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Pruning in Perturbation DSGE Models - Guidance from Nonlinear Moving Average Approximations

Hong Lan*
Alexander Meyer-Gohde*

* Humboldt-Universität zu Berlin, Germany

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Pruning in Perturbation DSGE Models

Guidance from Nonlinear Moving Average Approximations *

Hong Lan † Alexander Meyer-Gohde§

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Abstract

We derive recursive representations of nonlinear moving average (NLMA) perturbations of DSGE models. As the stability of higher order NLMA representations follows directly from stability at first order, these recursive representations provide rigorous support for the practice of pruning that is becoming widespread. Our recursive representation differs from pruned perturbations in that it centers the approximation and its coefficients at the approximation of the stochastic steady state consistent with the order of approximation. We compare our algorithm with six different pruning algorithms at second and third order, documenting the differences between these six algorithms and standard (non pruned) state space perturbations at first, second, and third order in a unified notation compatible with the popular software package Dynare. While our third order algorithm is the most accurate, the gains over two alternate algorithms are modest, suggesting that this choice is unlikely to be a potential source of error.

JEL classification: C52, C63, E30

Keywords: Perturbation; DSGE; nonlinear; pruning

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†Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 1466; E-Mail:lanhong@cms.hu-berlin.de

§Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 5720; E-Mail: alexander.meyer-gohde@wiwi.hu-berlin.de

1 Introduction

Locally approximated models that are stable at first order can produce explosive simulations when approximated at second or higher order. This is troublesome as higher order approximations are needed to capture salient features of the macroeconomy.[1] The instability induced by higher order simulations is caused by the accumulation of nonlinear terms higher than the order of approximation that add additional instable steady states to the approximation. Judd, Maliar, and Maliar (2011) offer one solution to generate stable simulations efficiently. Another solution offered by the literature is to maintain the local, perturbation approach, but to "prune" these higher order terms and restore the desired stability. This later approach has the additional advantage of enabling the application of GMM and SMM to these nonlinear settings[2] as well as a decomposition of theoretical moments into orders of approximation and risk adjustment.[3]

The nonlinear moving average perturbations of Lan and Meyer-Gohde (2012b) produce approximations that are stable at all orders of approximations when the first order approximation is stable. In this study, we derive recursive representations of infinite moving average approximations, providing endogenously pruned algorithms for nonlinear simulations. While the pruning of nonlinear perturbations introduced by Kim, Kim, Schaumburg, and Sims (2008), and indeed the different algorithms to implement pruning, has proliferated in the recent literature, Den Haan and De Wind (2012) and Lombardo (2012) have objected, calling this methodology ad hoc, and Ruge-Murcia (2012) has noted the nontrivial nature of extending Kim, Kim, Schaumburg, and Sims's (2008) second order algorithm to higher orders. We provide theoretical support for pruning algorithms, interpreting them as recursive formulations of nonlinear moving average approximations.

We compare our nonlinear moving average based recursive algorithm to the pruning algorithms of (at second order) Kim, Kim, Schaumburg, and Sims (2008) and Den Haan and De Wind (2012) and of (at third order) Andreasen (2012), Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011), Den Haan and De Wind (2012), and Dynare,[4] providing the literature

[1] As noted by Ruge-Murcia (2012), Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011), Andreasen (2012), and van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012), capturing the time-varying shifts in risk premia or precautionary behavior requires at least a third order approximation.

[2] See Ruge-Murcia (2012) and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2012).

[3] See Lan and Meyer-Gohde (2013).

[4] This is an undocumented algorithm at third order by Michel Juillard. On Dynare, see Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011).

with an overview of the various algorithms in a unified notation. Additionally, we compare all the algorithms with standard (non pruned) perturbations at first through third order and with the exact solution when known or a highly accurate projection solution when unknown.

We run three horse races to compare the various pruning algorithms beyond theoretical considerations. First, we choose the Brock and Mirman (1972) log preference and complete depreciation case of the stochastic neoclassical growth model.[5] Second, we evaluate the algorithms in Burnside's (1998) asset pricing model. Finally, we examine the performance of the different algorithms in Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao's (2012) model with recursive preferences and stochastic volatility. The first two models possess closed form solutions and we measure the distance of the various pruning algorithms as well as the unpruned perturbations to the exact solution in terms of average, mean square, and maximal error. While the last model has no closed form solution and needs to be approximated, we follow Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) and choose the Chebyshev polynomial approximation as the reference solution of the model to examine the performance of the different algorithms. The most accurate pruning algorithms are those that can be derived directly from a moving average approximation or Lombardo's (2012) matched perturbation, with our algorithm performing marginally better according to several criteria we use to compare the algorithms. Algorithms, however, that drop terms of the order of approximation or add higher order terms suffer in terms of accuracy.

The paper is organized as follows. The family of models we will be analyzing is presented with the nonlinear moving average solution form in section 2. We derive the recursive representation of the nonlinear moving average approximation in section 3, and present the various pruning algorithms in a unified notation in section 4. We examine Lombardo's (2012) matched perturbation algorithm separately in section 5. The numerical performance of the different algorithms are analyzed using Brock and Mirman's (1972) neoclassical stochastic growth model and Burnside's (1998) asset pricing model in section 6, and in section 6.3 we report the numerical performance of these algorithms in a neoclassical stochastic growth model with recursive preferences and stochastic volatility. Section 7 concludes.

[5] See McCallum (1989).

2 Model Class

We begin by introducing our class of models, a standard system of (nonlinear) second order expectational difference equations. We then present the solution as a policy function that directly maps from realization of the exogenous shocks to the endogenous variables of interest, and approximate the solution with a Taylor series. Adopting Dynare's typology of all the endogenous variables, we differ from Lan and Meyer-Gohde (2012b) and present the class of models and the approximations of its solution out to third order in a computationally efficient notation.

2.1 Problem Statement

We analyze a family of discrete-time rational expectations models given by

$$0 = E_t[f(y_{t+1}^{fwdendo}, y_t, y_{t-1}^{state}, \varepsilon_t)] \tag{1}$$

f is an $(neq \times 1)$ vector valued function, continuously M-times (the order of approximation to be introduced subsequently) differentiable in all its arguments; y_t is an $(ny \times 1)$ vector of endogenous variables divided following, e.g. Dynare,[6] additionally into two subvectors, $y_t^{fwdendo}$ and y_t^{state}, $(nfwdendo \times 1)$ and $(ns \times 1)$ respectively, commensurate with the presence of elements of y_t with subscripts $t+1$ and $t-1$ in the system of equations; the vector of exogenous shocks ε_t is of dimension $(ne \times 1)$ and it is assumed that there are as many equations as endogenous variables $(neq = ny)$. ε_t is assumed independently and identically distributed[7] such that $E(\varepsilon_t) = 0$ and $E({\varepsilon_t}^{\otimes[m]})$ exists and is finite for all m up to and including the order of approximation to be introduced subsequently.[8]

As is usual in perturbation methods, we introduce an auxiliary parameter $\sigma \in [0,\ 1]$ to scale the risk in the model. The value $\sigma = 1$ corresponds to the "true" stochastic model under study and $\sigma = 0$ represents the deterministic version of the model.[9] Following Anderson, Levin, and

[6] See Villemot (2011) and Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011).

[7] Thus in practice, any exogenous serial correlation must be incorporated into the vector y_t, which is why this vector might be more properly labeled endogenous and exogenous variables. We maintain this practice of the literature for brevity.

[8] The notation ${\varepsilon_t}^{\otimes[m]}$ represents Kronecker powers, ${\varepsilon_t}^{\otimes[m]}$ is the m'th fold Kronecker product of ε_t with itself: $\underbrace{\varepsilon_t \otimes \varepsilon_t \cdots \otimes \varepsilon_t}_{m\ times}$. For simulations, of course, more specific decisions regarding the distribution of the exogenous processes will have to be made. Kim, Kim, Schaumburg, and Sims (2008, p. 3402) emphasize that distributional assumptions like these are not entirely local assumptions. Dynare (Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot 2011) assumes normality of the underlying shocks.

[9] Our formulation follows Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare,

Swanson (2006, p. 4), we do not scale $\{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$ — the realizations of the exogenous shocks up to (including) t — with σ, as they are known with certainty at t. The perturbation parameter does not enter the problem statement explicitly, but only implicitly through the policy functions, and its role will become clear as we introduce the solution form and its approximation.

Fleming (1971) and Jin and Judd (2002) emphasize that the use of σ to transition from the deterministic to the stochastic model depends crucially on the two models being "close," in the sense that the underlying risk scaled by σ is "small," as a stochastic perturbation like this is singular in that it changes the underlying order of the problem, see Judd (1998, ch. 13). Kim, Kim, Schaumburg, and Sims (2008) note the importance of the "underlying assumption" of sufficient differentiability within a neighborhood of $\sigma = 0$ and Anderson, Levin, and Swanson (2006) simply make the explicit assumption that the policy function, the solution to be introduced the following subsection, is analytic within a domain that encompasses $\sigma = 0$ and $\sigma = 1$, enabling its representation in σ by a Taylor series evaluated anywhere within that domain. Deriving explicit conditions for the model with $\sigma = 1$ to be sufficiently close to the $\sigma = 0$ model is beyond the scope of our study here and we follow the literature by assuming that a local approach to σ remains valid as we transition to the stochastic model.

2.2 Nonlinear Moving Average Solution Form

Let the policy function take the causal one-sided infinite sequence of shocks as its state vector following Lan and Meyer-Gohde (2012b) given by

$$y_t = y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \ldots), \quad g : \mathbb{R}^+ \times \mathbb{R}^{ne} \times \mathbb{R}^{ne} \times \cdots \to \mathbb{R}^{ny} \tag{2}$$

Note that σ enters as a separate argument. As the scale of risk changes, so too will the policy function y itself change. Time invariance and scaling the risk associated with future shocks give us

$$y_{t-1} = y^-(\sigma, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots) \tag{3}$$

$$\widetilde{y}_{t+1} = y^+(\sigma, \widetilde{\varepsilon}_{t+1}, \varepsilon_t, \varepsilon_{t-1}, \ldots), \text{ where } \widetilde{\varepsilon}_{t+1} \equiv \sigma\varepsilon_{t+1} \tag{4}$$

The notation, y, y^-, and y^+, was adopted in Lan and Meyer-Gohde (2012b) to keep track of the source (through y_t, y_{t-1}, and y_{t+1} respectively) of any given partial derivative of the policy function

Anderson, Levin, and Swanson's (2006) PerturbationAIM and Juillard (2011). This is not the only way to perturb the model: Lombardo (2010), for example, scales the entire history of shocks $\sigma\{\varepsilon_t, \sigma\varepsilon_{t-1}, \ldots\}$ along with the unrealized future shocks. See section 5 for further discussion.

necessary in calculations. Likewise, we append a tilde to y at $t+1$ as we did ε at $t+1$ in deference to time t conditioning in the equilibrium conditions (1); from the time t perspective of (1), ε_{t+1} is a random variable, hence y_{t+1} as well, whereas $\varepsilon_t, \varepsilon_{t-1}, \ldots$ and hence y_t and y_{t-1} are realizations of random variables. These notational issues will play only a minor role here, as we will take the calculations of Lan and Meyer-Gohde (2012b) as given. Due to the assumption of time invariance, y, y^-, and y^+ are the same function differing only in the timing of their arguments. The term $\sigma\varepsilon_{t+1}$ in (4) is the source of risk, via ε_{t+1}, that we are perturbing with σ.

With the policy function of the form (2), (3) and (4), we can write (1) as

$$\begin{aligned} 0 &= E_t\left[f\left(y^{+fwdendo}(\sigma,\widetilde{\varepsilon}_{t+1},\varepsilon_t,\ldots),y(\sigma,\varepsilon_t,\varepsilon_{t-1},\ldots),y^{-state}(\sigma,\varepsilon_{t-1},\varepsilon_{t-2},\ldots),\varepsilon_t\right)\right] \\ &= F(\sigma,\varepsilon_t,\varepsilon_{t-1},\ldots) \end{aligned} \tag{5}$$

a function with arguments σ and and the infinite history of innovations $\{\varepsilon_{t-j}\}_{j=0}^{\infty}$.[10]

2.3 Nonlinear Moving Average Approximation

We will approximate the solution, (2), as a Taylor series in the infinite state vector (i.e., a Volterra series) expanded around a deterministic steady state, $\overline{y}$, the time invariant fixed point in y of (5), with all shocks, past and present, set to zero and all risk regarding the future eliminated ($\sigma = 0$)

Definition 2.1. *Deterministic Steady State*

Let $\overline{y} \in \mathbb{R}^{ny}$ be a vector such that

$$0 = f\left(\overline{y}^{fwdendo},\overline{y},\overline{y}^{state},0\right) = f(\overline{x}) = F(0,0,\ldots) \tag{6}$$

Furthermore, $\overline{y} = y(0,0,\ldots)$ is the solution (2) evaluated at the deterministic steady state.

Analogously, we define the stochastic or "risky" steady state as the stationary point in the absence of past and present shocks but the risk of future shocks[11]

Definition 2.2. *Stochastic Steady State*

[10] Note that ε_{t+1} is not an argument of F as it is the variable of integration inside the expectations. I.e.,

$$F(\sigma,,\varepsilon_t,\varepsilon_{t-1},\ldots) = \int_{\Omega} f\left(y^{+fwdendo}(\sigma,\widetilde{\varepsilon}_{t+1},\varepsilon_t,\ldots),y(\sigma,\varepsilon_t,\varepsilon_{t-1},\ldots),y^{-state}(\sigma,\varepsilon_{t-1},\varepsilon_{t-2},\ldots),\varepsilon_t\right)\phi(\varepsilon_{t+1})\,d\varepsilon_{t+1}$$

where Ω is the support and ϕ the p.d.f. of ε_{t+1}. Thus, when $\sigma = 0$, ε_{t+1} is no longer an argument of f and the integral (and hence the expectations operator) is superfluous, yielding the deterministic version of the model.

[11] This definition parallels to that of Coeurdacier, Rey, and Winant (2011) within the state space context. See section 4 for our state space definition of this concept.

Let $\overline{y}^{stoch} = y(1,0,0,\ldots) \in \mathbb{R}^{ny}$ *be a vector such that*

$$0 = E_t\left[f\left(y^{+fwdendo}(1,\varepsilon_{t+1},0,\ldots),\overline{y}^{stoch},\overline{y}^{stoch,state},0\right)\right] = F(1,0,\ldots) \tag{7}$$

Assuming the $\sigma = 1$ model is sufficient close to its deterministic, $\sigma = 0$ counterpart, the stochastic steady state can be approximated by expanding $0 = F(1,0,\ldots)$ in σ around the deterministic, that is, $\sigma = 0$, steady state in definition 2.1.

Since *y* is a vector valued function, its partial derivatives form a hypercube. We use the method of Lan and Meyer-Gohde (2012b) that differentiates conformably with the Kronecker product, allowing us to maintain standard linear algebraic structures to derive our results.

Definition 2.3. *Matrix Derivatives*

Let $A(B) : \mathbb{R}^{s\times 1} \to \mathbb{R}^{p\times q}$ *be a matrix-valued function that maps an* $s \times 1$ *vector B into an* $p \times q$ *matrix* $A(B)$*, the derivative structure of* $A(B)$ *with respect to B is defined as*

$$A_B \equiv \mathscr{D}_{B^T}\{A\} \equiv \begin{bmatrix}\frac{\partial}{\partial b_1} & \ldots & \frac{\partial}{\partial b_s}\end{bmatrix} \otimes A \tag{8}$$

where b_i *denotes i'th row of vector B,* ^{T} *indicates transposition; n'th derivatives are*

$$A_{B^n} \equiv \mathscr{D}_{(B^T)^n}\{A\} \equiv \left(\begin{bmatrix}\frac{\partial}{\partial b_1} & \ldots & \frac{\partial}{\partial b_s}\end{bmatrix}^{\otimes[n]}\right) \otimes A \tag{9}$$

Adopting the abbreviated notation above, we write $y_{\sigma^n i_1 i_2\cdots i_m}$ as the partial derivative, evaluated at the deterministic steady state, of *y* with respect to σ for *n* times and with respect to $\varepsilon_{t-i_1}^T, \varepsilon_{t-i_2}^T, \cdots, \varepsilon_{t-i_m}^T$. Thus, we can then write the *M*-th order Taylor approximation of the policy function (2) as

$$y_t = \sum_{m=0}^{M}\frac{1}{m!}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}\left[\sum_{n=0}^{M-m}\frac{1}{n!}y_{\sigma^n i_1 i_2\cdots i_m}\sigma^n\right](\varepsilon_{t-i_1}\otimes\varepsilon_{t-i_2}\otimes\cdots\otimes\varepsilon_{t-i_m}) \tag{10}$$

where we refer to Lan and Meyer-Gohde (2012b) for further details.

This nonlinear moving average, or Volterra series with kernels $\left[\sum_{n=0}^{M-m}\frac{1}{n!}y_{\sigma^n i_1\cdots i_m}\sigma^n\right]$, directly maps the exogenous innovations to endogenous variables up the *M*-th order. The kernels at *m* collects all the coefficients associated with the *m*'th fold Kronecker products of exogenous innovations i_1, i_2, ... and i_m periods ago. Importantly, the outer sum indicates that an approximation of any given order is linear in all the kernels up to and including the order in question; thus, the approximation is linearly recursive.[12] For a given set of indices, i_1, i_2, ... and i_m, the sum over *n* gathering terms in powers of the perturbation parameter σ, adjusts the kernel for risk up to the

[12]The terminology is Lombardo's (2010). See section 5 for a comparison with the method advocated by him.

n-th order,[13] thereby enabling a classification of the contributions of risk to the model alongside polynomial nonlinearity.

The nonlinear moving average constructs an approximation in the (countable) sequence space as opposed to the (measurable) function space sought in the standard state space set up. Thus, by construction, the approximation will be bounded for bounded sequences of inputs, whereas iterations on approximations in the standard function space in general cannot. Differently, the nonlinear moving average can be derived by "solving out" an "invertible" nonlinear state space representation following Priestly (1988, p. 25), which is only defined within the region of convergence of the state space representation. By jumping straight to the nonlinear moving average representation and allowing shocks from distributions with infinite support, we are, from this perspective, imposing a region of convergence with an infinite radius on the nonlinear state space policy function. That is, we achieve stability by assumption and the construction of our approximation is only valid when this assumption holds.[14]

3 Recursive Representation of Nonlinear Moving Averages

As shown in Lan and Meyer-Gohde (2012b), nonlinear moving average perturbations are linear in the kernels (or sums of product spaces in the history of shocks) which inherit the stability properties of the approximation at first order and whose coefficients can be expressed recursively similarly to the linear case explored by Taylor (1986). We will now show that the recursivity in parameters leads to recursive representations in the endogenous variables themselves, but in an order dependent manner consistent with pruning algorithms in the literature, as we will explore in detail in section 4.

3.1 First Order Recursive Approximation

The first order approximation of the policy function takes the form

$$y_t^{(1)} = \overline{y} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i},\ i = 0,1,2,\ldots \tag{11}$$

[13] A similar interpretation for standard state space policy functions can be found in section 4 and Lan and Meyer-Gohde (2012a) for multivariate and Judd and Mertens (2012) for univariate expansions.

[14] See Jin and Judd (2002) for an example of when this would not hold.

where the superscript $^{(1)}$ on y_t implies this is the first order of approximation. $\bar{y}$ denotes the deterministic steady state value of the vector y_t. The partial derivative y_i is a linear convergent recursion (See the Appendix.) with a saddle-stable matrix α as the coefficient on its homogenous part. For notational ease in deriving the recursive representation of the previous equation, we define

$$dy_t^{(1)} \equiv y_t^{(1)} - \bar{y} \tag{12}$$

It follows that

$$dy_t^{(1)} = \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} \tag{13}$$

Anticipating the derivations of higher order recursive representations, we first derive a recursive representation for the increment and then, using this increment, express the first order approximation recursively. This is obviously unnecessary at first order, as this recursive representation is a standard result, see, e.g., Uhlig (1999), but will fix ideas for the more involved higher order recursive representations.

The increment of the first order approximation $dy_t^{(1)}$ can be expressed recursively, as we summarize in the following

Proposition 3.1. *The first order increment, $dy_t^{(1)}$, can be expressed as a linear function of its own past and the current realizations of exogenous shocks*

$$dy_t^{(1)} = \alpha dy_{t-1}^{(1)state} + \beta_0 \varepsilon_t \tag{14}$$

Proof. See the Appendix. □

Accordingly, the first order approximation can likewise be expressed recursively

Proposition 3.2. *The first order approximation of y_t, denoted $y_t^{(1)}$, can be expressed recursively through*

$$y_t^{(1)} = \bar{y} + dy_t^{(1)} \tag{15}$$

where

$$dy_t^{(1)} = y_z dz_t^{(1)},\ dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{16}$$

Proof. This is an immediate consequence of the definition of the increment in (12). □

Thus recovering the state-space policy function in linear settings—see, e.g., Uhlig (1999)—and reiterating the equivalence at first order of moving average representations–see Taylor (1986)—with state space methods. Note the coefficient α in (14) is the homogenous coefficient of the recursion of y_i.

