)|]0, \infty[$ be twice continuously differentiable.*

Then for any $x \geq 0$ the mapping $g(x, \cdot) |]0, \infty[$ is differentiable satisfying

$$\phi(x, y) = \left[I_1 \left\{ x, \frac{\partial g(x, \cdot)}{\partial y} \Big|_y \right\}, \dots, I_r \left\{ x, \frac{\partial g(x, \cdot)}{\partial y} \Big|_y \right\} \right] \text{ for } y > 0.$$

Proof:

Let for $x \geq 0$ the mapping $F_x :]0, \infty[\times]0, \infty[\rightarrow \mathbb{R}$ be defined by $F_x(y, z) = \sum_{i=1}^r I_i(x, z) - y$.

Since the mappings $g_1(x, \cdot)|]0, \infty[, \dots, g_r(x, \cdot)|]0, \infty[$ are assumed to be strictly concave and twice continuously differentiable, their second derivatives are strictly negative. Then by local inverse theorem

the mappings $I_1(x, \cdot), \dots, I_r(x, \cdot)$ are continuously differentiable, having strictly negative derivatives. In particular F_x is continuously differentiable, satisfying

$$\frac{\partial F_x}{\partial z} \Big|_{(y,z)} \neq 0 \text{ for } y, z > 0.$$

Furthermore, since $I_1(x, \cdot), \dots, I_r(x, \cdot)$ are continuous and strictly decreasing mappings onto $]0, \infty[$, we may find for any $y > 0$ a unique $\varphi(y) > 0$ with $F(y, \varphi(y)) = 0$. Drawing on the implicit function theorem, $y \mapsto \varphi(y)$ defines a differentiable mapping $\varphi :]0, \infty[\rightarrow]0, \infty[$.

Moreover, for $y > 0$ and $y_1, \dots, y_r \geq 0$ with $\sum_{i=1}^r y_i \leq y$, we may conclude

$$\begin{aligned} \sum_{i=1}^r g_i(x, y_i) + \varphi(y)(y - \sum_{i=1}^r y_i) &= \varphi(y)y + \sum_{i=1}^r \{g_i(x, y_i) + \varphi(y)y_i\} \leq \\ &\varphi(y)y + \sum_{i=1}^r \sup_{z \geq 0} \{g_i(x, z) + \varphi(y)z\} = \\ &\varphi(y)y + \sum_{i=1}^r [g_i\{x, I_i(x, \varphi(y))\} + \varphi(y)I_i\{x, \varphi(y)\}] = \\ &\sum_{i=1}^r g_i[x, I_i\{x, \varphi(y)\}] - F_x\{y, \varphi(y)\} = \sum_{i=1}^r g_i[x, I_i\{x, \varphi(y)\}]. \end{aligned}$$

This means

$$g(x, y) = \sum_{i=1}^r g_i[x, I_i\{x, \varphi(y)\}],$$

and hence by Lemma B.1

$$(*) \quad \phi(x, y) = (I_1[x, \varphi(y)], \dots, I_r[x, \varphi(y)]).$$

As a further consequence $g(x, \cdot) |]0, \infty[$ is differentiable satisfying

$$\frac{dg(x, \cdot)}{dy} \Big|_{y=\varphi(y)} = \sum_{i=1}^r \varphi(y) \frac{dI_i(x, \cdot) \circ \varphi}{dy} \Big|_{y=\varphi(y)} \frac{d\left(\sum_{i=1}^r I_i(x, \cdot) \circ \varphi\right)}{dy} \Big|_{y=\varphi(y)}.$$

For the last equation notice that $\sum_{i=1}^r I_i(x, \cdot) \circ \varphi$ is just the identity on $]0, \infty[$. In view of (*) the proof is complete.

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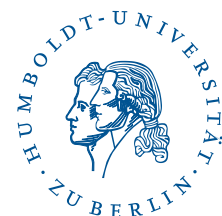
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