3.2 Second Order Recursive Approximation

The second order approximation of the policy function takes the form

$$y_t^{(2)} = \bar{y} + \frac{1}{2} y_{\sigma^2} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \tag{17}$$

For the derivation of the recursive representation of the previous equation, we define the second order increment as the difference between the first and second order approximation, subtracting the constant risk adjustment of the second order

$$dy_t^{(2)} \equiv y_t^{(2)} - \frac{1}{2} y_{\sigma^2} - y_t^{(1)} \tag{18}$$

Inserting (11) and (17) in the previous equation yields the moving averaging representation of the second order increment

$$dy_t^{(2)} = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \tag{19}$$

The increment of the second order approximation can be expressed recursively, as we summarize in the following

Proposition 3.3. *The second order increment, $dy_t^{(2)}$, can be expressed as a linear function of its own past and products of terms of lower order according to the following recursion*

$$dy_t^{(2)} - \alpha dy_{t-1}^{(2)state} = \frac{1}{2} \left[\beta_{22} dy_{t-1}^{state \otimes [2]} + 2\beta_{20} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right) + \beta_{00} \varepsilon_t^{\otimes [2]} \right] \tag{20}$$

Proof. See the Appendix. □

Combining the increment definitions and recursive representations at first and second order, we construct the following second order recursive formula for y_t

Proposition 3.4. *The second order approximation of y_t, denoted $y_t^{(2)}$, can be expressed recursively through*

$$y_t^{(2)} = \bar{y} + \frac{1}{2} y_{\sigma^2} + dy_t^{(1)} + dy_t^{(2)} \tag{21}$$

where

$$dy_t^{(1)} = y_z dz_t^{(1)}, \; dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{22}$$

$$dy_t^{(2)} = y_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2} y_{z^2} dz_t^{(1)^{\otimes [2]}} \tag{23}$$

Proof. Combine (18) and (12) to express $y_t^{(2)}$ as a linear function of the constants $\bar{y}$ and $\frac{1}{2} y_{\sigma^2}$ and the first and second order increments $dy_t^{(1)}$ and $dy_t^{(2)}$. Expressing the the first order increment in terms of the vector $dz_t^{(1)}$ and rearranging the coefficient matrices accordingly[15] yields the desired

[15] This can be implemented using Koning, Neudecker, and Wansbeek's (1991) block Kronecker product. See the

result. □

The second order recursive approximation (21) preserves the natural decomposition into order of approximation embedded in its nonlinear moving average counterpart (17) — Moving to the second order, y_{σ^2} adjusts the first order approximation for the variance of future shocks, and $dy_t^{(2)}$ for the second order effects of the realized shocks. While (21) is an equivalent rewriting of (17) and therefore accordingly stable, its stability can be seen by examining the linearly recursive structure of the second order increment. As a recursion in the variables, $dy_t^{(2)}$ in (20) shares the same coefficient with (14) on its homogenous part. The inhomogeneous part, consisting of the first order increment and the shocks only, inherits the stability from the previous order of approximation.

Besides stability, the second order recursive approximation (21) is centered at the second order approximation of stochastic steady state in definition 2.2 given by

$$\overline{y}^{(2)stoch} = \overline{y} + \frac{1}{2} y_{\sigma^2} \tag{24}$$

To see this, note that in the absence of the past and present shocks, (13) and (19) imply that both the first and second order increments are zero, leaving the approximation centered at the deterministic steady state value plus the risk adjustment for the variance of future shocks. Likewise, $F(\sigma, 0, \ldots) \approx F(0, 0, \ldots) + \frac{1}{2} F_{\sigma^2}(0, 0, \ldots)$ has two nonzero terms up to second order that are solved by $\overline{y}$ and $\frac{1}{2} y_{\sigma^2}$ respectively.

3.3 Third Order Recursive Approximation

The third order approximation of the policy function takes the form

$$\begin{aligned} y_t^{(3)} = & \overline{y} + \frac{1}{2} y_{\sigma^2} + \frac{1}{6} y_{\sigma^3} + \frac{1}{2} \sum_{i=0}^{\infty} \left(y_i + y_{\sigma^2, i} \right) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\ & + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k,j,i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \end{aligned} \tag{25}$$

To derive the recursive representation at third order, we define the third order increment as the difference between the second and third order approximation, subtracting the constant risk adjustment of the third order

$$dy_t^{(3)} = y_t^{(3)} - \frac{1}{6} y_{\sigma^3} - y_t^{(2)} \tag{26}$$

Inserting (25) and (17) in the previous equation yields the moving average representation of

Appendix.

the third order increment

$$dy_t^{(3)} = \frac{1}{2}\sum_{i=0}^{\infty} y_{\sigma^2,i}\varepsilon_{t-i} + \frac{1}{6}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} y_{k,j,i}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \tag{27}$$

The increment of the third order approximation can be expressed recursively, as we summarize in the following

Proposition 3.5. *The third order increment, $dy_t^{(3)}$, can be expressed as a linear function of its own past and products of terms of lower order according to the following recursion*

$$\begin{aligned} dy_t^{(3)} - \alpha dy_{t-1}^{(3)state} =& \frac{1}{6}\left[\beta_{333,1} dy_{t-1}^{(1)state\otimes[3]} + \beta_{000}\varepsilon_t^{\otimes[3]}\right] + \beta_{22}\left(dy_{t-1}^{(2)state}\otimes dy_{t-1}^{(1)state}\right) + \beta_{20}\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right) \\ &+ \frac{1}{2}\left[\beta_{300}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t^{\otimes[2]}\right) + \beta_{330,1}\left(dy_{t-1}^{(1)state\otimes[2]}\otimes\varepsilon_t\right) + \beta_{\sigma^2 0}\varepsilon_t + \beta_{\sigma^2 1} dy_{t-1}^{(1)state}\right] \end{aligned} \tag{28}$$

Proof. See the Appendix. □

Combining the increment definitions and recursive representations at first, second and third order, we construct the following third order recursive formula for y_t

Proposition 3.6. *The third order approximation of y_t, denoted $y_t^{(3)}$, can be expressed recursively through*

$$y_t^{(3)} = \bar{y} + \frac{1}{2}y_{\sigma^2} + \frac{1}{6}y_{\sigma^3} + dy_t^{(1)} + dy_t^{(2)} + dy_t^{(3)} \tag{29}$$

where

$$dy_t^{(1)} = y_z dz_t^{(1)},\ dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{30}$$

$$dy_t^{(2)} = y_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2}y_{z^2} dz_t^{(1)^{\otimes[2]}} \tag{31}$$

$$dy_t^{(3)} = y_{y^{state}} dy_{t-1}^{(3)state} + \frac{1}{6}y_{z^3} dz_t^{(1)^{\otimes[3]}} + \frac{1}{2}y_{\sigma^2 z} dz_t^{(1)} + y_{y^{state}z}\left(dy_{t-1}^{(2)state}\otimes dz_{t-1}^{(1)}\right) \tag{32}$$

Proof. Combine (26), (18), and (12) to express $y_t^{(3)}$ as a linear function of the constants $\bar{y}$ and $\frac{1}{2}y_{\sigma^2}$ and the first through third order increments $dy_t^{(1)}$, $dy_t^{(2)}$, and $dy_t^{(3)}$. Expressing the the first order increment in terms of the vector $dz_t^{(1)}$ and rearranging the coefficient matrices accordingly[16] yields the desired result. □

The third order recursive approximation (29) follows the pattern of lower orders and can be decomposed into order of approximation and risk adjustment. In the third order, y_{σ^3} adjusts the second order approximation for the skewness of future shocks. The third order increment, $dy_t^{(3)}$, adjusts the approximation for the third order effects of the realized shocks and opens the first order increment to the variance of future shocks, delivering a time-varying risk adjustment term.

[16] This can be done using the Block Kronecker product, see footnote 15.

As in the second order case, (29) is an equivalent rewriting of its moving average counterpart (25) and accordingly stable. The stability is also implied by the linearly recursive structure of the third order increment $dy_t^{(3)}$ in (28). This recursion shares the same homogenous coefficient with the recursions of the first and second order increments. Its inhomogeneous part, consisting of shocks and the increments of the first and second order only, inherits the stability from the previous order of approximation.

As was the case in the second order, (29) is centered at the third order approximation of stochastic steady state in definition 2.2

$$\bar{y}^{(3)stoch} = \bar{y} + \frac{1}{2} y_{\sigma^2} + \frac{1}{6} y_{\sigma^3} \tag{33}$$

as can be verified analogously to the second order case.

The third order increment can be decomposed into a time varying risk adjustment increment, $dy_t^{(3)risk}$, and a third order amplification increment, $dy_t^{(3)amp}$. Both of which can be expressed recursively, as we summarize in the following

Corollary 3.7. *The third order increment can be decomposed into two separate increments,* $dy_t^{(3)} \equiv dy_t^{(3)risk} + dy_t^{(3)amp}$*, both of which can be expressed as linear functions of their own past and products of terms of lower order according to the following recursions*

$$dy_t^{(3)risk} = \alpha dy_{t-1}^{(3)state,risk} + \frac{1}{2}\left(\beta_{\sigma^2 0}\varepsilon_t + \beta_{\sigma^2 1} dy_{t-1}^{(1)state}\right) \tag{34}$$

$$\begin{aligned} dy_t^{(3)amp} =& \alpha dy_{t-1}^{(3)state,amp} + \frac{1}{6}\left[\beta_{333,1} dy_{t-1}^{(1)state\otimes[3]} + \beta_{000}\varepsilon_t^{\otimes[3]}\right] \\ &+ \beta_{22}\left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state}\right) + \beta_{20}\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t\right) \end{aligned} \tag{35}$$

$$+ \frac{1}{2}\left[\beta_{300}\left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes[2]}\right) + \beta_{330,1}\left(dy_{t-1}^{(1)state\otimes[2]} \otimes \varepsilon_t\right)\right] \tag{36}$$

Proof. See the Appendix. □

This decomposition clearly separates the nonlinear time varying effects in a third order approximation that arise from higher order (quadratic and cubic) deterministic terms and the linear time varying risk adjustment terms. Thus, enabling a readily identifiable differentiation between, e.g., time varying precautionary motives and asymmetric responses to shocks.

4 Pruning Abounds

In this section, we present the state space solution as a policy function that maps from the endogenous variable in the past and the realization of current shock into the endogenous variable itself, to the class of models introduced in section 2 and approximate the solution with a Taylor series.[17] Simulating such an approximation of second or higher order may generate explosive time paths as noted by Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006, p. 2479) and Kim, Kim, Schaumburg, and Sims (2008, p. 3408), as additional, higher order nonlinear terms accumulate. While various pruning algorithms for the second and third order approximation have been provided by the literature to restore the desired stability, a comparison between these algorithms has yet to be made.[18] We present these pruning algorithms in a unified notation, and compare them to the nonlinear moving average based recursive algorithm derived in section 3.

4.1 State Space Perturbation Foundations

The state space counterpart[19] to the nonlinear moving average solution form of section 2 is given by

$$y_t = g(\sigma, z_t), \quad g : \mathbb{R}^+ \times \mathbb{R}^{nz} \to \mathbb{R}^{ny} \tag{37}$$

where σ scales risk and the state vector z_t given by[20]

$$z_t = \begin{bmatrix} y_{t-1}^{state} \\ \varepsilon_t \end{bmatrix} \in \mathbb{R}^{n_z \times 1}, \text{ where } nz = ny + ne \tag{38}$$

Assuming time invariance of the policy function and using $\tilde{g}$ to denote y_{t+1}, inserting into the problem statement (1), and scaling risk give

$$0 = E_t \left[f\left(\tilde{g}\left(\sigma, \begin{bmatrix} g(\sigma, z_t) \\ \sigma\varepsilon_{t+1} \end{bmatrix} \right), g(\sigma, z_t), z_t \right) \right] = F(\sigma, z_t) \tag{39}$$

a function with arguments σ and z_t.[21] The Taylor series approximation of the state space solution (2) will be developed around a deterministic steady state, which alternatively but equivalently to

[17] This nonlinear state space perturbation literature was initiated by Gaspar and Judd (1997), Judd and Guu (1997), and Judd (1998, ch. 13).

[18] Den Haan and De Wind (2012) compare their version of the pruning algorithm with standard perturbations and their own "perturbation plus' algorithm, yet do not compare to other pruning algorithms.

[19] Our formulation follows Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare, Anderson, Levin, and Swanson's (2006) PerturbationAIM and Juillard (2011). Jin and Judd's (2002) or Schmitt-Grohé and Uribe's (2004) model classes can be rearranged to fit (5) as we will discuss below.

[20] Note that we are recycling notation from the previous section by using z_t in analogy to dz_t there.

[21] Note that ε_{t+1} is not an argument of F as discussed in section 2.

definition 2.1 can be defined as

Definition 4.1. *Deterministic Steady State*

Let $\overline{y} \in \mathbb{R}^{ny}$ be a vector such that

$$0 = F(0,\overline{z}), \text{ where } \overline{z} = \begin{bmatrix} \overline{y} \\ 0 \end{bmatrix} \tag{40}$$

solving (39) in the absence of both risk ($\sigma = 0$) and shocks ($\varepsilon_t = 0$).

The policy function evaluated at the deterministic steady state is thus $\overline{y} = g(0,\overline{z})$ and, assuming (37) is C^M with respect to all its arguments, we can write a Taylor series approximation of $y_t = g(\sigma, z_t)$ at a deterministic steady state as

$$y_t = \sum_{j=0}^{M} \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^j \sigma^i} \sigma^i \right] (z_t - \overline{z})^{\otimes[j]} \tag{41}$$

where $g_{z^j \sigma^i} \in \mathbb{R}^{ny \times nz^j}$ is the partial derivative of the vector function g with respect to the state vector z_t j times and the perturbation parameter σ i times evaluated at the deterministic steady state. Here $\left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i\right]$ collects all the coefficients associated with the j'th fold Kronecker product of the state vector, $(z_t - \overline{z})$. Higher orders of σ adjust the Taylor series coefficients for risk by successively opening the coefficients to higher moments in the distribution of future shocks. Out to third order and for $\sigma = 1$, (41) is given, where only terms with nonzero coefficients have been included, by

$$y_t^{(1)} = \overline{y} + g_z \left(z_t^{(1)} - \overline{z} \right) \tag{42}$$

at first,

$$y_t^{(2)} = \overline{y} + \frac{1}{2} g_{\sigma^2} + g_z \left(z_t^{(2)} - \overline{z} \right) + \frac{1}{2} g_{z^2} \left(z_t^{(2)} - \overline{z} \right)^{\otimes[2]} \tag{43}$$

at second, and

$$y_t^{(3)} = \overline{y} + \frac{1}{2} g_{\sigma^2} + \frac{1}{6} g_{\sigma^3} + \left[g_z + \frac{1}{2} g_{\sigma^2 z} \right] \left(z_t^{(3)} - \overline{z} \right) + \frac{1}{2} g_{z^2} \left(z_t^{(3)} - \overline{z} \right)^{\otimes[2]} + \frac{1}{6} g_{z^3} \left(z_t^{(3)} - \overline{z} \right)^{\otimes[3]} \tag{44}$$

at third.

Stationary points, or steady states, of y in approximations will play a key role in understanding the differences between many of the pruning algorithms we will examine. Standard linear approximations are certainty equivalent and their stationary points are the deterministic steady states of definition 2.1 (or equivalently definition 4.1). By extension, one might expect or desire[22] an M'th order pruned perturbation to have as a stationary point the M'th order approximation of the

[22] See Evers (2010) and Den Haan and De Wind (2012).

stochastic steady state. Accordingly and analogously to definition 2.2, we define the stochastic or "risky" steady state as the stationary point in the absence of past shocks but the risk of future shocks, which in the state space setting here is given by

Definition 4.2. *Stochastic Steady State*

Let $\overline{y}^{stoch} = g(1,\overline{z}^{stoch}) \in \mathbb{R}^{ny}$ *be a vector such that*

(45)

$$0 = E_t\left[f\left(\tilde{g}^{fwdendo}\left(1,\begin{bmatrix} g^{state}(1,\overline{z}^{stoch}) \\ \varepsilon_{t+1}\end{bmatrix}\right), g(1,\overline{z}^{stoch}), \overline{z}\right)\right] = F(1,\overline{z}^{stoch}),\ \textit{where}\ \overline{z}^{stoch} = \begin{bmatrix}\overline{y}^{stoch} \\ 0\end{bmatrix}$$

As in section 2 for the nonlinear moving average representation, the stochastic steady state can be approximated by expanding $0 = F(\sigma,\widetilde{\overline{z}})$ in σ around the deterministic steady state, assuming the $\sigma = 1$ model is sufficient close to its deterministic, $\sigma = 0$ counterpart. Notice that unlike the nonlinear moving average, the state space formulation, $0 = F(\sigma,\widetilde{\overline{z}})$, is complicated by the additional argument, $\widetilde{\overline{z}}$ the steady state of the state vector—itself a function of σ, being equal to the deterministic steady state when $\sigma = 0$ and the stochastic steady state when $\sigma = 1$.

4.2 Second Order Pruning

When iterating on the second order approximation of (43), the quadratic term will generate nonlinear terms of successively higher order, see Kim, Kim, Schaumburg, and Sims (2008). These accumulated terms can lead to explosive time paths and Kim, Kim, Schaumburg, and Sims (2008) suggested pruning these higher order terms by operating the quadratic on the first order simulated time path, restoring stability. The algorithms presented here all agree on this point, but differ on the the constant risk adjustment term that enters in the approximation. Throughout the rest of this section and in section 5, we recycle the notation $dy_t^{(n)}$ (where n denotes the order of approximation) and $dz_t^{(1)}$ from section 3 and redefine them in each and every pruning algorithm we will be introducing.

4.2.1 Kim, Kim, Schaumburg, and Sims's (2008) Pruning Algorithm

Kim, Kim, Schaumburg, and Sims (2008) were the first to formulate a pruning algorithm for the second order approximation, (43).[23]

[23] In Dynare—see Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011), the initial value of the first order term, say $dy_0^{(1)state}$, need not be set equal to the deterministic steady state and can be set to any arbitrary value. Whether this corresponds to a second order accurate approximation of an arbitrary initial value has not, to our

Lemma 4.3 (Kim, Kim, Schaumburg, and Sims's (2008) Second Order Pruning Algorithm)**.**

$$y_t^{(2)} = \overline{y} + dy_t^{(1)} + dy_t^{(2)} \tag{46}$$

where

$$dy_t^{(1)} = g_z dz_t^{(1)},\ dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{47}$$

$$dy_t^{(2)} = g_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2}\left[g_{\sigma^2} + g_{z^2} dz_t^{(1)^{\otimes[2]}} \right] \tag{48}$$

Apart from replacing the second order base of the two-fold Kronecker power with its first order counterpart $dz_t^{(1)}$ to restore stability in simulation, this algorithm transitions deterministically to a second order approximation to the stochastic steady state of definition 4.2. I.e., setting the initial value of y_0 to its deterministic steady state value $\overline{y}$ and simulating forward with all shock realizations set to zero, the constant risk correction term g_{σ^2} is accumulated at each iteration as it is a component of $dy_t^{(2)}$, and therefore keeps accumulating along with the iteration, pushing the algorithm away from $\overline{y}$, past $\overline{y} + \frac{1}{2} g_{\sigma^2}$, and to $\overline{y} + (I - g_y)^{-1} \frac{1}{2} g_{\sigma^2}$.

4.2.2 Den Haan and De Wind's (2012) Second Order Pruning Algorithm

Den Haan and De Wind (2012) formulated the following alternative second order pruning algorithm motivated by the observation that the steady state of the second order approximation does not coincide with the second order approximation of the (stochastic) steady state,[24]

Lemma 4.4 (Den Haan and De Wind's (2012) Second Order Pruning Algorithm)**.**

$$y_t^{(2)} = \overline{y} + \frac{1}{2} g_{\sigma^2} + dy_t^{(2)} \tag{49}$$

where

$$dy_t^{(1)} = g_z dz_t^{(1)},\ dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{50}$$

$$dy_t^{(2)} = g_z \begin{bmatrix} dy_{t-1}^{(2)state} \\ \varepsilon_t \end{bmatrix} + \frac{1}{2} g_{z^2} dz_t^{(1)^{\otimes[2]}} \tag{51}$$

While pruning the quadratic term in the same way as Kim, Kim, Schaumburg, and Sims's (2008) algorithm does, this algorithm does not transition deterministically, but remains at $\overline{y}^{stoch(2)} = \overline{y} + \frac{1}{2} g_{\sigma^2}$. It restores this consistency by excluding g_{σ^2} from its $dy_t^{(2)}$, and therefore prevents g_{σ^2} from accumulating in simulation. However, this point is not a second order approximation of the

knowledge, been proven.

[24] See also Evers (2010) for more on this and other consistency points.

stochastic steady state in definition 4.2 and its appropriateness as a centering point of the algorithm is unclear.

4.2.3 Comparison of Second Order Pruning Algorithms

As noted also by Den Haan and De Wind (2012), Kim, Kim, Schaumburg, and Sims's (2008) pruning algorithm transitions from $\overline{y} + \frac{1}{2} g_{\sigma^2}$ to some other steady state when $dy_t^{(1)}$ is initialized at zero.[25] As we state in the following proposition, Kim, Kim, Schaumburg, and Sims's (2008) pruning algorithm transitions to $\overline{y} + \frac{1}{2} y_{\sigma^2}$, the second order approximation of the stochastic steady state (see definition 2.2) using nonlinear moving average policy functions. Additionally, all other coefficients (and hence all coefficients that are not partials with respect to σ) are identical in all three algorithms.

Proposition 4.5 (Deterministic Equivalence, Risk Sensitive Nonequivalence with Section 3)**.** *The algorithms in lemmata 4.3 and 4.4 and in proposition 3.4 are identical in all coefficients except for the constant term involving $\frac{1}{2} g_{\sigma^2}$ (or $\frac{1}{2} y_{\sigma^2}$).*

As a consequence, when all shock realizations are zero in all periods,

- *the algorithm in lemma 4.4 will remain at $\overline{y} + \frac{1}{2} g_{\sigma^2}$*
- *the algorithm in lemma 4.3 will transition from $\overline{y} + \frac{1}{2} g_{\sigma^2}$ to $\overline{y} + (I - g_y)^{-1} \frac{1}{2} g_{\sigma^2}$*
- *the algorithm in proposition 3.4 will remain at $\overline{y} + \frac{1}{2} y_{\sigma^2}$*
- $(I - g_y)^{-1} g_{\sigma^2} = \frac{1}{2} y_{\sigma^2}$

Proof. See the Appendix. □

Thus, asymptotically, Kim, Kim, Schaumburg, and Sims's (2008) pruning algorithm and our second order recursive nonlinear moving average (see proposition 3.4) converge deterministically, as the former converges to the latter.

4.3 Higher Order Pruning

The third order approximation (44) contains quadratic and cubic terms, both of which are sources of potential instability. As noted by Ruge-Murcia (2012), the pruning concept proposed by Kim,

[25] That is, when the first order approximation is started at the deterministic steady state. It is noteworthy that Kim, Kim, Schaumburg, and Sims's (2008) pruning algorithm as implemented by Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare lets the user initialize $dy_t^{(1)}$ arbitrarily, whether this translates to second order accurate initial values is relegated to future study.

Kim, Schaumburg, and Sims (2008) at second order does not generalize straightforwardly to higher orders. Indeed, at third order, we find discrepancies between pruning algorithms in how they prune the cubic term. While these differences are in line with Lombardo (2012) and Den Haan and De Wind's (2012) critique that pruning is an ad hoc procedure, our nonlinear moving average based recursive algorithm can be viewed as a theoretical support for pruning and guidance in terms of choosing the way of reconstructing the potentially instable nonlinear terms consistent with the original, unpruned nonlinear approximation.

4.3.1 Andreasen's (2012) Algorithm[26]

This algorithm[27] chooses to keep both the quadratic and cubic term in the unpruned third order approximation, (44). It prunes the quadratic term by replacing it with the Kronecker product of the first order approximation. The cubic term is replaced by the first order approximation raising to the three-fold Kronecker power, and the Kronecker product of the pruned quadratic term and the first order approximation.

Lemma 4.6 (Andreasen's (2012) and Third Order Pruning Algorithm)**.**

$$y_t^{(3)} = \overline{y} + dy_t^{(1)} + dy_t^{(2)} + dy_t^{(3)} \tag{52}$$

where

$$dy_t^{(1)} = g_z dz_t^{(1)},\ dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{53}$$

$$dy_t^{(2)} = g_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2}\left[g_{\sigma^2} + g_{z^2} dz_t^{(1)^{\otimes[2]}}\right] \tag{54}$$

$$dy_t^{(3)} = g_{y^{state}} dy_{t-1}^{(3)state} + \frac{1}{6}\left[g_{\sigma^3} + g_{z^3} dz_t^{(1)^{\otimes[3]}}\right] + \frac{1}{2} g_{\sigma^2 z} dz_t^{(1)} + g_{y^{state} z}\left(dy_{t-1}^{(2)state} \otimes dz_{t-1}^{(1)}\right) \tag{55}$$

This algorithm is, we argue, the third order equivalent to Kim, Kim, Schaumburg, and Sims (2008), because its differences to our nonlinear moving average algorithm are third order analogs (owing to cumulative risk sensitive adjustments) to the differences between Kim, Kim, Schaum-

[26] Downloaded on January 11, 2013 as `ForWeb_NewKeynesianModel.zip` from `http://ideas.repec.org/c/red/ccodes/11-84.html` as linked through `http://www.economicdynamics.org/RED15.htm`. The file `simulate_3rd_kron.m` contains the the following algorithm and is preceded by the header

```
% By Martin M. Andreasen, April 22 2010
% This function simulates the model when solved up to third order.
% The pruning scheme is used.
```

[27] See also Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2012), for an implementation to time series properties and further documentation of this algorithm.

burg, and Sims's (2008) and our algorithm at second order.

Proposition 4.7 (Deterministic Equivalence, Risk Sensitive Nonequivalence with Section 3)**.** *The algorithms in lemma 4.6 and in 3.6 are identical in all coefficients except for terms involving* σ, $\frac{1}{2}g_{\sigma^2}$ *(or* $\frac{1}{2}y_{\sigma^2}$*),* $\frac{1}{6}g_{\sigma^3}$ *(or* $\frac{1}{6}y_{\sigma^3}$*), and* $\frac{1}{2}g_{\sigma^2 z}$ *(or* $\frac{1}{2}y_{\sigma^2 z}$*).*

As a consequence, when all shock realizations are zero in all periods,

- *the algorithm in lemma 4.6 will transition from* $\overline{y}+\frac{1}{2}g_{\sigma^2}+\frac{1}{6}g_{\sigma^3}$ *to* $\overline{y}+(I-g_y)^{-1}\left(\frac{1}{2}g_{\sigma^2}+\frac{1}{6}g_{\sigma^3}\right)$
- *the algorithm in proposition 3.4 will remain at* $\overline{y}+\frac{1}{2}y_{\sigma^2}+\frac{1}{6}y_{\sigma^3}$
- $(I-g_y)^{-1}\left(\frac{1}{2}g_{\sigma^2}+\frac{1}{6}g_{\sigma^3}\right)=\frac{1}{2}y_{\sigma^2}+\frac{1}{6}y_{\sigma^3}$

Proof. See the Appendix. □

Skewed risk adjustments deterministically accumulate along with the second order risk adjustments for variance. At third order, the differences in instantaneous second order risk adjustments for variance are interacted with the vector of states, leading to differences in the time varying response to risk posited by the two algorithms.

4.3.2 Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe's (2011) Algorithm[28]

This algorithm keeps both the quadratic and cubic term in the unpruned third order approximation, (44), as well. While it again prunes the quadratic term by replacing it with the Kronecker product of the first order approximation, this algorithm prunes the cubic term by replacing it with the first order approximation raising to the three-fold Kronecker power only, and does not include the Kronecker product of the pruned quadratic term and the first order approximation like the Andreasen's (2012) algorithm does.

Lemma 4.8 (Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe's (2011) Third Order Pruning Algorithm)**.**

$$y_t^{(3)}=\overline{y}+dy_t^{(3)} \tag{56}$$

where

$$dy_t^{(1)}=g_z dz_t^{(1)},\ dz_t^{(1)}=\begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{57}$$

[28] Downloaded on January 11, 2013 as `20090428_data.zip` from http://www.aeaweb.org/articles.php?doi=10.1257/aer.101 The file `code_AERirf_moments.m` contains the the following algorithm and is dated December, 2010.

$$dy_t^{(3)} = g_z \begin{bmatrix} dy_{t-1}^{(3)state} \\ \varepsilon_t \end{bmatrix} + \frac{1}{2}\left[g_{\sigma^2} + g_{z^2} dz_{t-1}^{(1)\otimes[2]}\right] + \frac{1}{6}\left[g_{\sigma^3} + g_{z^3} dz_t^{(1)\otimes[3]} + 3g_{\sigma^2 z} dz_t^{(1)}\right] \tag{58}$$

This algorithm, like the previous one, will transition from $\overline{y} + \frac{1}{2}g_{\sigma^2} + \frac{1}{6}g_{\sigma^3}$ to $\overline{y} + (I - g_y)^{-1}\left(\frac{1}{2}g_{\sigma^2} + \frac{1}{6}g_{\sigma^3}\right)$ as the two constant risk adjustment terms, g_{σ^2} and g_{σ^3}, are included in its $dy_t^{(3)}$ and therefore will keep accumulating in iteration.

4.3.3 Michel Juillard's Algorithm[29]

This algorithm keeps both the quadratic and cubic term of the unpruned third order approximation, (44), pruning the quadratic term by replacing it with the Kronecker product of the first order approximation just like the previous two algorithms. When pruning the cubic term, it raises the first order approximation to the three-fold Kronecker power as the previous two algorithms do. However, this algorithm then multiplies (in Kronecker) its pruned second order term with the endogenous state space of the first order approximation, differing from Andreasen's (2012) algorithm who multiplies (in Kronecker) its pruned second order term with the exogenous state space (vector of shocks ε_t) as well.

Lemma 4.9 (Michel Juillard's Third Order Pruning Algorithm)**.**

$$y_t^{(3)} = \overline{y} + dy_t^{(1)} + dy_t^{(2)} + dy_t^{(3)} \tag{59}$$

where

$$dy_t^{(1)} = g_z dz_t^{(1)},\ dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{60}$$

$$dy_t^{(2)} = g_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2}\left[g_{\sigma^2} + g_{z^2} dz_t^{(1)\otimes[2]}\right] \tag{61}$$

(62)

$$dy_t^{(3)} = g_{y^{state}} dy_{t-1}^{(3)state} + \frac{1}{6}\left[g_{\sigma^3} + g_{z^3} dz_t^{(1)\otimes[3]}\right] + \frac{1}{2} g_{\sigma^2 z} dz_t^{(1)} + g_{(y^{state})^2}\left(dy_{t-1}^{(1)state} \otimes dy_{t-1}^{(2)state}\right)$$

This algorithm, like the previous two, will transition from $\overline{y} + \frac{1}{2}g_{\sigma^2} + \frac{1}{6}g_{\sigma^3}$ to $\overline{y} + (I - g_y)^{-1}\left(\frac{1}{2}g_{\sigma^2} + \frac{1}{6}g_{\sigma^3}\right)$ as the two constant risk adjustment terms, g_{σ^2} and g_{σ^3}, are included in its $dy_t^{(3)}$ and therefore will keep accumulating in iteration.

[29] Downloaded as `dynare-2013-01-10-win.exe` from http://www.dynare.org/snapshot/windows/ on January 11, 2013. Thank you to Michel Juillard for drawing our attention to this undocumented feature in Dynare.

4.3.4 Den Haan and De Wind's (2012) Third Order Pruning Algorithm

This algorithm keeps both the quadratic and cubic term of the unpruned third order approximation, (44), pruning the quadratic term by replacing it with the Kronecker product of the first order approximation just like the previous three algorithms. When pruning the cubic term, it raises the first order approximation to the three-fold Kronecker power as the previous three algorithms do. However, unlike Michel Juillard's algorithm who multiplies (in Kronecker) its pruned quadratic term with the endogenous state space of the first order approximation, and Andreasen's (2012) algorithm who multiplies (in Kronecker) its pruned quadratic term with the first order approximation, this algorithm raises the pruned second order term to the second-fold Kronecker power. This introduces terms of fourth order, which is responsible for the relative reduction in accuracy compared to the other third order algorithms, as we shall document. Additionally, the time-varying risk adjustment at third order is applied retroactively to the first order approximation, see Den Haan and De Wind (2012, p. 1490) and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2012, p. 9). It is conceivable that a large enough risk adjustment could thus introduce instability into their first order approximation.

Lemma 4.10 (Den Haan and De Wind's (2012) Third Order Pruning Algorithm)**.**

$$y_t^{(3)} = \overline{y} + \frac{1}{2} g_{\sigma^2} + \frac{1}{6} g_{\sigma^3} + dy_t^{(3)} \tag{63}$$

where

$$dy_t^{(1)} = \left(g_{y^{state}} + \frac{1}{2} g_{\sigma^2 y^{state}} \right) dy_{t-1}^{(1)state} + \left(g_{\varepsilon} + \frac{1}{2} g_{\sigma^2 \varepsilon} \right) \varepsilon_t \tag{64}$$

$$dy_t^{(2)} = \left(g_{y^{state}} + \frac{1}{2} g_{\sigma^2 y^{state}} \right) dy_{t-1}^{(2)state} + \frac{1}{2} g_{z^2} dz_t^{(1)^{\otimes[2]}}, \; dz_t^{(1)} = \begin{bmatrix} dy_{t-1}^{(1)state} \\ \varepsilon_t \end{bmatrix} \tag{65}$$

$$\begin{aligned} dy_t^{(3)} = & \left(g_{y^{state}} + \frac{1}{2} g_{\sigma^2 y^{state}} \right) dy_{t-1}^{(3)state} + \left(g_{\varepsilon} + \frac{1}{2} g_{\sigma^2 \varepsilon} \right) \varepsilon_t \\ & + \frac{1}{2} g_{z^2} dz_t^{(2)^{\otimes[2]}} + \frac{1}{6} g_{z^3} dz_t^{(1)^{\otimes[3]}}, \; dz_t^{(2)} = \begin{bmatrix} dy_{t-1}^{(2)state} \\ \varepsilon_t \end{bmatrix} \end{aligned} \tag{66}$$

Unlike the previous three algorithms, this algorithm like its second order counterpart does not have a deterministic transition, remaining at $\overline{y} + \frac{1}{2} g_{\sigma^2} + \frac{1}{6} g_{\sigma^3}$. Again, this point is not a third order approximation of the stochastic steady state in definition 4.2 and its appropriateness as a centering point of the algorithm is unclear.

5 Lombardo's (2012) Matched Perturbation Algorithm

Lombardo (2012) presents a method based on "matched perturbations," see Holmes (1995), that delivers higher order stable recursive state space approximations that are linearly recursive in the order of nonlinear terms. All of these features are shared by our method based on nonlinear moving averages as presented in section 3 as well as many of the various pruning algorithms examined in section 4. In this section, we will determine whether Lombardo's (2012) method justifies a particular pruning method of section 4 or whether it produces an independent method as did our nonlinear moving average in section 3. To match his setup, we must redefine the problem statement above slightly[30] by defining

$$\tilde{z}_t \equiv \begin{bmatrix} y_{t-1}^{state} \\ \sigma\varepsilon_t \end{bmatrix} \tag{67}$$

and replacing z_t in (39)

$$0 = E_t\left[f\left(\tilde{g}\left(\sigma, \begin{bmatrix} g(\sigma,\tilde{z}_t) \\ \sigma\varepsilon_{t+1} \end{bmatrix}\right), g(\sigma,\tilde{z}_t), \tilde{z}_t\right)\right] = \tilde{F}(\sigma, z_t) \tag{68}$$

still a function with arguments σ and z_t.[31] Essentially Lombardo (2012) uses σ to expand from the deterministic steady state to the stochastic dynamic solution, whereas the formulation we have used above following Jin and Judd (2002) and others uses σ to expand the deterministic dynamic solution to the stochastic dynamic solution;[32] when $\sigma = 1$, however, both approaches are equivalent.

To third order, the Taylor series approximation or standard perturbation of the solution to (68), where only terms with nonzero coefficients have been included, is given by

$$y_t = \overline{y} + \frac{1}{2}g_{\sigma^2} + \frac{1}{6}g_{\sigma^3} + \left[g_z + \frac{1}{2}g_{\sigma^2 z}\right](\tilde{z}_t - \overline{z}) + \frac{1}{2}g_{z^2}(\tilde{z}_t - \overline{z})^{\otimes[2]} + \frac{1}{6}g_{z^3}(\tilde{z}_t - \overline{z})^{\otimes[3]} \tag{69}$$

Lombardo (2012) gives the following procedure for deriving matched perturbations or series expansions of the foregoing: guess that the solution is of the linearly recursive (in order) form, where we have adapted his procedure to our notation,

$$y_t - \overline{y} = \sigma dy_t^{(1)} + \sigma^2 dy_t^{(2)} + \sigma^3 dy_t^{(3)} + \ldots \tag{70}$$

[30] Note, the following perturbation setup is widely used, see Schmitt-Grohé and Uribe (2004) and others, but identical to the statement used above for $\sigma = 1$. Unlike the other methods presented above, however, Lombardo's (2012) matched perturbation method cannot be readily adapted to alternative problem statements.

[31] Note that ε_{t+1} is not an argument of F as discussed previously.

[32] See also Den Haan and De Wind (2012), who state in their supplemental Appendix that Lombardo's (2012) method "does not describe any transition dynamics" when $\sigma = 0$.

insert the guess into (69)

$$\sigma dy_t^{(1)} + \sigma^2 dy_t^{(2)} + \sigma^3 dy_t^{(3)} = \frac{1}{2} g_{\sigma^2}\sigma^2 + \frac{1}{6} g_{\sigma^3}\sigma^3 + \left[g_z + \frac{1}{2} g_{\sigma^2 z}\sigma^2 \right] \left[\begin{matrix} \sigma dy_{t-1}^{(1)state} + \sigma^2 dy_{t-1}^{(2)state} + \sigma^3 dy_{t-1}^{(3)state} + \dots \\ \sigma\varepsilon_t \end{matrix} \right]$$

$$+ \frac{1}{2} g_{z^2} \left[\begin{matrix} \sigma dy_{t-1}^{(1)state} + \sigma^2 dy_{t-1}^{(2)state} + \sigma^3 dy_{t-1}^{(3)state} + \dots \\ \sigma\varepsilon_t \end{matrix} \right]^{\otimes[2]}$$

$$+ \frac{1}{6} g_{z^3} \left[\begin{matrix} \sigma dy_{t-1}^{(1)state} + \sigma^2 dy_{t-1}^{(2)state} + \sigma^3 dy_{t-1}^{(3)state} + \dots \\ \sigma\varepsilon_t \end{matrix} \right]^{\otimes[3]} \tag{71}$$

and "equat[e] like powers" (Holmes 1995, p. 27) in σ, which gives

$$dy_t^{(1)} = g_z z_t^{(1)},\ dz_t^{(1)} = \left[dy_{t-1}^{(1)state} \varepsilon_t \right] \tag{72}$$

$$dy_t^{(2)} = g_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2} \left(g_{\sigma^2} + g_{z^2} dz_t^{(1)\otimes[2]} \right) \tag{73}$$

$$dy_t^{(3)} = g_{y^{state}} dy_{t-1}^{(3)state} + \frac{1}{6} \left(g_{\sigma^3} + g_{z^3} dz_t^{(1)\otimes[3]} \right) + \frac{1}{2} g_{\sigma^2 z} z_t^{(1)} + g_{y^{state} z} \left(dy_{t-1}^{(2)state} \otimes dz_t^{(1)} \right) \tag{74}$$

and Lombardo's (2012) second order series expansion approximation for the stochastic ($\sigma = 1$) case is

$$y_t = \overline{y} + dy_t^{(1)} + dy_t^{(2)} \tag{75}$$

and at third order

$$y_t = \overline{y} + dy_t^{(1)} + dy_t^{(2)} + \sigma^3 dy_t^{(3)} \tag{76}$$

Lombardo's (2012) method recovers Kim, Kim, Schaumburg, and Sims's (2008) pruning algorithm at second order and Andreasen's (2012) algorithm at third order, as we summarize in the following

Proposition 5.1 (Equivalence of Series Expansion and Pruning)**.** *Lombardo's (2012) method of series expansion is identical to*

- *the algorithm in lemma 4.3 at second order*
- *the algorithm in lemma 4.6 at third order*

Proof. By inspection. □

While Lombardo (2012) identifies the first equivalence, the equivalence at third order is apparently new. Indeed, Lombardo (2012, p. 12) seems to imply that his series expansion at third order would yield the algorithm in lemma 4.8, which does not include the cross product term $dy_{t-1}^{(2)state} \otimes dz_t^{(1)}$ as in the algorithm of lemma 4.6. This would be mistaken, as we have shown above. We conclude that Lombardo's (2012) method provides a rigorous foundation for the vari-

ants of pruning that are complete up to the order of approximation. Yet, as shown in propositions 4.5 and 4.7, these pruning algorithms (and hence Lombardo's (2012) method as well) differ from the recursive algorithms in section 3 in terms that adjust for risk, centering the approximation at the deterministic model and its lack of risk adjustment.

6 Applications to Production and Asset Pricing

In this section, we compare the numerical performance of the various pruning algorithms presented in section 4. A version of the stochastic neoclassical growth model and the asset pricing model in Burnside (1998) are chosen as the benchmarks to run the horse races, as both of the two models possess closed-form solution and widely used in evaluating the numerical performance of solution methods for DSGE models.

We employ three criteria for comparing models

$$E_1 = \frac{1}{T}\sum_{t=1}^{T}\left|\frac{x_t^{approx.} - x_t^{true}}{x_t^{true}}\right| \tag{77}$$

$$E_2 = \frac{1}{T}\sum_{t=1}^{T}\left(x_t^{approx.} - x_t^{true}\right)^2 \tag{78}$$

$$E_\infty = \max\left\{\left|\frac{x_t^{approx.} - x_t^{true}}{x_t^{true}}\right|\right\} \tag{79}$$

measuring the distance of the various pruning algorithms, including the nonlinear moving average based recursive algorithm, as well as the unpruned perturbations to the true solution in terms of average, mean square and maximal error at second and third order.

6.1 The Discrete Brock and Mirman (1972) Neoclassical Growth Model

In this section, we examine a version of the stochastic neoclassical growth model, case of log preferences in consumption and full depreciation, with a known solution to compare methods. This model has been used in numerous studies comparing numerical techniques and is a natural benchmark.

The model is populated by an infinitely lived representative household seeking to maximize its expected discounted lifetime utility given by

$$E_0\left[\sum_{t=0}^{\infty}\beta^t ln\left(C_t\right)\right] \tag{80}$$

where C_t is consumption, and $\beta \in (0,1)$ the discount factor, subject to

$$C_t + K_t = e^{Z_t} K_{t-1}^{\alpha} \tag{81}$$

where K_t is the capital stock accumulated today for productive purposes tomorrow, Z_t a stochastic productivity process, $\alpha \in [0,1]$ the capital share, and note that we have assumed complete depreciation. Maximization delivers the following first order condition

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \alpha e^{Z_{t+1}} K_t^{\alpha-1} \right] \tag{82}$$

an intertemporal Euler condition equalizing the expected present-discounted utility value of postponing consumption one period to its utility value today.

In this log preferences and complete capital depreciation case, a well-known closed-form solution for the policy functions exists given by

$$\begin{aligned} K_t &= \alpha\beta e^{Z_t} K_{t-1}^{\alpha} \\ C_t &= (1-\alpha\beta)\, e^{Z_t} K_{t-1}^{\alpha} \end{aligned} \tag{83}$$

Additionally, we will assume that productivity is described by

$$Z_t = \rho_Z Z_{t-1} + \varepsilon_{Z,t}, \;\; \varepsilon_{Z,t} \sim \mathcal{N}\left(0, (\chi\sigma_Z)^2\right) \tag{84}$$

with $|\rho_Z| < 1$ and $\varepsilon_{Z,t}$ the innovation with standard deviation $\chi\sigma_Z$. We use χ as a scaling factor that when equal to one, gives the standard deviation of the technology process as σ_Z, which we set to a standard calibration value.

As the model is loglinear, we could redefine the variables in terms of logarithms—e.g., $\exp(\hat{c}_t) \equiv C_t$—and a first-order approximation of either the state space or infinite moving average policy function, see Lan and Meyer-Gohde (2012b), would deliver (83). However, to study the properties of simulations generated by the methods compared above, we will compute perturbations in the the level variables using our method derived in section 3 and compare it with the standard state space perturbation and the "pruned" state space perturbations of Kim, Kim, Schaumburg, and Sims (2008) for second order and Andreasen (2012) for third order summarized in the previous section.

[Table 1 about here.]

In figures 1, 2, and 3, we plot the E_1, E_2, and E_∞ accuracy of the different perturbation and pruning methods out to third order for K_t measured relative to (83) for values of χ, thereby scaling up the standard deviation of the technology process, from one to fifty. We run 100 simulations of 10,000 periods and report the average result for E_1 and E_2 and the maximum for E_∞.

[Figure 1 about here.]

[Table 2 about here.]

For E_1 and E_2, a clear patter emerges. Increasing the order of approximation increases the accuracy of the approximation. The exceptions are provided by the second and third order perturbations after χ equal to seven and forty respectively reflecting explosive simulations after these values, as well as the third order pruning algorithm in lemma 4.10 that disappointingly is roughly as accurate as the first order approximation. Tables 2 and 4 confirm the results in Den Haan and De Wind (2012), regarding the accuracy of perturbation and pruning in the log preference and full depreciation special case of the neoclassical growth model.[33]

[Figure 2 about here.]

[Table 3 about here.]

At second order, all three pruning algorithms deliver numerically identical simulations. This follows directly from proposition 4.5, recognizing that the model of this section is certainty equivalent in its nonlinear form. Consequently at third order, our method in 3.6 is numerically identical to the method in lemma 4.6.

[Figure 3 about here.]

[Table 4 about here.]

Figures 4 and 5 display subsets of two simulations with large differences in different algorithms. Note that both the first order perturbation and the third order algorithm of Den Haan and De Wind (2012) yield negative values for capital in these cases.[34] The second order perturbation and pruning algorithms fall above and the third order perturbation and other pruning algorithms slightly below the exact value.

[33]They report in their Table 1 (Den Haan and De Wind 2012, p. 1492) that for $\sigma_Z = 0.1$ and otherwise identical calibration as we have chosen here E_1 and E_∞ errors for a first order of 8.00E-1 and 7.61E-1, second order perturbation of 1.90E-2 and 3.10E-1, and second order pruning of 2.00E-2 and 4.79E-1, which corresponds to a factor of $\chi = 14$ and lines up roughly with the results we report. Likewise their E_1 and E_∞ errors with $\sigma_Z = 0.2$ for a first order of 8.00E-1 and 7.61E-1, second order perturbation of 1.90E-2 and 3.10E-1, and second order pruning of 2.00E-2 and 4.79E-1 are comparable to our results.

[34]Though, all the algorithms we compare here are capable of the same due to their local nature.

[Figure 4 about here.]

[Figure 5 about here.]

Figure 6 shows an example explosive time path that the pruning algorithms guard against. A shock around the 70th period pushes the third order perturbation beyond a threshold, setting it on an unrecoverable upward explosion. This inaccuracy obviously dominates all other differences between the varying algorithms in this simulation.

[Figure 6 about here.]

6.2 The Asset Pricing Model of Burnside (1998)

An agent maximizes her expected discounted lifetime utility from consumption

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\gamma}}{1-\gamma}\right] \tag{85}$$

subject to the period budget constraint

$$C_t + P_t S_t = (D_t + P_t)\,S_{t-1} \tag{86}$$

where S_t is the end of period holding of the single asset, which is priced P_t at t and pays D_t dividends per unit held at the beginning of the period. Combining the agent's first order condition with market clearing delivers

$$v_t = \beta E_t\left[e^{(1-\gamma)x_{t+1}}\left(1+v_{t+1}\right)\right] \tag{87}$$

where $v_t \equiv P_t/D_t$ is the price dividend ratio and $x_t \equiv \ln\left(D_t/D_{t-1}\right)$ is the log dividend difference. Assuming that

$$x_t = (1-\rho)\,\mu - \rho x_{t-1} + \xi_t,\ \xi_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0,\sigma^2\right) \tag{88}$$

Burnside (1998) derives a closed form solution given by[35]

$$v_t = \sum_{i=1}^{\infty}\beta^i \exp\left[a_i + b_i\left(x_t-\mu\right)\right] \tag{90}$$

where

$$a_i = \alpha i\mu + \frac{1}{2}\alpha^2\frac{\sigma^2}{(1-\rho)^2}\left[i - 2\frac{\rho}{1-\rho}\left(1-\rho^i\right) + \rho^2\frac{1-\rho^{2i}}{1-\rho^2}\right] \tag{91}$$

[35] To ensure convergence

$$\beta\exp\left[\alpha\mu + \frac{1}{2}\alpha^2\frac{\sigma^2}{(1-\rho)^2}\right] < 1 \tag{89}$$

and

$$b_i = \alpha \frac{\rho}{1-\rho}\left(1-\rho^i\right) \tag{92}$$

where $\alpha \equiv (1-\gamma)$.

We compare the different pruning algorithms relative to this closed form solution for the different parameterizations used in Collard and Juillard (2001), corresponding to different levels of patience, of persistence and volatility of the log dividend difference process, and of curvature in the utility function. For each parameterization, we run 100 simulations of 10,000 periods each and present the relative errors of v_t according to the three criteria—average (E_1), mean square (E_2), and maximum (E_∞)—in tables 6 through 8.

[Table 5 about here.]

As Collard and Juillard (2001) observed for the linear approximation, all algorithms tend to deteriorate in accuracy as the log dividend difference process becomes more highly persistent (ρ increases) or volatile (σ increases), or risk aversion is increased (α decreases). This follows naturally from the local nature of all the approximations considered here, as increasing either of the two shock process parameters increases the cumulative variance of the process and increasing risk aversion makes the agent's policy functions more sensitive to the exogenous process.

[Table 6 about here.]

In general, increasing the order of approximation increases the accuracy of approximation. According to the E_1 criterium, see table 6, increasing the order of approximation (here from second to third order) can, however, lead to a deterioration in the quality of approximation in the case of very risk averse ($\alpha = -5$ and $\alpha = -10$) or very patient ($\beta = 0.99$) agents. While this result is not robust to the choice of criteria (the E_2 and E_∞ criteria do not display a loss in accuracy with an increase in order), this reiterates that there is no guarantee that a Taylor approximation will converge monotonically to the true policy function, even if the latter is analytic such that convergence is assured in the limit of an infinite order Taylor expansion.[36]

[Table 7 about here.]

[36] See Judd (1998) and Lombardo (2010).

For a given order of approximation, most algorithms perform identically at each order of approximation. This is due to the lack of endogenous propagation in the fully forward looking model of Burnside (1998), making an accumulation of risk adjustments in steady states and slope coefficients impossible. The exception is again the third order algorithm of Den Haan and De Wind (2012) in lemma 4.10, which was only as accurate as the first order approximation as measured with E_∞. Indeed, when the log dividend difference process is highly persistent ($\rho = 0.9$) or the agent is highly impatient ($\beta = 0.5$), it is even less accurate than the first order approximation according to E_∞. More interesting is that the algorithm of lemma 4.10 is identical to the other third order algorithms for all three measures (E_1, E_2, and E_∞) when either log dividend growth is not serially correlated ($\rho = 0$) or the agent has an intertemporal elasticity of substitution of unity ($\gamma = 1 \rightarrow \alpha = 0$). In both of these cases, the true policy function is a constant[37] and even all second and third order approximations coincide. This follows as the shock, ξ_t, was assumed normally distributed, leading to $y_{\sigma^3} = 0$ and the second order term y_{σ^2} is identical for all algorithms, following proposition 4.5, due to the absence of propagation ($g_y = 0$) in this case.

[Table 8 about here.]

Both Burnside's (1998) and Brock and Mirman's (1972) models admit known closed form solutions, enabling a precise investigation of the properties of the different pruning algorithms. However both lack important features of nonlinear models (internal propagation in Burnside's (1998) case and certainty nonequivalence in Brock and Mirman's (1972)) that one would like these pruning algorithms to cover. Accordingly, we will turn to our final model, a highly nonlinear variant of the neoclassical growth model due to Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), and abandon a closed form baseline solution, as none is known, for a highly accurate global projection solution as a baseline.

[37] Substituting either $\rho = 0$ or $\alpha = 0$ into (92) delivers $b_i = 0$, $\forall i$ and, hence, (90) becomes

$$v_t = \sum_{i=1}^{\infty} \beta^i \exp a_i \tag{93}$$

where a_i is as given in (91).

6.3 Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao's (2012) Model with Recursive Preferences and Stochastic Volatility

In this section, we examine Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao's (2012) stochastic neoclassical growth model with recursive preferences and stochastic volatility. We do so as the previous two models have lacked either risk sensitivity (the model of section 6.1 is certainty equivalent)[38] or endogenous state variables to propagate risk adjustments (the model of section 6.2 is entirely forward looking in endogenous variables). As noted by Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), the model incorporates more nonlinearities and therefore imposes a challenge on different solution algorithms. Due to the absence of closed-form solution, the model needs to be approximated. We choose the Chebyshev polynomial approximation as the true solution to run the three horse races again since it achieves a very high level of accuracy as reported by Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012).

As the first two welfare theorems hold in their model, we move right to the social planner's problem, in which the planner maximizes the expected discounted lifetime utility of a representative household given by the recursive preferences

$$U_t = \max_{c_t, l_t} \left[(1-\beta) \left(c_t^{\nu} (1-l_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}} \tag{95}$$

where c_t is consumption, l_t labor, $\beta \in (0,1)$ the discount factor, ν a labor supply parameter, γ risk aversion, and

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} \tag{96}$$

where ψ is the elasticity of intertemporal substitution. The social planner faces the resource constraint

$$c_t + k_t = e^{z_t} k_{t-1}^{\xi} l_t^{1-\xi} + (1-\delta) k_{t-1} \tag{97}$$

with k_t being capital, ξ its share and δ its depreciation rate, and z_t a mean reverting productivity process given by

$$z_t = \rho_z z_{t-1} + \sigma_z e^{\sigma_t} \varepsilon_t^z, \; \varepsilon_t^z \sim \mathcal{N}(0,1) \tag{98}$$

[38] Judd, Maliar, and Maliar (2011, p. 197) rearrange (82) as

$$K_t = E_t \left[\beta \frac{C_t}{C_{t+1}} \alpha e^{Z_{t+1}} K_t^{\alpha} \right] \tag{94}$$

and note that the *integrand* under the conditional expectations on the left hand side is equal to K_t for all values of Z_{t+1}.

with $|\rho_z| < 1$ a persistence parameter, σ_z the homoskedastic volatility of z_t, and σ_t a stochastic volatility process contributing conditional heteroskedasticity to z_t given by

$$\sigma_t = \rho_\sigma \sigma_{t-1} + \sigma_\sigma \varepsilon_t^\sigma,\ \varepsilon_t^\sigma \sim \mathcal{N}(0,1) \tag{99}$$

with $|\rho_\sigma| < 1$ a persistence parameter and σ_σ the standard deviation of innovations to the volatility process, σ_t.

The first order conditions are the intratemporal condition

$$\frac{1-\nu}{\nu}\frac{c_t}{1-l_t} = (1-\xi)\, e^{z_t} k_{t-1}^{\xi} l_t^{-\xi} \tag{100}$$

and the intertemporal condition

$$1 = E_t\left[m_{t+1}\left(\xi e^{z_{t+1}} k_t^{\xi-1} l_t^{1-\xi} + 1 - \delta\right)\right] \tag{101}$$

where the pricing kernel is given by

$$m_{t+1} \doteq \frac{\partial V_t/\partial c_{t+1}}{\partial V_t/\partial c_t} = \beta \frac{c_t}{c_{t+1}} \frac{\left(c_{t+1}^{\nu}\left(1-l_{t+1}\right)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}}}{\left(c_t^{\nu}\left(1-l_t\right)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}}} \left(\frac{U_{t+1}^{1-\gamma}}{E_t\left[U_{t+1}\right]^{1-\gamma}}\right)^{1-\frac{1}{\theta}} \tag{102}$$

The presence of U_{t+1} in the pricing kernel necessitates the inclusion of the value function evaluated at the optimum

$$U_t = \left[(1-\beta)\left(c_t^{\nu}\left(1-l_t\right)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}} + \beta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\theta}{1-\gamma}} \tag{103}$$

along with the first order conditions, the resource constraint (97), and the exogenous driving force (98) and its volatility (99) to characterize an equilibrium.

Following Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), we will also track two asset prices, the gross return on capital

$$R_t = \xi e^{z_t} k_{t-1}^{\xi-1} l_t^{1-\xi} + 1 - \delta \tag{104}$$

and the gross risk-free rate

$$1 = E_t\left[m_{t+1} R_t^f\right] \tag{105}$$

In contrast to the first two models, Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao's (2012) model has no known analytic solution to serve as a baseline for comparing the different pruning algorithms. However, they show that a projection solution with Chebyshev polynomial basis functions consistently achieves a high degree of accuracy across different parameterizations and for a large range in the state space. With this result, we take their Chebyshev projection as our baseline for comparison.

[Table 9 about here.]

We parameterize the model as in Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) and will examine a baseline and an extreme calibration. The parameters that stay fixed across both calibrations are in table 9 and are standard values that reflect post-war US data.[39] The differences between the baseline and extreme parameterizations can be found in table 10 and are in the value of risk aversion ($\gamma = 5$ versus 40), in the homoskedastic volatility in the productivity process ($\sigma_z = 0.007$ versus 0.021), and in the standard deviation of the stochastic volatility process ($\eta = 0.06$ versus 0.1). The values for the extreme parameterization are purposely set at the edge of credulity to introduce a very large amount of nonlinearity into the model to test the different algorithms.

[Table 10 about here.]

For each calibration, we run 100 simulations of 10,000 periods each and present the relative errors of k_t, c_t, l_t, i_t, y_t, R_t^f, and R_t, according to the three criteria—average (E_1), mean square (E_2), and maximum (E_∞). For the baseline calibration, the results can be found in tables 11 through 13 and for the extreme calibration, the results can be found in tables 14 through 16.

[Table 11 about here.]

Broadly speaking, increasing the order of approximation increases the accuracy of the approximation. This is not, however, true for the non pruned perturbations, which frequently perform worse at third than at second order (see l_t through R_t in tables 11 and 12) Under the same two E_1 and E_2 criteria, the pruned algorithms actually perform better than the non pruned algorithms. This stands in contrast to the results reported in Den Haan and De Wind (2012) and is likewise a combination of the different models and their choice of pruning algorithm; the latter is consistently outperformed by the other pruning algorithms.

[Table 12 about here.]

[39] Note that the value of ν here yields a deterministic steady state value of $l = 1/3$, correcting Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao's (2012, p. 197) Table 1, which mistakenly reported ν equal to 0.357, the value of θ stated on the same page.

As the model of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) is risk sensitive and has internal propagation, the three second order algorithms will differ, see proposition 4.5. The second order pruned series are more accurate than their non pruned counterparts, with Den Haan and De Wind's (2012) second order algorithm performing worst. At third order, time varying risk corrections enter the algorithms, which are crucial for the dynamics under stochastic volatility and recursive preferences, see, e.g., Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012).

[Table 13 about here.]

For the third order, all of the pruning algorithms perform comparably except for that of Den Haan and De Wind (2012), detailed in lemma 4.10, which performs markedly poorer. To blame are the terms of fourth order introduced into their third order algorithm and the imposition of third order risk correction on the first order transition, which comprise the major differences to the other algorithms. The algorithm of lemma 4.8—Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe's (2011) algorithm, while more accurate than Den Haan and De Wind's (2012), is inferior according to all three criteria and for all the variables considered here.Thus, the cross terms (products of the second and first order approximations) mentioned in Lombardo (2012) are important contributors to the accuracy of third order pruning algorithms. The algorithm of lemma 4.9 sheds some light on which cross terms might be most important; it contains only the product of the second order approximation of endogenous variables with the first order endogenous state space—neglecting the cross products with the first order exogenous state space—yet is generally only marginally worse than the two top performing third order algorithms and, for some cases, is even the most accurate algorithm (k_t in table 11 and c_t and i_t in table 13).

[Table 14 about here.]

Our nonlinear moving average (see proposition 3.6) and Andreasen's (2012) third order pruning algorithm (see lemma 4.6) are the two top performing algorithms. Simply enumerating the cases where one or the other performs better as displayed in tables 11 through 13, our nonlinear moving average displays superior performance 50% more often. Interestingly, in those cases where these two algorithms display different average mean squared deviations (the measure E_2 displayed in table 12), it is always our nonlinear moving average that is on top. Indeed, as measured at full

double precision,[40] our nonlinear moving average is uniformly superior according to the mean square criterium. This must be tempered, however, as the differences in accuracy between the two algorithms for the model here are marginal.

[Table 15 about here.]

The results for the extreme calibration parallel those of our baseline calibration, higher order leads to more accuracy, Den Haan and De Wind's (2012) pruning algorithms are generally the least accurate second order and the least accurate third order algorithms, and the inclusion of more cross products in third order pruning algorithms improves accuracy. For the average and maximum criteria, tables 14 and 16 respectively, all algorithms are about one order of magnitude less accurate than under the baseline calibration (for the mean square criterium in table 15 the loss is about two orders of magnitude). The evidence in favor of our nonlinear moving average is now more clear cut: it is the most accurate algorithm in 16 cases in tables 14 through 16 (compared to Andreasen's (2012) 3, the next most accurate) and is the most accurate for all variables according to the mean square criterium, see table 15.

[Table 16 about here.]

In sum for all three models we have examined here, there is compelling evidence that the third order nonlinear moving average, expressed recursively in proposition 3.6, is the highest performing algorithm among the perturbation and pruning algorithms we have examined here. Yet, the gains are modest at best compared with the third order algorithms in lemmata 4.6 and 4.9 and, e.g., in the absence of risk sensitivity or endogenous propagation, the algorithms coincide.

7 Conclusion

We have derived a recursive representation from the nonlinear moving average approximation of Lan and Meyer-Gohde (2012b). That this recursive algorithm inherits stability from first order invites comparison with so-called pruning algorithms in the literature that purport to do the same. We document six different pruning algorithms from the literature at second and third order and

[40] The full tables, along with the algorithms, are available online. We stopped at two digit accuracy here to minimize clutter.

show that even with its closest counterparts, at second order the algorithm of Kim, Kim, Schaumburg, and Sims (2008) and at third the algorithm of Andreasen (2012), differences remain in that our algorithm centers the approximation and its coefficients at the stochastic steady state as approximated up to the order in question. Hence our algorithm gives a stable approximation taking into account steady state risk adjustments, whereas our closest counterparts center their algorithms at the deterministic steady state making the interpretation of the risk adjustment components more difficult.

Numerically, we compare the six algorithms with our second and third order recursive representations and the first through third order standard perturbations for accuracy. We choose three models to test the algorithms in: the Brock and Mirman (1972) special case of the stochastic neoclassical growth model, Burnside's (1998) asset pricing model, and the model of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) with recursive preferences and stochastic volatility. The first two have known closed form solutions but are not rich enough to capture the differences from the propagation of risk adjustments—the Brock and Mirman's (1972) is certainty equivalent and Burnside's (1998) lacks endogenous state variables—leading many algorithms to be identical; the last model is highly risk sensitive and has endogenous propagation, but does not possess a known closed form solution forcing us to rely on another approximation as a baseline.

In general, the differences are modest, with the major difference coming with the increase of order of approximation. The exception is the algorithm of Den Haan and De Wind (2012), which at third order performs more comparably to a first order approximation. We do not find evidence that much accuracy in simulations is lost by choosing a pruning algorithm to guarantee stability. On the contrary, pruned series are often more accurate than the standard perturbation. This is not surprising as the two most accurate algorithms are not ad-hoc pruning algorithms, but theoretically justified nonlinear moving average perturbations (see Lan and Meyer-Gohde (2012b)) or matched perturbations (see Lombardo (2012)).

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Appendix

A Dimension Reduction Typology

We adopt Dynare's typology of all the endogenous variables[41] to minimize superfluous calculations.

Definition A.1. *Typology of Endogenous Variables*

- ***Static endogenous variables:*** *those that appears only at period t. Their number is $nstatic$*
- ***Purely forward endogenous variables:*** *those that appear only at period $t+1$, possibly at period t, but not at period $t-1$. Their number is $nfwd$*
- ***Purely backward endogenous variables:*** *those that appear only at period $t-1$, possibly at period t, but not period $t+1$. Their number is $nbwd$*
- ***Mixed endogenous variables:*** *those that appear both at period $t-1$ and $t+1$, and possibly at period t. Their number is $nmix$*

These four types variables, abbreviated as st, fwd, bwd and mix respectively, form a partition of the endogenous variables with the identity

$$nstatic + nbwd + nmix + nfwd = ny$$

For notational ease in derivations, we also define

- ***Forward endogenous variables:*** *the union of mixed and purely forward endogenous variables. Their number therefore is $nfwdendo = nfwd + nmix$*
- ***Backward endogenous variables:*** *the union of static and purely backward endogenous variables. Their number therefore is $nbwdendo = nstatic + nbwd$*
- ***Endogenous state variables:*** *the union of the purely backward and the mixed endogenous variables. Their number therefore is $ns = nbwd + nmix$*

with the abbreviation $fwdendo$, $bwdendo$ and $state$ respectively.

Note that, the last two types of variables, i.e., the backward endogenous and endogenous state variables in definition (A.1), are different from those defined by Dynare: (i) the backward endogenous variables in Dynare refers to the union of the purely backward and the mixed endogenous variables, which is the endogenous state variables in our case, (ii) the state variables in Dynare

[41] See again Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011)

refer to the union of our endogenous state variables and the exogenous variables of the model.

Based on the definition (A.1), the entries of the vector of the endogenous variables are ordered[42] such that the vector admits the partitions

$$y_t = \begin{bmatrix} y_t^{static} \\ y_t^{bwd} \\ y_t^{mix} \\ y_t^{fwd} \end{bmatrix} = \begin{bmatrix} y_t^{static} \\ y_t^{state} \\ y_t^{fwd} \end{bmatrix} = \begin{bmatrix} y_t^{static} \\ y_t^{bwd} \\ y_t^{fwdendo} \end{bmatrix} = \begin{bmatrix} y_t^{bwdendo} \\ y_t^{fwdendo} \end{bmatrix}$$

While all the partitions in y_t are superscripted with the abbreviated names of the variable type, these superscripts can be considered as the indicator for the number of rows of that partition, for example, y_t^{static} is of dimension $nstatic \times 1$.

The definition (A.1) and the ordering of y_t in the previous equation implies that the derivatives of the f function with respect to y^-, y and y^+ have the structure

$$\begin{aligned} f_{y^-} &= \begin{bmatrix} 0_{ny \times nstatic} & f_{y^- bwd} & f_{y^- both} & 0_{ny \times nfwd} \end{bmatrix} \\ &= \begin{bmatrix} 0_{ny \times nstatic} & f_{y^- state} & 0_{ny \times nfwd} \end{bmatrix} \\ f_y &= \begin{bmatrix} f_{ystatic} & f_{ybwd} & f_{ymix} & f_{yfwd} \end{bmatrix} \\ &= \begin{bmatrix} f_{ystatic} & f_{ystate} & f_{yfwd} \end{bmatrix} \\ &= \begin{bmatrix} f_{ybwdendo} & f_{yfwdendo} \end{bmatrix} \\ f_{y^+} &= \begin{bmatrix} 0_{ny \times nstatic} & 0_{ny \times nbwd} & f_{y^+ mix} & f_{y^+ fwd} \end{bmatrix} \\ &= \begin{bmatrix} 0_{ny \times nstatic} & 0_{ny \times nbwd} & f_{y^+ fwdendo} \end{bmatrix} \end{aligned}$$

where the abbreviated names as subscripts can be considered as the indictor of the number of columns of that partition, for example, $f_{y^- bwd}$ is of dimension $ny \times nbwd$.

B Coefficients of Nonlinear Moving Averages Recursive in the Minimal State Representation

Here we apply the dimension reduction to the equations of Lan and Meyer-Gohde (2012b). As

(B-1) $$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)]$$

can be rewritten

(B-2) $$0 = E_t[f(y_{t+1}^{fwdendo}, y_t, y_{t-1}^{state}, \varepsilon_t)]$$

[42] This is the decision rule order of Dynare. See again Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011).

For notational ease in derivations, we will define the vector x_t, containing the complete set of variables

$$x_t = \begin{bmatrix} y_{t+1}^{fwdendo} \\ y_t \\ y_{t-1}^{state} \\ \varepsilon_t \end{bmatrix} \tag{B-3}$$

with the dimension $nx \times 1$ with $(nx = nfwdendo + ny + ns + ne)$. This differs from the vector of total variables in Lan and Meyer-Gohde (2012b) by allowing for the possibility, mentioned above, that only a subset of the variables in y_t is present in $t+1$, $y_{t+1}^{fwdendo}$, and only a subset in $t-1$, y_{t-1}^{state}.

With the policy function of the form (2), (3) and (4), we can write x_t as

$$x_t = x(\sigma, \widetilde{\varepsilon}_{t+1}, \varepsilon_t, \varepsilon_{t-1}, \ldots) \tag{B-4}$$

B.1 First Order Coefficients

At first order, the approximation is

$$y_t = \overline{y} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} \tag{B-5}$$

where we have already removed coefficients equal to zero.[43] Accordingly, the task is to pin down y_i.

As it is serially uncorrelated vector of innovations, ε_t can be represented by trivial infinite moving average with the first or impact coefficient the identity matrix and all other coefficients zero. This makes the relation between endogenous variables and the underlying innovations different upon impact than in subsequent periods after a realization from the vector of innovations. Accordingly, we split the problem in two: indices, i, greater than zero and $i = 0$. Accordingly, the first-order equation of Lan and Meyer-Gohde (2012b) becomes

$$f_{y^- state} y_{i-1}^{state} + f_y y_i + f_{y^+ fwdendo} y_{i+1}^{fwdendo} = 0 \tag{B-6}$$

for positive i and

$$f_y y_0 + f_{y^+ fwdendo} y_1^{fwdendo} + f_\varepsilon = 0 \tag{B-7}$$

otherwise. We summarize the solutions in the following

Proposition B.1. *The solution to (B-6) and (B-7) takes the form*

$$y_i = \alpha y_{i-1}^{state} \; \forall \; i > 0 \tag{B-8}$$

[43] Here y_σ is zero, see Lan and Meyer-Gohde (2012b) and more generally in state space contexts, Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and Lan and Meyer-Gohde (2012a).

(B-9) $$y_0 = \beta_0$$

Proof. y_i solves the deterministic system (B-6) as studied by Anderson (2010) for positive i, with a unique solution α such that all the eigenvalues of α^{state} are inside the unit circle.[44] Substituting this for y_1 in (B-7) yields a linear equation in y_0, whose solution we call β_0. See the Appendix. □

B.2 Second Order Coefficients

At second order, the approximation is

(B-10) $$y_t = \overline{y} + \frac{1}{2} y_{\sigma^2} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

where we have already removed coefficients equal to zero.[45] Accordingly, the task is to pin down $y_{j,i}$ and y_{σ^2} and we shall proceed in that order.

The equation from Lan and Meyer-Gohde (2012b) for $y_{j,i}$ is now

(B-11) $$f_{y^- state} y^{state}_{j-1,i-1} + f_y y_{j,i} + f_{y^+ fwdendo} y^{fwdendo}_{j+1,i+1} + f_{x^2}(x_j \otimes x_i) = 0$$

From (B-8), we rewrite the derivative of x_t with respect to ε_{t-i} as the product of a constant matrix and the vector of state variable coefficients from the first order

$$x_i = \begin{bmatrix} y^{state\prime}_{i-1} & y'_i & y^{fwdendo\prime}_{i+1} & \varepsilon'_i \end{bmatrix}' = \gamma_1 y^{state}_{i-1} \qquad \forall\, i > 0$$

(B-12) $$\text{where } \gamma_1 = \begin{bmatrix} I_{ns} & \alpha' & \left(\alpha^{fwdendo} \alpha^{state}\right)' & \underset{ns \times ne}{0} \end{bmatrix}'$$

This reduces (B-11) to a difference equation system with inhomogenous terms in the first order coefficients of the endogenous state variables and homogenous coefficients identical to (B-6), the equation at first order. This is in line with the so-called pruning algorithm of Kim, Kim, Schaumburg, and Sims (2008), though they are not entirely identical as we will show in section 4.

Eliminating redundant calculations, $y_{j,i}$ can be split into three difference equations according to the different values that the indices j and i take on. The initial values (when j and/or i are zero) are handled separately, as in the first order case, by recognizing that the inhomogenous component associated with the zero index is a known constant

(B-13) $$x_0 = \begin{bmatrix} \underset{ne \times ns}{0} & \beta'_0 & \left(\alpha^{fwdendo} \beta^{state}_0\right)' & I_{ne} \end{bmatrix}'$$

We summarize the solutions in the following

[44] See Villemot (2011) for details on solving the first order homogenous problem with the variable typology we have adapted here from Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare.

[45] Here $y_{\sigma,i}$ is zero, see footnote 43.

Proposition B.2. *The solution to (B-11) takes the form*

$$y_{j,i} = \alpha y^{state}_{j-1,i-1} + \beta_{22}(y^{state}_{j-1} \otimes y^{state}_{i-1}) \qquad \forall\, i \& j > 0 \tag{B-14}$$

$$y_{j,0} = \beta_{20}(y^{state}_{j-1} \otimes I_{ne}) \qquad \forall\, j > 0 \tag{B-15}$$

$$y_{0,0} = \beta_{00} \tag{B-16}$$

Proof. See the Appendix. □

The coefficient $y_{0,i}$ follows from the commutability of the matrix derivative operator and upon application of Magnus and Neudecker's (1979) commutation matrix, K, to reverse the order of Kronecker tensors. Accordingly

$$y_{0,i} = \beta_{02}(I_{ne} \otimes y^{state}_{j-1}), \text{ where } \beta_{02} \equiv \beta_{20} K_{ns,ne} \tag{B-17}$$

The second order approximation also contains a constant correction for risk that is generically nonzero, see, e.g., Collard and Juillard (2001) or Schmitt-Grohé and Uribe (2004),

Proposition B.3. y_{σ^2} *solves*

$$\begin{aligned} y_{\sigma^2} = &\left[f_{ystatic} \quad f_{y^- bwd} + f_{ybwd} \quad f_{y^- mix} + f_{ymix} + f_{y^+ mix} \quad f_{yfwd} + f_{y^+ fwd} \right]^{-1} \\ &\left[f_{y^+ fwdendo} \beta^{fwdendo}_{00} + f_{(y^+ fwdendo)^2} \beta^{fwdendo \otimes [2]}_{0} \right] E_t \left(\varepsilon^{\otimes[2]}_{t+1} \right) \end{aligned} \tag{B-18}$$

Proof. Direct verification of Lan and Meyer-Gohde's (2012b) equation for y_{σ^2}

$$y_{\sigma^2} = -(f_{y^-} + f_y + f_{y^+})^{-1}[f_{y^+} y_{0,0} + f_{y^{+2}} y_0^{\otimes[2]}] E_t \left(\varepsilon^{\otimes[2]}_{t+1} \right) \tag{B-19}$$

upon application of the variable typology here yields the desired result. □

This set of coefficients corrects for the risk of future shocks as captured by $E_t \left(\varepsilon^{\otimes[2]}_{t+1} \right)$, the (column vectorized) variance of next period's shocks.

B.3 Third Order Coefficients

The third order approximation of the policy function takes the form

$$\begin{aligned} y^{(3)}_t = &\bar{y} + \frac{1}{2} y_{\sigma^2} + \sum_{i=0}^{\infty} \left(y_i + \frac{1}{2} y_{\sigma^2,i} \right) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\ &+ \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k,j,i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \end{aligned} \tag{B-20}$$

where again we have already removed coefficients equal to zero.[46] Accordingly, the task is to pin down $y_{k,j,i}$ and $y_{\sigma^2,i}$.

[46] Here $y_{\sigma,i,j}$ and y_{σ^3} are zero, see again footnote 43. The latter follows from our assumption of normality, see Andreasen (2012) for a third order perturbation with nonnormal shocks and, consequently, nonzero third order constant risk corrects like y_{σ^3}

Lan and Meyer-Gohde's (2012b) equation for $y_{k,j,i}$ with nonzero k, j, and i is now given by[47]

$$f_{y^- state} y^{state}_{k-1,j-1,i-1} + f_y y_{k,j,i} + f_{y^+ fwdendo} y^{fwdendo}_{k+1,j+1,i+1} + \gamma_{333} s_{k,j,i} = 0 \tag{B-21}$$

where the inhomogenous terms consists of the state spaces of all lower orders

$$s_{k,j,i} = \begin{bmatrix} y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes y^{state}_{i-1} \\ y^{state}_{k-1,j-1} \otimes y^{state}_{i-1} + \left(y^{state}_{k-1,i-1} \otimes y^{state}_{j-1}\right)(I_{ne} \otimes K_{ne,ne}) + \left(y^{state}_{j-1,i-1} \otimes y^{state}_{k-1}\right) K_{ne^2,ne} \end{bmatrix} \tag{B-22}$$

mapped into (B-21) with the following constant matrix

$$\gamma_{333} = \begin{bmatrix} f_{x^3}\gamma_1^{\otimes[3]} + f_{x^2}(\gamma_{22} \otimes \gamma_1)(I_{ns^3} + I_{ns} \otimes K_{ns,ns} + K_{ns^2,ns}) & f_{x^2}\gamma_1^{\otimes[2]} \end{bmatrix} \tag{B-23}$$

where

$$\gamma_{22} = \begin{bmatrix} 0_{ns^2 \times ns} & \beta'_{22} & \left(\alpha^{fwdendo}\beta^{state}_{22} + \beta^{fwdendo}_{22}\alpha^{state\otimes[2]}\right)' & 0_{ns^2 \times ne} \end{bmatrix}' \tag{B-24}$$

and where K again is Magnus and Neudecker's (1979) commutation matrix.

To eliminate redundant calculations, we split $y_{k,j,i}$ into four difference equations according to the different values that the indices k, j, and i take on and replace repeated coefficients with their lower order predecessors. The initial values (when k, j, and/or i are zero) are again handled separately by recognizing that the inhomogenous component associated with the zero index is a known constant.[48] We summarize the solutions in the following

Proposition B.4. *The solution to (B-11) takes the form*

$$y_{k,j,i} = \alpha y^{state}_{k-1,j-1,i-1} + \begin{bmatrix} \beta_{333} & \beta_{22} \end{bmatrix} s_{k,j,i} \qquad \forall\, i,j,\&k > 0 \tag{B-25}$$

$$y_{k,j,0} = \begin{bmatrix} \beta_{330} & \beta_{20} \end{bmatrix} \begin{bmatrix} y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes I_{ne} \\ y^{state}_{k-1,j-1} \otimes I_{ne} \end{bmatrix} \qquad \forall\, k\&j > 0 \tag{B-26}$$

$$y_{k,0,0} = \beta_{300}\left(y^{state}_{k-1} \otimes I_{ne^2}\right) \qquad \forall\, k > 0 \tag{B-27}$$

$$y_{0,0,0} = \beta_{000} \tag{B-28}$$

Proof. See the Appendix. □

The third order approximation also contains a time varying correction for risk that is generically nonzero, see, e.g., Andreasen (2012), Ruge-Murcia (2012), or Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012). Analogously to the first order, $y_{\sigma^2,i}$ must be split into two equations to respect the nonzero value of the shocks at impact. For nonzero i, the source equation can

[47] See appendix H for the problem statement with zero k's, j's, and/or i's, which reduces the underlying state space to products of lower order state spaces.

[48] As the calculation are rather onerous, the reader is directed to the Appendix for details and the second order calculations above for an example.

be written

$$
\begin{aligned}
(\text{B-29}) \qquad & f_{y^- state} y^{state}_{\sigma^2,i-1} + f_y y_{\sigma^2,i} + f_{y^+ fwdendo} y^{fwdendo}_{\sigma^2,i+1} \\
& + \left\{ \left[f_{x^3} \left(x^{\otimes[2]}_{\tilde{\varepsilon}} \otimes \gamma_1 \right) + 2 f_{x^2} \left(x_{\tilde{\varepsilon}} \otimes \left(\gamma_4 \beta^{fwdendo}_{02} \right) \right) \left(I_{ne^2} \otimes \alpha^{state} \right) + f_{x^2} \left(x_{\tilde{\varepsilon},\tilde{\varepsilon}} \otimes \gamma_1 \right) \right. \right. \\
& \left. \left. + f_x \left(\gamma_4 \beta^{fwdendo}_{003} \left(I_{ne^2} \otimes \alpha^{state} \right) \right) \right] \left(E_t \varepsilon^{\otimes[2]}_{t+1} \otimes I_{ns} \right) + f_{x^2} \left(x_{\sigma^2} \otimes \gamma_1 \right) \right\} y^{state}_{i-1} = 0
\end{aligned}
$$

where $x_{\tilde{\varepsilon}}$, $x_{\tilde{\varepsilon},\tilde{\varepsilon}}$, x_{σ^2}, γ_4, $x_{\tilde{\varepsilon},i}$, and $x_{\tilde{\varepsilon}^2,i}$ are constant matrices and coefficients from previous calculations. For $i = 0$, the source equation is

$$
\begin{aligned}
& f_y y_{\sigma^2,0} + f_{y^+ fwdendo} y^{fwdendo}_{\sigma^2,1} + f_{x^2} \left(x_{\sigma^2} \otimes x_0 \right) \\
(\text{B-30}) \qquad & + \left[f_{x^3} \left(x^{\otimes[2]}_{\tilde{\varepsilon}} \otimes x_0 \right) + f_{x^2} \left(x_{\tilde{\varepsilon},\tilde{\varepsilon}} \otimes x_0 \right) + 2 f_{x^2} \left(x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon},0} \right) + f_x x_{\tilde{\varepsilon}^2,0} \right] \left(E_t \varepsilon^{\otimes[2]}_{t+1} \otimes I_{ne} \right) = 0
\end{aligned}
$$

where $x_{\tilde{\varepsilon},0}$ and $x_{\tilde{\varepsilon}^2,0}$ are coefficients from previous calculation

We summarize the solutions in the following

Proposition B.5. *The solution to (B-29) and (B-30) takes the form*

$$
(\text{B-31}) \qquad y_{\sigma^2,i} = \alpha y^{state}_{\sigma^2,i-1} + \beta_{\sigma^2 1} y^{state}_{i-1} \qquad \forall\, i > 0
$$

$$
(\text{B-32}) \qquad y_{\sigma^2,0} = \beta_{\sigma^2 0}
$$

Proof. The first equation follows directly as the homogenous component is identical to that of the first order with the first order itself being the inhomogenous component. The second follows from inserting the first into (B-30) and solving the resulting linear problem. See the Appendix. □

This set of coefficients corrects (up to first order) for the time varying conditional risk of future shocks as captured by $E_t \left(\varepsilon^{\otimes[2]}_{t+1} \right)$, the (column vectorized) variance of next period's shocks.

C First Order Recursive Approximation Appendix

We define

$$
(\text{C-33}) \qquad dy^{(1)}_t \equiv y^{(1)}_t - \bar{y}
$$

It follows that

$$
(\text{C-34}) \qquad dy^{(1)}_t = \sum_{i=0}^{\infty} y_i \varepsilon_{t-i}
$$

Evaluating and rearranging $dy^{(1)}_t - \alpha dy^{(1)state}_{t-1}$ yields

$$
dy^{(1)}_t - \alpha dy^{(1)state}_{t-1} = \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} - \alpha \sum_{i=0}^{\infty} y^{state}_i \varepsilon_{t-i-1}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \left(y_i - \alpha y_{i-1}^{state}\right) \varepsilon_{t-i} \\
&= \beta_0 \varepsilon_t + \sum_{i=1}^{\infty} \left(y_i - \alpha y_{i-1}^{state}\right) \varepsilon_{t-i} \\
&= \beta_0 \varepsilon_t
\end{aligned} \tag{C-35}
$$

where the last line follows as $y_i = \alpha y_{i-1}^{state}$, $\forall i > 0$ from the first order solution in proposition B.1. Therefore the state space representation of the first order approximation (11) takes the form

$$
dy_t^{(1)} = \alpha dy_{t-1}^{(1)state} + \beta_0 \varepsilon_t \tag{C-36}
$$

D Second Order Recursive Approximation Appendix

Inserting the first order approximation (11) into the definition of the second order increment, (18), yields

$$
dy_t^{(2)} = y_t^{(2)} - y_t^{(1)} - \frac{1}{2} y_{\sigma^2} \tag{D-37}
$$

It follows

$$
dy_t^{(2)} = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \tag{D-38}
$$

Evaluating and rearranging $dy_t^{(2)} - \alpha dy_{t-1}^{(2)state}$ yields

$$
\begin{aligned}
dy_t^{(2)} - \alpha dy_{t-1}^{(2)state} =& \frac{1}{2} \left[\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) - \alpha \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j-1,i-1}^{state} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \right] \\
=& \frac{1}{2} \left[\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left(y_{j,i} - \alpha y_{j-1,i-1}^{state}\right) (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) + \sum_{i=1}^{\infty} y_{0,i} (\varepsilon_t \otimes \varepsilon_{t-i}) \right. \\
&\left. + \sum_{j=1}^{\infty} y_{j,0} (\varepsilon_{t-j} \otimes \varepsilon_i) + y_{0,0} \varepsilon_t^{\otimes[2]} \right]
\end{aligned} \tag{D-39}
$$

applying the second order solutions $y_{j,i} = \alpha y_{j-1,i-1}^{state} + \beta_{22} (y_{j-1}^{state} \otimes y_{i-1}^{state})$, $\forall\, j,i > 0$, $y_{0,i} = \beta_{02} (I_{ne} \otimes y_{i-1}^{state})$, $\forall\, j = 0,\ i > 0$, $y_{j,0} = \beta_{20} (y_{j-1}^{state} \otimes I_{ne})$, $\forall\, j > 0,\ i = 0$ and noting $\beta_{00} = y_{0,0}$

$$
\begin{aligned}
dy_t^{(2)} - \alpha dy_{t-1}^{(2)state} =& \frac{1}{2} \left[\beta_{22} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left(y_{j-1}^{state} \otimes y_{i-1}^{state}\right) (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) + \beta_{02} \sum_{i=0}^{\infty} (I_{ne} \otimes y_{i-1}^{state}) (\varepsilon_t \otimes \varepsilon_{t-i}) \right. \\
&\left. + \beta_{20} \sum_{j=0}^{\infty} (y_{j-1}^{state} \otimes I_{ne}) (\varepsilon_{t-j} \otimes \varepsilon_i) + \beta_{00} \varepsilon_t^{\otimes[2]} \right]
\end{aligned} \tag{D-40}
$$

which, using the mixed product rule can be rewritten as

$$
dy_t^{(2)} - \alpha dy_{t-1}^{(2)state} = \frac{1}{2} \left[\beta_{22} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left(y_{j-1}^{state} \varepsilon_{t-j} \otimes y_{i-1}^{state} \varepsilon_{t-i}\right) + \beta_{02} \sum_{i=0}^{\infty} (I_{ne} \varepsilon_t \otimes y_{i-1}^{state} \varepsilon_{t-i}) \right.
$$

$$
\begin{aligned}
&+\beta_{20}\sum_{j=0}^{\infty}\left(y_{j-1}^{state}\varepsilon_{t-j}\otimes I_{ne}\varepsilon_i\right)+\beta_{00}\varepsilon_t^{\otimes[2]}\Big] \\
=&\frac{1}{2}\Big[\beta_{22}\left(y_0^{state}\varepsilon_{t-1}+y_1^{state}\varepsilon_{t-2}+\ldots\right)\otimes\left(y_0^{state}\varepsilon_{t-1}+y_1^{state}\varepsilon_{t-2}+\ldots\right) \\
&+\beta_{02}\varepsilon_t\otimes\left(y_0^{state}\varepsilon_{t-1}+y_1^{state}\varepsilon_{t-2}+\ldots\right) \\
&+\beta_{20}\left(y_0^{state}\varepsilon_{t-1}+y_1^{state}\varepsilon_{t-2}+\ldots\right)\otimes\varepsilon_t+\beta_{00}\varepsilon_t^{\otimes[2]}\Big]
\end{aligned}
\tag{D-41}
$$

and from (11)

$$
dy_t^{(2)}-\alpha dy_{t-1}^{(2)state}=\frac{1}{2}\left[\beta_{22}dy_{t-1}^{state\otimes[2]}+\beta_{02}\left(\varepsilon_t\otimes dy_{t-1}^{state}\right)+\beta_{20}\left(dy_{t-1}^{state}\otimes\varepsilon_t\right)+\beta_{00}\varepsilon_t^{\otimes[2]}\right]
\tag{D-42}
$$

The previous equation (17) can be further simplified by using $\left(y_{t-1}^{state}\otimes\varepsilon_t\right)=K_{ns,ne}\left(\varepsilon_t\otimes y_{t-1}^{state}\right)$ and $\beta_{20}=\beta_{02}K_{ne,ns}$

$$
dy_t^{(2)}-\alpha dy_{t-1}^{(2)state}=\frac{1}{2}\left[\beta_{22}dy_{t-1}^{state\otimes[2]}+2\beta_{20}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t\right)+\beta_{00}\varepsilon_t^{\otimes[2]}\right]
\tag{D-43}
$$

D.1 Block Kronecker Expression of Second Order Coefficients

Following Koning, Neudecker, and Wansbeek (1991), we define the block Kronecker product, denoted by $\boxtimes$, as

$$
A\boxtimes B\equiv\begin{bmatrix}A\otimes B_{11} & \ldots & A\otimes B_{1t}\\ \vdots & & \vdots\\ A\otimes B_{s1} & \ldots & A\otimes B_{st}\end{bmatrix}
\tag{D-44}
$$

for a $u\times v$ matrix B consisting of blocks B_{kl} of size $u_k\times v_l$, where $u=\sum_{k=1}^{s}u_k$ and $v=\sum_{l=1}^{t}v_l$. This contrasts with the standard Kronecker product

$$
B\otimes A\equiv\begin{bmatrix}B_{11}\otimes A & \ldots & B_{1t}\otimes A\\ \vdots & & \vdots\\ B_{s1}\otimes A & \ldots & B_{st}\otimes A\end{bmatrix}=\begin{bmatrix}b_{11}A & \ldots & b_{1v}A\\ \vdots & & \vdots\\ b_{u1}A & \ldots & b_{uv}A\end{bmatrix}
\tag{D-45}
$$

where b is used to distinguish the individual elements of B from the blocks defined above.

Applying the properties of the block Kronecker product, we can connect y_{z^2} and

$$
\begin{bmatrix}y_{y^{state2}} & y_{\varepsilon y^{state}} & y_{y^{state}\varepsilon} & y_{\varepsilon^2}\end{bmatrix}
$$

through operations with Magnus and Neudecker's (1979) commutation matrix, denoted here by $K_{a,b}$, as follows

$$
\begin{aligned}
y_{z^2}dz_t^{(1)^{\otimes[2]}}&=y_{z^2}\underbrace{\begin{bmatrix}K_{ns,ns+ne} & 0_{ns(ns+ne)\times ne(ns+ne)}\\ 0_{ne(ns+ne)\times ns(ns+ne)} & K_{ne,ns+ne}\end{bmatrix}}_{\equiv G_2}\begin{bmatrix}dy_{t-1}^{(1)state}\\ \varepsilon_t\end{bmatrix}\boxtimes\begin{bmatrix}dy_{t-1}^{(1)state}\\ \varepsilon_t\end{bmatrix}\\
&=\begin{bmatrix}y_{y^{state2}} & y_{\varepsilon y^{state}} & y_{y^{state}\varepsilon} & y_{\varepsilon^2}\end{bmatrix}dz_t^{(1)^{\boxtimes[2]}}
\end{aligned}
\tag{D-46}
$$

Accordingly,

$$y_{z^2}G_2 = \begin{bmatrix} y_{y^{state2}} & y_{\varepsilon y^{state}} & y_{y^{state}\varepsilon} & y_{\varepsilon^2} \end{bmatrix} \tag{D-47}$$

Hence, the block Kronecker product, through G_2, allows us to extract the individual block second derivatives with respect to y_{t-1}^{state} and e_t from the matrix of second derivatives with respect to the entire state vector, z_t.

E Third Order Recursive Approximation Appendix

Using the second order approximation (17), the definition of the third order increment, (26), can be written as

$$dy_t^{(3)} = y_t^{(3)} - y_t^{(2)} \tag{E-48}$$

It follows

$$dy_t^{(3)} = \frac{1}{2}\sum_{i=0}^{\infty} y_{\sigma^2,i}\varepsilon_{t-i} + \frac{1}{6}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} y_{k,j,i}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \tag{E-49}$$

Rearranging and evaluating $dy_t^{(3)} - \alpha dy_{t-1}^{(3)state}$ yields

$$\begin{aligned} dy_t^{(3)} - \alpha dy_{t-1}^{(3)state} =& \frac{1}{2}\sum_{i=0}^{\infty}\left(y_{\sigma^2,i}\sigma^2 - \alpha y_{\sigma^2,i-1}^{state}\sigma^2\right)\varepsilon_{t-i} \\ &+ \frac{1}{6}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left(y_{k,j,i} - \alpha y_{k-1,j-1,i-1}^{state}\right)(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \end{aligned} \tag{E-50}$$

The first term on the right hand side of the previous equation can be written as

$$\sum_{i=0}^{\infty}\left(y_{\sigma^2,i}\sigma^2 - \alpha y_{\sigma^2,i-1}^{state}\sigma^2\right)\varepsilon_{t-i} = \beta_{\sigma^2 0}\varepsilon_t + \beta_{\sigma^2 1}dy_{t-1}^{(1)state} \tag{E-51}$$

The second term can be written as

$$\begin{aligned} &\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left(y_{k,j,i} - \alpha y_{k-1,j-1,i-1}^{state}\right)(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \\ &= \sum_{k=1}^{\infty} y_{k,0,0}(\varepsilon_{t-k}\otimes\varepsilon_t\otimes\varepsilon_t) + \sum_{j=1}^{\infty} y_{0,j,0}(\varepsilon_t\otimes\varepsilon_{t-j}\otimes\varepsilon_t) + \sum_{i=1}^{\infty} y_{0,0,i}(\varepsilon_t\otimes\varepsilon_t\otimes\varepsilon_{t-i}) \\ &+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} y_{k,j,0}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_t) + \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} y_{k,0,i}(\varepsilon_{t-k}\otimes\varepsilon_t\otimes\varepsilon_{t-i}) + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} y_{0,j,i}(\varepsilon_t\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \\ &+ \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left(y_{k,j,i} - \alpha y_{k-1,j-1,i-1}^{state}\right)(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) + y_{0,0,0}(\varepsilon_t\otimes\varepsilon_t\otimes\varepsilon_t) \end{aligned} \tag{E-52}$$

Each term can be converted into the corresponding state space representation. We will proceed

one by one

$$y_{0,0,0}(\varepsilon_t \otimes \varepsilon_t \otimes \varepsilon_t) = \beta_{000}\left(\varepsilon_t^{\otimes[3]}\right) \tag{E-53}$$

The triple sum, by commuting, can be written as

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left(y_{k,j,i} - \alpha y_{k-1,j-1,i-1}^{state}\right)(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \\
=&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\beta_{333}s_{k,j,i}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \\
=&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\beta_{333,1}\left[\left(y_{k-1}^{state}\varepsilon_{t-k}\right)\otimes\left(y_{j-1}^{state}\varepsilon_{t-j}\right)\otimes\left(y_{i-1}^{state}\varepsilon_{t-i}\right)\right] \\
&+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\beta_{22}\left[\left(y_{k-1,j-1}^{state}\otimes y_{i-1}^{state}\right)(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i})\right. \\
&+\left(y_{k-1,i-1}^{state}\otimes y_{j-1}^{state}\right)(I_{ne}\otimes K_{ne,ne})(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \\
&\left.+\left(y_{j-1,i-1}^{state}\otimes y_{k-1}^{state}\right)K_{ne^2,ne}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i})\right] \\
=&\beta_{333,1}\left(dy_{t-1}^{(1)state^{\otimes[3]}}\right)+6\beta_{22}\left(dy_{t-1}^{(2)state}\otimes dy_{t-1}^{(1)state}\right)
\end{aligned} \tag{E-54}$$

The following applies to, by commuting, all three of the single sums

$$\begin{aligned}
&\sum_{k=1}^{\infty}y_{k,0,0}(\varepsilon_{t-k}\otimes\varepsilon_t\otimes\varepsilon_t) \\
=&\sum_{k=1}^{\infty}\beta_{300}\left(y_{k-1}^{state}\otimes\varepsilon_{t-k}\right)(I_{ne^2}\otimes\varepsilon_t\otimes\varepsilon_t) \\
=&\beta_{300}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t^{\otimes[2]}\right)
\end{aligned} \tag{E-55}$$

The following applies to, by commuting, all three of the double sums

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}y_{k,j,0}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_t) \\
=&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\beta_{330}\begin{bmatrix}y_{k-1}^{state}\otimes y_{j-1}^{state}\otimes I_{ne}\\ y_{k-1,j-1}\otimes I_{ne}\end{bmatrix}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_t) \\
=&\beta_{330,1}\left(dy_{t-1}^{(1)state^{\otimes[2]}}\otimes\varepsilon_t\right)+2\beta_{20}\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)
\end{aligned} \tag{E-56}$$

Combining the above

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left(y_{k,j,i} - \alpha y_{k-1,j-1,i-1}^{state}\right)(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \\
=&\beta_{333,1}\left(dy_{t-1}^{(1)state^{\otimes[3]}}\right)+\beta_{000}\left(\varepsilon_t^{\otimes[3]}\right) \\
&+3\beta_{330,1}\left(dy_{t-1}^{(1)state^{\otimes[2]}}\otimes\varepsilon_t\right)+3\beta_{300}\left(dy_{t-1}^{(1)state}\otimes\varepsilon_t^{\otimes[2]}\right)
\end{aligned}$$

$$+6\beta_{22}\left(dy_{t-1}^{(2)state}\otimes dy_{t-1}^{(1)state}\right)+6\beta_{20}\left(dy_{t-1}^{(2)state}\otimes\varepsilon_t\right)$$

E.1 Block Kronecker Expression of Third Order Coefficients

Similarly to the derivations at second order, we can connect y_{z^3} and

$$\begin{bmatrix} y_{y^{state3}} & y_{\varepsilon y^{state2}} & y_{y^{state}\varepsilon y^{state}} & y_{\varepsilon^2 y^{state}} & y_{y^{state2}\varepsilon} & y_{\varepsilon y^{state}\varepsilon} & y_{y^{state}\varepsilon^2} & y_{\varepsilon^3}\end{bmatrix}$$

with operations involving Magnus and Neudecker's (1979) commutation matrix, denoted here by $K_{a,b}$, following the definition of the block Kronecker product of Koning, Neudecker, and Wansbeek (1991) as follows

$$\begin{aligned} y_{z^3}dz_t^{(1)^{\otimes[3]}} &= y_{z^3}\underbrace{\begin{bmatrix} K_{ns,(ns+ne)^2} & 0_{ns(ns+ne)^2\times ne(ns+ne)^2} \\ 0_{ne(ns+ne)^2\times ns(ns+ne)^2} & K_{ne,(ns+ne)^2}\end{bmatrix}}_{\equiv G_{31}}\left(dz_t^{(1)^{\otimes[2]}}\right)\boxtimes dz_t^{(1)} \\ &= y_{z^3}G_{31}\left(G_2 dz_t^{(1)^{\boxtimes[2]}}\right)\boxtimes dz_t^{(1)} \\ &= y_{z^3}G_{31}\left[G_2\boxtimes I_{ns+ne}\right]dz_t^{(1)^{\boxtimes[3]}} \\ &= y_{z^3}\left(I_{ns+ne}\otimes G_2\right)G_{31}dz_t^{(1)^{\boxtimes[3]}} \\ &= \begin{bmatrix} y_{y^{state3}} & y_{\varepsilon y^{state2}} & y_{y^{state}\varepsilon y^{state}} & y_{\varepsilon^2 y^{state}} & y_{y^{state2}\varepsilon} & y_{\varepsilon y^{state}\varepsilon} & y_{y^{state}\varepsilon^2} & y_{\varepsilon^3}\end{bmatrix}dz_t^{(1)^{\boxtimes[3]}} \end{aligned} \tag{E-57}$$

where G_2 was defined in the proof of the block Kronecker formulation of the second order approximation. Accordingly

(E-58)

$$y_{z^3}\left(I_{ns+ne}\otimes G_2\right)G_{31}=\begin{bmatrix} y_{y^{state3}} & y_{\varepsilon y^{state2}} & y_{y^{state}\varepsilon y^{state}} & y_{\varepsilon^2 y^{state}} & y_{y^{state2}\varepsilon} & y_{\varepsilon y^{state}\varepsilon} & y_{y^{state}\varepsilon^2} & y_{\varepsilon^3}\end{bmatrix}$$

As in the second order case, the block Kronecker product, through G_2 and G_{31}, allows us to extract the individual block third derivatives with respect to y_{t-1}^{state} and e_t from the matrix of third derivatives with respect to the entire state vector, z_t.

E.2 Proof of Proposition 4.5

Our assumption of the existence of a nonlinear moving average policy function (2)

$$y_t = y(\sigma,\varepsilon_t,\varepsilon_{t-1},\ldots) \tag{E-59}$$

requires that the state space representation (37)

$$y_t = g(\sigma,\varepsilon_t,y_{t-1}^{state}) \tag{E-60}$$

can be "inverted" in the sense that recursive substitution of (37) in itself will deliver (2)

$$y_t = g(\sigma, \varepsilon_t, \underbrace{g^{state}(\sigma, \varepsilon_{t-1}, \ldots)}_{y_{t-1}^{state}}) \equiv y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \ldots) \tag{E-61}$$

Thus, we can rewrite (37) by replacing with y_t and y_{t-1}^{state} with (2), appropriately lagged and with the subvector of states selected for the latter. This gives

$$y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \ldots) = g(\sigma, \varepsilon_t, y^{state}(\sigma, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots)) \tag{E-62}$$

By differentiating (E-62) with respect to the arguments of the nonlinear moving average policy function (2), we will demonstrate the equivalence or difference of the coefficients in the recursive algorithms of section 3 with those of the pruning algorithms in section 4.

At first order, we differentiate with respect to σ and the sequence of shocks $\{\varepsilon_{t-i}\}_{i=0}^{\infty}$. Accordingly

$$y_\sigma = g_\sigma + g_{y^{state}} y_\sigma^{state} \tag{E-63}$$

which when evaluated at the deterministic steady state confirms[49]

$$g_\sigma = 0 \rightarrow y_\sigma = g_\sigma = 0 \tag{E-64}$$

and with respect to the sequence of shocks

$$y_i = \begin{cases} g_{y^{state}} y_{i-1}^{state} & , \text{for } i > 0 \\ g_\varepsilon & , \text{for } i = 0 \end{cases} \tag{E-65}$$

comparing with (B-8), it follows by inspection that $g_{y^{state}} = \alpha \equiv y_{y^{state}}$ and $g_\varepsilon = \beta_0 \equiv y_\varepsilon$

At second order, we differentiate with respect to σ twice, σ and the sequence of shocks $\{\varepsilon_{t-i}\}_{i=0}^{\infty}$, and with respect to two sequences of shocks, $\{\varepsilon_{t-i}\}_{i=0}^{\infty}$ and $\{\varepsilon_{t-j}\}_{j=0}^{\infty}$.

Beginning with the derivative with respect to σ twice,

$$y_{\sigma^2} = g_{\sigma^2} + 2 g_{\sigma y^{state}} y_\sigma^{state} + g_{y^{state2}} y_\sigma^{state \otimes [2]} + g_{y^{state}} y_{\sigma^2}^{state} \tag{E-66}$$

evaluating at the deterministic steady state yields

$$y_{\sigma^2} = g_{\sigma^2} + g_{y^{state}} y_{\sigma^2}^{state} \tag{E-67}$$

or, reexpressing the second term on the r.h.s in terms of the full vector of endogenous variables,

$$y_{\sigma^2} = (I_{ny} - g_y)^{-1} g_{\sigma^2} \tag{E-68}$$

as was claimed.

[49] See Schmitt-Grohé and Uribe (2004), Jin and Judd (2002), and Lan and Meyer-Gohde (2012a).

With respect to σ and the sequence of shocks $\{\varepsilon_{t-i}\}_{i=0}^{\infty}$, we obtain

$$y_{\sigma i} = \begin{cases} g_{\sigma y^{state}} y_{i-1}^{state} + g_{y^{state2}} \left(y_{\sigma}^{state} \otimes y_{i-1}^{state} \right) + g_{y^{state}} y_{\sigma i-1}^{state} & \text{, for } i > 0 \\ g_{y^{state}\varepsilon} y_{\sigma}^{state} + g_{\sigma\varepsilon} & \text{, for } i = 0 \end{cases} \tag{E-69}$$

evaluating at the deterministic steady state, $g_{\sigma y^{state}} = 0$ and $g_{\sigma\varepsilon} = 0$ and[50] recalling the results from the first order above

$$y_{\sigma i} = 0 \tag{E-70}$$

With respect to two sequences of shocks

$$y_{ji} = \begin{cases} g_{y^{state}} y_{j-1,i-1}^{state} + g_{y^{state2}} \left(y_{j-1}^{state} \otimes y_{i-1}^{state} \right) & \text{, for } j, i > 0 \\ g_{y^{state}\varepsilon} \left(y_{j-1}^{state} \otimes I_{ne} \right) & \text{, for } j > 0, i = 0^{51} \\ g_{\varepsilon^2} & \text{, for } j, i = 0 \end{cases} \tag{E-71}$$

comparing with (B-14), it follows by inspection that $g_{y^{state2}} = \beta_{22} \equiv y_{y^{state2}}$, $g_{y^{state}\varepsilon} = \beta_{20} \equiv y_{y^{state}\varepsilon}$, and $g_{\varepsilon^2} = \beta_{00} \equiv y_{\varepsilon^2}$.

This completes the proof that all coefficients in the second order pruning algorithms and recursive formulation of the nonlinear moving average are identical, except for the constant risk adjustment terms y_{σ^2} and g_{σ^2}. The transitions follow immediately when setting all shock realizations to zero. For example, Kim, Kim, Schaumburg, and Sims's (2008) algorithm in lemma 4.3 in the absence of shocks is

$$y_t^{(2)} = \overline{y} + dy_t^{(2)} \tag{E-72}$$

where

$$dy_t^{(2)} = g_{y^{state}} dy_{t-1}^{(2)state} + \frac{1}{2} g_{\sigma^2} \tag{E-73}$$

with $dy_0^{(1)}$ and $dy_0^{(2)}$ initialized to zero. $dy_t^{(2)}$ transitions from zero to $(I_{ny} - g_y)^{-1} g_{\sigma^2}$ and the same follows for $y_t^{(2)}$ due to its linearity.

E.3 Proof of Proposition 4.7

Here we follow the proof of proposition 4.5 above. At third order, we have four derivatives: σ thrice, σ twice and a sequence of shocks, σ once and two sequences of shocks, and three sequences of shocks. In our derivations, we will jump right to the equations evaluated in the deterministic steady state.

[50] See again Schmitt-Grohé and Uribe (2004), Jin and Judd (2002), and Lan and Meyer-Gohde (2012a).

[51] The case $i > 0, j = 0$ follows symmetrically.

With respect to σ thrice at the deterministic steady state

$$y_{\sigma^3} = g_{\sigma^3} + g_{y^{state}} y_{\sigma^3}^{state} \tag{E-74}$$

or, reexpressing the second term on the r.h.s in terms of the full vector of endogenous variables,

$$y_{\sigma^3} = (I_{ny} - g_y)^{-1} g_{\sigma^3} \tag{E-75}$$

as was claimed.

With respect to σ twice and a sequence of shocks and evaluating at the deterministic steady state

$$y_{\sigma^2 i} = \begin{cases} g_{\sigma^2 y^{state}} y_{i-1}^{state} + g_{y^{state^2}} \left(y_{\sigma^2}^{state} \otimes y_{i-1}^{state} \right) + g_{y^{state}} y_{i-1}^{\sigma^2 state} & \text{, for } i > 0 \\ g_{\sigma^2 \varepsilon} + g_{y^{state} \varepsilon} \left(y_{\sigma^2}^{state} \otimes I_{ne} \right) & \text{, for } i = 0 \end{cases} \tag{E-76}$$

comparing with (B-31) $g_{\sigma^2\varepsilon} + g_{y^{state}\varepsilon}\left(y_{\sigma^2}^{state} \otimes I_{ne}\right) = \beta_{\sigma^2 0} \equiv y_{\sigma^2\varepsilon}$ and $g_{\sigma^2 y^{state}} + g_{y^{state^2}}\left(y_{\sigma^2}^{state} \otimes I_{ns}\right) = \beta_{\sigma^2 1} \equiv y_{\sigma^2 y^{state}}$ and, clearly, $g_{\sigma^2\varepsilon} \neq y_{\sigma^2\varepsilon}$ and $g_{\sigma^2 y^{state}} \neq y_{\sigma^2 y^{state}}$.

Derivatives with respect to σ once and two sequences of shocks are zeros in both representations,[52] $g_{\sigma y^{state^2}} = y_{\sigma y^{state^2}} = 0$, $g_{\sigma y^{state}\varepsilon} = y_{\sigma y^{state}\varepsilon} = 0$, and $g_{\sigma\varepsilon^2} = y_{\varepsilon^2} = 0$.

Finally, the terms with respect to three sequences of shocks,

$$y_{kji} = \begin{cases} g_{y^{state^2}} \left[y_{k-1,j-1}^{state} \otimes y_{i-1}^{state} + \left(y_{k-1,i-1}^{state} \otimes y_{j-1}^{state} \right) (I_{ne} \otimes K_{ne,ne}) + y_{j-1,i-1}^{state} \otimes y_{k-1}^{state} \right] & \\ + g_{y^{state}} y_{k-1,j-1,i-1}^{state} + g_{y^{state^3}} \left(y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes y_{i-1}^{state} \right) & \text{, for } k,j,i > 0 \\ g_{y^{state}\varepsilon} \left(y_{k-1,j-1}^{state} \otimes I_{ne} \right) + g_{y^{state^2}\varepsilon} \left(y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes I_{ne} \right) & \text{, for } k,j > 0, i = 0 \text{[53]} \\ g_{y^{state}\varepsilon^2} \left(y_{k-1}^{state} \otimes I_{ne^2} \right) & \text{, for } k > 0, j,i = 0 \text{[54]} \\ g_{\varepsilon^3} & \text{, for } k,j,i = 0 \end{cases} \tag{E-77}$$

comparing with (B-25), $g_{y^{state^3}} = \beta_{333} \equiv y_{y^{state^3}}$, $g_{y^{state^2}\varepsilon} = \beta_{330} \equiv y_{y^{state^2}\varepsilon}$, $g_{y^{state}\varepsilon^2} = \beta_{300} \equiv y_{y^{state}\varepsilon^2}$, and $g_{\varepsilon^3} = \beta_{000} \equiv y_{\varepsilon^3}$.

This completes the proof that all coefficients in the third order pruning algorithm in lemma 4.6 and recursive formulation of the nonlinear moving average are identical, except for the risk adjustment terms y_{σ^3} and g_{σ^3} as well as $y_{\sigma^2 y^{state}}$ and $g_{\sigma^2 y^{state}}$. The transitions follow immediately when setting all shock realizations to zero, see the second order case.

[52] See Andreasen (2012), Jin and Judd (2002), and Lan and Meyer-Gohde (2012a).

[54] The cases $k,i > 0, j = 0$ and $i,j > 0, k = 0$ follow symmetrically.

[54] The cases $i > 0, k,j = 0$ and $j > 0, k,i = 0$ follow symmetrically.

Online Appendix

F First Order Coefficients Appendix

We divide the problem into two cases, as the exogenous shocks are nonzero only upon impact.

F.0.1 Case 1: $i > 0$

Inserting (B-8) into (B-6), noting that $\varepsilon_i = 0$ for all positive i. The coefficient matrix α solves a matrix quadratic problem and as our typology of variables follows that of Dynare, we refer to Villemot (2011) for details on how this problem can be solved efficiently.

α is partitioned as

$$\underset{ny\times ns}{\alpha} = \begin{bmatrix} \alpha^{st} \\ \alpha^{bwd} \\ \alpha^{mix} \\ \alpha^{fwd} \end{bmatrix} = \begin{bmatrix} \alpha^{st} \\ \alpha^{state} \\ \alpha^{fwd} \end{bmatrix} = \begin{bmatrix} \alpha^{st} \\ \alpha^{bwd} \\ \alpha^{fwdendo} \end{bmatrix} = \begin{bmatrix} \alpha^{bwdendo} \\ \alpha^{fwdendo} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^{st}_{bwd} & \alpha^{st}_{mix} \\ \alpha^{bwd}_{bwd} & \alpha^{bwd}_{mix} \\ \alpha^{mix}_{bwd} & \alpha^{mix}_{mix} \\ \alpha^{fwd}_{bwd} & \alpha^{fwd}_{mix} \end{bmatrix} = \begin{bmatrix} \alpha^{bwdendo}_{bwd} & \alpha^{bwdendo}_{mix} \\ \alpha^{fwdendo}_{bwd} & \alpha^{fwdendo}_{mix} \end{bmatrix}$$

For stability, we assume that the square partition α^{state} has eigenvalues all inside the unit circle.

F.0.2 Case 2: $i = 0$

The impact effect of shocks on y_t is β_0 which can be partitioned as

$$\beta_0 = \begin{bmatrix} \beta_0^{st} \\ \beta_0^{state} \\ \beta_0^{fwd} \end{bmatrix} \tag{F-1}$$

When $i = 0$, the source equation reduces to

$$f_y y_0 + f_{y^+ fwdendo} y_1^{fwdendo} + f_\varepsilon = 0 \tag{F-2}$$

inserting (B-8) in the previous equation and collecting terms yields

$$\underbrace{\begin{bmatrix} f_{yst} & f_{ystate} + f_{y^+ fwdendo}\alpha^{fwdendo} & f_{yfwd} \end{bmatrix}}_{\widetilde{A}} \begin{bmatrix} \beta_0^{st} \\ \beta_0^{state} \\ \beta_0^{fwd} \end{bmatrix} = -f_\varepsilon \tag{F-3}$$

Solving for β_0 therefore is a standard linear problem

(F-4) $$\beta_0 = -\widetilde{A}^{-1} f_\varepsilon$$

G Second Order Coefficients Appendix

G.1 Solving the Unknown Coefficient $y_{j,i}$

To avoid unnecessary repetitive calculation, we split the derivation of $y_{j,i}$ into three parts according to the different values that the indices j and i take on. This enables us to use smaller state spaces to construct the solutions.

G.1.1 Case 1: $j > 0$ and $i > 0$

Note that the derivative of x_t with respect to ε_{t-i} can be written as the product of a constant matrix and the vector of state variables

(G-5) $$x_i = \gamma_1 y_{i-1}^{state}$$

(G-6) $$\text{where } x_i = \begin{bmatrix} y_{i-1}^{state} \\ y_i \\ y_{i+1}^{fwdendo} \\ \varepsilon_i \end{bmatrix}, \ \gamma_1 = \begin{bmatrix} I_{ns} \\ \alpha \\ \alpha^{fwdendo}\alpha^{state} \\ 0_{ne\times ns} \end{bmatrix}$$

Using the previous equation, the source equation takes the form

(G-7) $$f_{y^- state} y_{j-1,i-1}^{state} + f_y y_{j,i} + f_{y^+ fwdendo} y_{j+1,i+1}^{fwdendo} + f_{x^2}\gamma_1^{\otimes[2]}(y_{j-1}^{state} \otimes y_{i-1}^{state}) = 0$$

The solution takes the form

(G-8) $$y_{j,i} = \alpha y_{j-1,i-1}^{state} + \beta_{22}(y_{j-1}^{state} \otimes y_{i-1}^{state})$$

where β_{22} can be partitioned as

(G-9) $$\beta_{22} = \begin{bmatrix} \beta_{22}^{st} \\ \beta_{22}^{bwd} \\ \beta_{22}^{mix} \\ \beta_{22}^{fwd} \end{bmatrix}$$

With this partition, the recursion of $y_{j+1,i+1}^{fwdendo}$ takes the form

(G-10) $$y_{j+1,i+1}^{fwdendo} = \alpha^{fwdendo}\alpha^{state} y_{j-1,i-1}^{state} + \left[\alpha^{fwdendo}\beta_{22}^{state} + \beta_{22}^{fwdendo}\alpha^{state\otimes[2]}\right](y_{j-1}^{state} \otimes y_{i-1}^{state})$$

Inserting the solution (G-8) and (G-10) in the source equation and matching coefficients yields the following

(G-11) $$f_y\beta_{22} + f_{y^+ fwdendo}\left[\alpha^{fwdendo}\beta_{22}^{state} + \beta_{22}^{fwdendo}\alpha^{state\otimes[2]}\right] + f_{x^2}\gamma_1^{\otimes[2]} = 0$$

Again using the partition of β_{22} and collecting terms yields the following equation in two unknowns

$$\underbrace{\left[f_{yst} \quad f_{ybwd} + f_{y^+fwdendo}\alpha_{bwd}^{fwdendo}\right]}_{A}\beta_{22}^{bwdendo} + \underbrace{\left[f_{ymix} + f_{y^+fwdendo}\alpha_{mix}^{fwdendo} \quad f_{yfwd}\right]}_{B}\beta_{22}^{fwdendo}$$

(G-12)

$$+\underbrace{\left[f_{y^+mix} \quad f_{y^+fwd}\right]}_{C}\beta_{22}^{fwdendo}\alpha^{state\otimes[2]} + f_{x^2}\gamma_1^{\otimes[2]} = 0$$

Using A^O to denote the null space of A and pre-multiplying the previous equation by A^O yields the following Sylvester equation in $\beta_{22}^{fwdendo}$

$$(A^OB)\beta_{22}^{fwdendo} + (A^OC)\beta_{22}^{fwdendo}\alpha^{state\otimes[2]} + A^O f_{x^2}\gamma_1^{\otimes[2]} = 0 \tag{G-13}$$

With $\beta_{22}^{fwdendo}$ in hand, solving $\beta_{22}^{bwdendo}$ is a standard linear problem

$$\beta_{22}^{bwdendo} = -pinv(A)\left[B\beta_{22}^{fwdendo} + C\beta_{22}^{fwdendo}\alpha^{state\otimes[2]} + f_{x^2}\gamma_1^{\otimes[2]}\right] \tag{G-14}$$

where $pinv(A)$ represents the Moore-Penrose inverse of A.

G.1.2 Case 2: $j > 0$ and $i = 0$

Notice that

$$x_0 = \begin{bmatrix} 0 \\ ns\times ne \\ \beta_0 \\ y_1^{fwdendo} \\ I_{ne} \end{bmatrix} = \begin{bmatrix} 0 \\ ns\times ne \\ \beta_0 \\ \alpha^{fwdendo}\beta_0^{state} \\ I_{ne} \end{bmatrix} \tag{G-15}$$

which is a known constant matrix given the results from the first order results,[55] and the source equation takes the form

$$f_y y_{j,0} + f_{y^+fwdendo}y_{j+1,1}^{fwdendo} + f_{x^2}(x_j \otimes x_0) = 0 \tag{G-16}$$

The solution (G-8) implies $y_{1,i+1}^{fwdendo}$ takes the form

$$y_{j+1,1}^{fwdendo} = \alpha^{fwdendo}y_{j,0}^{state} + \beta_{22}^{fwdendo}(y_j^{state} \otimes \beta_0^{state}) \tag{G-17}$$

Inserting the previous equation in the source equation (G-16) and collecting terms yields

$$\underbrace{\left[f_{yst} \quad f_{ystate} + f_{y^+fwdendo}\alpha^{fwdendo} \quad f_{yfwd}\right]}_{\widetilde{A}}\underbrace{\begin{bmatrix} y_{j,0}^{st} \\ y_{j,0}^{state} \\ y_{j,0}^{fwd} \end{bmatrix}}_{y_{j,0}}$$

[55] While the first zero block should be removed from x_0 in order to further reduce the size of the state space in this case, we choose to keep it as otherwise the dimension of x_0 is different from that of x_i. This difference requires additional efforts in indexing the variables when coding the method.

(G-18) $$+\left[f_{y^+fwdendo}\beta_{22}^{fwdendo}(\alpha^{state}\otimes\beta_0^{state})+f_{x^2}(\gamma_1\otimes x_0)\right](y_{j-1}^{state}\otimes I_{ne})=0$$

The solution of $y_{0,i}$ takes the form

(G-19) $$y_{j,0}=\underset{ny\times(ne*ns)}{\beta_{20}}(y_{j-1}^{state}\otimes I_{ne})$$

Inserting the previous equation in the source equation (G-18) and matching coefficient yields

(G-20) $$\widetilde{A}\beta_{20}=-\left[f_{y^+fwdendo}\beta_{22}^{fwdendo}(\alpha^{state}\otimes\beta_0^{state})+f_{x^2}(\gamma_1\otimes x_0)\right]$$

which is a standard linear equation in the unknown coefficient β_{20}

(G-21) $$\beta_{20}=-\widetilde{A}^{-1}\left[f_{y^+fwdendo}\beta_{22}^{fwdendo}(\alpha^{state}\otimes\beta_0^{state})+f_{x^2}(\gamma_1\otimes x_0)\right]$$

The coefficient, $y_{0,i}$, can be computed by exploiting the commutability of the matrix derivative operator

$$\begin{aligned}y_{0,i}=&y_{j,0}K_{ne,ne}=\beta_{20}(y_{j-1}^{state}\otimes I_{ne})K_{ne,ne}\\=&\beta_{20}K_{ns,ne}(I_{ne}\otimes y_{j-1}^{state})K_{ne,ne}K_{ne,ne}\\=&\beta_{02}(I_{ne}\otimes y_{j-1}^{state})\end{aligned}\tag{G-22}$$

with

(G-23) $$\beta_{02}=\beta_{20}K_{ns,ne}$$

G.1.3 Case 3: $j=0$ and $i=0$

In this case the source equation (B-11) takes the form

(G-24) $$f_y y_{0,0}+f_{y^+fwdendo}y_{1,1}^{fwdendo}+f_{x^2}x_0^{\otimes[2]}=0$$

The solution (G-8) implies $y_{1,1}^{fwdendo}$ takes the form

(G-25) $$y_{1,1}^{fwdendo}=\alpha^{fwdendo}y_{0,0}^{state}+\beta_{22}^{fwdendo}\beta_0^{state\otimes[2]}$$

Inserting the previous equation in the source equation (G-24) and collecting terms yields

(G-26)
$$\underbrace{\begin{bmatrix}f_{yst} & f_{ystate}+f_{y^+fwdendo}\alpha^{fwdendo} & f_{yfwd}\end{bmatrix}}_{\widetilde{A}}\underbrace{\begin{bmatrix}y_{0,0}^{st}\\y_{0,0}^{state}\\y_{0,0}^{fwd}\end{bmatrix}}_{y_{0,0}}+f_{y^+fwdendo}\beta_{22}^{fwdendo}\beta_0^{state\otimes[2]}+f_{x^2}x_0^{\otimes[2]}=0$$

Solving $y_{0,0}$ therefore is a standard linear problem

(G-27) $$y_{0,0}=-\widetilde{A}^{-1}\left[f_{y^+fwdendo}\beta_{22}^{fwdendo}\beta_0^{state\otimes[2]}+f_{x^2}x_0^{\otimes[2]}\right]$$

For the consistency of notation between the moving average and state space representations of

the second order approximation of the policy function, we let

$$\beta_{00} = y_{0,0} \tag{G-28}$$

G.2 Solving the Unknown Coefficient y_{σ^2}

The source equation takes the form

$$y_{\sigma^2} = -(f_{y^-} + f_y + f_{y^+})^{-1}[f_{y^+} y_{0,0} + f_{y^{+2}} y_0^{\otimes[2]}] E_t\left(\varepsilon_{t+1}^{\otimes[2]}\right) \tag{G-29}$$

Making use of the special structure of f_{y^-}, f_y and f_{y^+} and collecting terms yields

$$\begin{aligned} y_{\sigma^2} = &\left[f_{yst} \quad f_{y^- bwd} + f_{ybwd} \quad f_{y^- mix} + f_{ymix} + f_{y^+ mix} \quad f_{yfwd} + f_{y^+ fwd}\right]^{-1} \\ &\left[f_{y^+ fwdendo}\beta_{00}^{fwdendo} + f_{(y^+ fwdendo)^2}\beta_0^{fwdendo\otimes[2]}\right] E_t\left(\varepsilon_{t+1}^{\otimes[2]}\right) \end{aligned} \tag{G-30}$$

H Third Order Coefficients Appendix

Given the results from lower orders, including that terms linear in the perturbation parameter are zero, the third order approximation of the policy function takes the form

$$\begin{aligned} y_t^{(3)} = &\overline{y} + \frac{1}{2} y_{\sigma^2} + \sum_{i=0}^{\infty}\left(y_i + \frac{1}{2} y_{\sigma^2,i}\right)\varepsilon_{t-i} + \frac{1}{2}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} y_{j,i}(\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\ &+ \frac{1}{6}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} y_{k,j,i}(\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \end{aligned} \tag{H-31}$$

The task at hand is to pin down some third derivatives of the policy function, including $y_{k,j,i}$, $y_{\sigma^2,i}$.

H.1 Solving the Unknown Coefficient $y_{k,j,i}$

As in the second order case, to avoid redundant calculations, we split the derivation of $y_{k,j,i}$ into four parts according to the different values that the indices k, j and i take on.

H.1.1 Case 1: $k > 0$, $j > 0$ and $i > 0$

Note the second derivative of x_t vector, $x_{j,i}$ can be written as the product of a constant matrix and the second order state space $S_{j,i}$

$$x_{j,i} = \gamma_2 S_{j,i} \tag{H-32}$$

$$\text{where } x_{j,i} = \begin{bmatrix} y^{state}_{j-1,i-1} \\ y_{j,i} \\ y^{fwdendo}_{j+1,i+1} \\ 0_{ne\times ne^2} \end{bmatrix},\ S_{j,i} = \begin{bmatrix} y^{state}_{j-1,i-1} \\ y^{state}_{j-1} \otimes y^{state}_{i-1} \end{bmatrix}$$

(H-33)
$$\gamma_2 = \begin{bmatrix} I_{ns} & 0_{ns\times ns^2} \\ \alpha & \beta_{22} \\ \alpha^{fwdendo}\alpha^{state} & \alpha^{fwdendo}\beta^{state}_{22} + \beta^{fwdendo}_{22}\alpha^{state\otimes[2]} \\ 0_{ne\times ns} & 0_{ne\times ns^2} \end{bmatrix}$$

In particular, let

(H-34)
$$\gamma_{22} = \begin{bmatrix} 0_{ns\times ns^2} \\ \beta_{22} \\ \alpha^{fwdendo}\beta^{state}_{22} + \beta^{fwdendo}_{22}\alpha^{state\otimes[2]} \\ 0_{ne\times ns^2} \end{bmatrix}$$

then γ_2 can be written as

(H-35)
$$\gamma_2 = \begin{bmatrix} \gamma_1 & \gamma_{22} \end{bmatrix}$$

which implies

(H-36)
$$\gamma_2 \otimes \gamma_1 = \begin{bmatrix} \gamma_1^{\otimes[2]} & \gamma_{22} \otimes \gamma_1 \end{bmatrix}$$

This is a very useful property for avoiding redundant computations in solving for the coefficients of the third order approximation. The third order state space consists of the state spaces of all lower orders

(H-37)
$$S_{k,j,i} = \begin{bmatrix} y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes y^{state}_{i-1} \\ S_{k,j} \otimes y^{state}_{i-1} \\ \left(y^{state}_{j-1} \otimes S_{k,i}\right)(K_{ne,ne} \otimes I_{ne}) \\ y^{state}_{k-1} \otimes S_{j,i} \end{bmatrix}$$

By constructing the following constant matrix

(H-38)
$$\gamma_3 = \begin{bmatrix} \gamma_1^{\otimes[3]} & 0 & 0 & 0 \\ 0 & \gamma_2 \otimes \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_1 \otimes \gamma_2 & 0 \\ 0 & 0 & 0 & \gamma_1 \otimes \gamma_2 \end{bmatrix}$$

the source equation can be written as

(H-39)
$$f_{y^- state} y^{state}_{k-1,j-1,i-1} + f_y y_{k,j,i} + f_{y^+ fwdendo} y^{fwdendo}_{k+1,j+1,i+1} + \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \gamma_3 S_{k,j,i} = 0$$

The state space for the third order approximation, $S_{k,j,i}$, can be further reduced using (H-36),

the partition of γ_2. Multiplying out the last term of the previous equation yields[56]

$$\begin{aligned}\left[f_{x^3} \quad f_{x^2} \quad f_{x^2} \quad f_{x^2}\right]\gamma_3 S_{k,j,i} = {} & f_{x^3}\gamma_1^{\otimes[3]}\left(y_{k-1}^{state}\otimes y_{j-1}^{state}\otimes y_{i-1}^{state}\right) + f_{x^2}(\gamma_2\otimes\gamma_1)(S_{k,j}\otimes y_{i-1}^{state}) \\ & + f_{x^2}(\gamma_1\otimes\gamma_2)\left[\left(y_{j-1}^{state}\otimes S_{k,i}\right)K_{ne,ne^2}(I_{ne}\otimes K_{ne,ne})\right] \\ & + f_{x^2}(\gamma_1\otimes\gamma_2)\left(y_{k-1}^{state}\otimes S_{j,i}\right)\end{aligned} \tag{H-40}$$

Using (H-35) and (H-36), terms on the right hand side of the previous equation can be written as

(H-41)

$$\begin{aligned}& f_{x^2}(\gamma_2\otimes\gamma_1)(S_{k,j}\otimes y_{i-1}^{state}) \\ = {} & f_{x^2}\left[\gamma_1^{\otimes[2]}\left(y_{k-1,j-1}^{state}\otimes y_{i-1}^{state}\right) + (\gamma_{22}\otimes\gamma_1)\left(y_{k-1}^{state}\otimes y_{j-1}^{state}\otimes y_{i-1}^{state}\right)\right]\end{aligned}$$

(H-42)

$$\begin{aligned}& f_{x^2}(\gamma_1\otimes\gamma_2)\left[\left(y_{j-1}^{state}\otimes S_{k,i}\right)K_{ne,ne^2}(I_{ne}\otimes K_{ne,ne})\right] \\ = {} & f_{x^2}\left[\gamma_1^{\otimes[2]}\left(y_{k-1,i-1}^{state}\otimes y_{j-1}^{state}\right)(I_{ne}\otimes K_{ne,ne}) + (\gamma_{22}\otimes\gamma_1)(I_{ns}\otimes K_{ns,ns})\left(y_{k-1}^{state}\otimes y_{j-1}^{state}\otimes y_{i-1}^{state}\right)\right]\end{aligned}$$

(H-43)

$$\begin{aligned}& f_{x^2}(\gamma_1\otimes\gamma_2)\left(y_{k-1}^{state}\otimes S_{j,i}\right) \\ = {} & f_{x^2}\left[\gamma_1^{\otimes[2]}\left(y_{j-1,i-1}^{state}\otimes y_{k-1}^{state}\right)K_{ne^2,ne} + (\gamma_{22}\otimes\gamma_1)K_{ns^2,ns}\left(y_{k-1}^{state}\otimes y_{j-1}^{state}\otimes y_{i-1}^{state}\right)\right]\end{aligned}$$

therefore

$$\begin{aligned}& \left[f_{x^3} \quad f_{x^2} \quad f_{x^2} \quad f_{x^2}\right]\gamma_3 S_{k,j,i} \\ = {} & \left[f_{x^3}\gamma_1^{\otimes[3]} + f_{x^2}(\gamma_{22}\otimes\gamma_1) + f_{x^2}(\gamma_{22}\otimes\gamma_1)(I_{ns}\otimes K_{ns,ns}) + f_{x^2}(\gamma_{22}\otimes\gamma_1)K_{ns^2,ns}\right]\left(y_{k-1}^{state}\otimes y_{j-1}^{state}\otimes y_{i-1}^{state}\right) \\ & + f_{x^2}\gamma_1^{\otimes[2]}\left[y_{k-1,j-1}^{state}\otimes y_{i-1}^{state} + \left(y_{k-1,i-1}^{state}\otimes y_{j-1}^{state}\right)(I_{ne}\otimes K_{ne,ne}) + \left(y_{j-1,i-1}^{state}\otimes y_{k-1}^{state}\right)K_{ne^2,ne}\right]\end{aligned} \tag{H-44}$$

$$= \gamma_{333} s_{k,j,i}$$

with

$$\gamma_{333} = \left[f_{x^3}\gamma_1^{\otimes[3]} + f_{x^2}(\gamma_{22}\otimes\gamma_1)(I_{ns^3} + I_{ns}\otimes K_{ns,ns} + K_{ns^2,ns}) \quad f_{x^2}\gamma_1^{\otimes[2]}\right] \tag{H-45}$$

where $s_{k,j,i}$ is the state space for the third order approximation defined in (B-22) that replaces the larger $S_{k,j,i}$, and the source equation (H-39) can therefore be written as

$$f_{y^-state}y_{k-1,j-1,i-1}^{state} + f_y y_{k,j,i} + f_{y^+fwdendo}y_{k+1,j+1,i+1}^{fwdendo} + \gamma_{333}s_{k,j,i} = 0 \tag{H-46}$$

The solution takes the form

$$y_{k,j,i} = \alpha y_{k-1,j-1,i-1}^{state} + \beta_{333}s_{k,j,i} \tag{H-47}$$

[56] We will make repeated use of the fact that $K_{ne,ne}\otimes I_{ne} = (K_{ne,ne}\otimes I_{ne})(I_{ne}\otimes K_{ne,ne})(I_{ne}\otimes K_{ne,ne}) = K_{ne,ne^2}(I_{ne}\otimes K_{ne,ne})$, see Lan and Meyer-Gohde (2012b), as this last representation will prove better suited to our needs.

which implies

(H-48) $$y^{fwdendo}_{k+1,j+1,i+1} = \alpha^{fwdendo}\alpha^{state}y^{state}_{k-1,j-1,i-1} + \left(\alpha^{fwdendo}\beta^{state}_{333}\right)s_{k,j,i} + \beta^{fwdendo}_{333}s_{k+1,j+1,i+1}$$

where

(H-49)

$$s_{k+1,j+1,i+1} = \begin{bmatrix} \alpha^{state\otimes[3]}\left(y^{state}_{k-1}\otimes y^{state}_{j-1}\otimes y^{state}_{i-1}\right) \\ \alpha^{state\otimes[2]}\left[y^{state}_{k-1,j-1}\otimes y^{state}_{i-1} + \left(y^{state}_{k-1,i-1}\otimes y^{state}_{j-1}\right)(I_{ne}\otimes K_{ne,ne}) + \left(y^{state}_{j-1,i-1}\otimes y^{state}_{k-1}\right)K_{ne^2,ne}\right] \\ + \left(\beta^{state}_{22}\otimes\alpha^{state}\right)(I_{ns^3} + I_{ns}\otimes K_{ns,ns} + K_{ns^2,ns})\left(y^{state}_{k-1}\otimes y^{state}_{j-1}\otimes y^{state}_{i-1}\right) \end{bmatrix}$$

With β_{333} conformably partitioned, the last term in (H-48) takes the form

(H-50) $$\beta^{fwdendo}_{333}s_{k+1,j+1,i+1} = \begin{bmatrix} \beta^{fwdendo}_{333,1}\alpha^{state\otimes[3]} \\ +\beta^{fwdendo}_{333,2}\left(\beta^{state}_{22}\otimes\alpha^{state}\right)(I_{ns^3} + I_{ns}\otimes K_{ns,ns} + K_{ns^2,ns}) \qquad \beta^{fwdendo}_{333,2}\alpha^{state\otimes[2]} \end{bmatrix} s_{k,j,i}$$

therefore (H-48) can be written as

(H-51)

$$\begin{aligned} & y^{fwdendo}_{k+1,j+1,i+1} \\ =& \alpha^{fwdendo}\alpha^{state}y^{state}_{k-1,j-1,i-1} \\ &+ \begin{bmatrix} \alpha^{fwdendo}\beta^{state}_{333,1} + \beta^{fwdendo}_{333,1}\alpha^{state\otimes[3]} & \alpha^{fwdendo}\beta^{state}_{333,2} \\ +\beta^{fwdendo}_{333,2}\left(\beta^{state}_{22}\otimes\alpha^{state}\right)(I_{ns^3} + I_{ns}\otimes K_{ns,ns} + K_{ns^2,ns}) & +\beta^{fwdendo}_{333,2}\alpha^{state\otimes[2]} \end{bmatrix} s_{k,j,i} \end{aligned}$$

Inserting the solution (H-47) and (H-51) in the source equation (B-21) and matching coefficients yields

(H-52)

$$\begin{aligned} & \begin{bmatrix} f_y\beta_{333,1} & f_y\beta_{333,2} \end{bmatrix} \\ & + f_{y^+fwdendo}\begin{bmatrix} \alpha^{fwdendo}\beta^{state}_{333,1} + \beta^{fwdendo}_{333,1}\alpha^{state\otimes[3]} & \alpha^{fwdendo}\beta^{state}_{333,2} \\ +\beta^{fwdendo}_{333,2}\left(\beta^{state}_{22}\otimes\alpha^{state}\right)(I_{ns^3} + I_{ns}\otimes K_{ns,ns} + K_{ns^2,ns}) & +\beta^{fwdendo}_{333,2}\alpha^{state\otimes[2]} \end{bmatrix} \\ & + \gamma_{333} = 0 \end{aligned}$$

which consists of two blocks. The second block takes the form

(H-53) $$f_y\beta_{333,2} + f_{y^+fwdendo}\left(\alpha^{fwdendo}\beta^{state}_{333,2} + \beta^{fwdendo}_{333,2}\alpha^{state\otimes[2]}\right) + f_{x^2}\gamma_1^{\otimes[2]} = 0$$

Partitioning $\beta_{333,2}$ conformably (in rows) and collecting terms yields the following equation in two unknowns

(H-54)

$$\underbrace{\begin{bmatrix} f_{yst} & f_{ybwd} + f_{y^+fwdendo}\alpha^{fwdendo}_{bwd} \end{bmatrix}}_{A}\beta^{bwdendo}_{333,2} + \underbrace{\begin{bmatrix} f_{ymix} + f_{y^+fwdendo}\alpha^{fwdendo}_{mix} & f_{yfwd} \end{bmatrix}}_{B}\beta^{fwdendo}_{333,2}$$

$$+\underbrace{\left[f_{y^+mix} \quad f_{y^+fwd}\right]}_{C}\beta_{333,2}^{fwdendo}\alpha^{state\otimes[2]}+f_{x^2}\gamma_1^{\otimes[2]}=0$$

noting that the coefficients in (H-54) are identical to those in (G-12). As in section G.1.1, we pre-multiply the null space A^O through the previous equation to obtain the Sylvester equation in $\beta_{333,2}^{fwdendo}$

$$(A^OB)\beta_{333,2}^{fwdendo}+(A^OC)\beta_{333,2}^{fwdendo}\alpha^{state\otimes[2]}+A^Of_{x^2}\gamma_1^{\otimes[2]}=0 \tag{H-55}$$

As the coefficients in the previous Sylvester equation are identical to those in (G-13), it follows immediately that

$$\beta_{333,2}^{fwdendo}=\beta_{22}^{fwdendo} \tag{H-56}$$

Given $\beta_{333,2}^{fwdendo}$, solving $\beta_{333,2}^{bwdendo}$ is a standard linear problem

$$\beta_{333,2}^{bwdendo}=-pinv(A)\left[B\beta_{333,2}^{fwdendo}+C\beta_{333,2}^{fwdendo}\alpha^{state\otimes[2]}+f_{x^2}\gamma_1^{\otimes[2]}\right] \tag{H-57}$$

It follows

$$\beta_{333,2}^{bwdendo}=\beta_{22}^{bwdendo} \tag{H-58}$$

Hence

$$\beta_{333,2}=\beta_{22} \tag{H-59}$$

Given $\beta_{333,2}$, the first block of (H-52) takes the form

$$f_y\beta_{333,1}+f_{y^+fwdendo}\left[\alpha^{fwdendo}\beta_{333,1}^{state}+\beta_{333,1}^{fwdendo}\alpha^{state\otimes[3]}\right]+D_3=0 \tag{H-60}$$

where

$$D_3=f_{x^3}\gamma_1^{\otimes[3]}+\left[f_{x^2}(\gamma_{22}\otimes\gamma_1)+f_{y^+fwdendo}\beta_{22}^{fwdendo}\left(\beta_{22}^{state}\otimes\alpha^{state}\right)\right]\left(I_{ns^3}+I_{ns}\otimes K_{ns,ns}+K_{ns^2,ns}\right) \tag{H-61}$$

Partitioning $\beta_{333,1}$ conformably (in rows) and collecting terms yields the following equation in two unknowns

$$\underbrace{\left[f_{yst} \quad f_{ybwd}+f_{y^+fwdendo}\alpha_{bwd}^{fwdendo}\right]}_{A}\beta_{333,1}^{bwdendo}+\underbrace{\left[f_{ymix}+f_{y^+fwdendo}\alpha_{mix}^{fwdendo} \quad f_{yfwd}\right]}_{B}\beta_{333,1}^{fwdendo}$$

$$+\underbrace{\left[f_{y^+mix} \quad f_{y^+fwd}\right]}_{C}\beta_{333,1}^{fwdendo}\alpha^{state\otimes[3]}+D_3=0 \tag{H-62}$$

Pre-multiplying the null space A^O through the previous equation yields a Sylvester equation in $\beta_{333,1}^{fwdendo}$

$$(A^OB)\beta_{333,1}^{fwdendo}+(A^OC)\beta_{333,1}^{fwdendo}\alpha^{state\otimes[3]}+A^OD_3=0 \tag{H-63}$$

Given $\beta_{333,1}^{fwdendo}$, solving $\beta_{333,1}^{bwdendo}$ is a standard linear problem

$$\beta_{333,1}^{bwdendo} = -pinv(A)\left[B\beta_{333,1}^{fwdendo} + C\beta_{333,1}^{fwdendo}\alpha^{state\otimes[3]} + D_3\right] \tag{H-64}$$

H.1.2 Case 2: $k = 0$, $j = 0$ and $i = 0$

Note that

$$x_{0,0} = \begin{bmatrix} 0_{ns\times ne^2} \\ y_{0,0} \\ y_{1,1}^{fwdendo} \\ 0_{ne\times ne^2} \end{bmatrix} \tag{H-65}$$

which is a known constant matrix given the lower order results. The source equation takes the form

$$\begin{aligned} &f_y y_{0,0,0} + f_{y^+fwdendo} y_{1,1,1}^{fwdendo} + f_{x^3} x_0^{\otimes[3]} + f_{x^2}\left(x_{0,0}\otimes x_0\right) \\ &+ f_{x^2}\left(x_0\otimes x_{0,0}\right)\left(K_{ne,ne}\otimes I_{ne}\right) + f_{x^2}\left(x_0\otimes x_{0,0}\right) = 0 \end{aligned} \tag{H-66}$$

Note that rolling the solution (H-47) one period forward and taking only the forward endogenous variables part yields

$$y_{1,1,1}^{fwdendo} = \alpha^{fwdendo} y_{0,0,0}^{state} + \beta_{333}^{fwdendo} s_{1,1,1} \tag{H-67}$$

where $s_{1,1,1}$ can be obtained by setting $k = j = i = 1$ in (B-22)

$$s_{1,1,1} = \begin{bmatrix} y_0^{state\otimes[3]} \\ \left(y_{0,0}^{state}\otimes y_0^{state}\right)\left(I_{ne^3} + I_{ne}\otimes K_{ne,ne} + K_{ne^2,ne}\right) \end{bmatrix} \tag{H-68}$$

As all the terms on the right hand side of the previous equation are known, $s_{1,1,1}$ is a known constant matrix. Inserting (H-67) in the source equation (H-66) and collecting terms yields

$$\begin{aligned} &\underbrace{\left[f_{yst} \quad f_{ystate} + f_{y^+fwdendo}\alpha^{fwdendo} \quad f_{yfwd}\right]}_{\widetilde{A}} y_{0,0,0} \\ &= -\left[f_{y^+fwdendo}\beta_{333}^{fwdendo} s_{1,1,1} + f_{x^3} x_0^{\otimes[3]} + f_{x^2}\left(x_{0,0}\otimes x_0\right)\left(I_{ne^3} + I_{ne}\otimes K_{ne,ne} + K_{ne^2,ne}\right)\right] \end{aligned} \tag{H-69}$$

Solving $y_{0,0,0}$ is therefore a standard linear problem

$$y_{0,0,0} = -\widetilde{A}^{-1}\left[f_{y^+fwdendo}\beta_{333}^{fwdendo} s_{1,1,1} + f_{x^3} x_0^{\otimes[3]} + f_{x^2}\left(x_{0,0}\otimes x_0\right)\left(I_{ne^3} + I_{ne}\otimes K_{ne,ne} + K_{ne^2,ne}\right)\right] \tag{H-70}$$

For notational consistency, we let

$$\beta_{000} = y_{0,0,0} \tag{H-71}$$

H.1.3 Case 3: $k > 0$, $j = 0$ and $i = 0$

Note that

$$x_{j,0} = \begin{bmatrix} 0 \\ {\scriptstyle ns \times ne^2} \\ y_{j,0} \\ y_{j+1,1}^{fwdendo} \\ 0 \\ {\scriptstyle ne \times ne^2} \end{bmatrix} \tag{H-72}$$

and from the solution (G-10) and (B-17)

$$y_{j+1,1}^{fwdendo} = \left[\alpha^{fwdendo}\beta_{20}^{state} + \beta_{22}^{fwdendo}\left(\alpha^{state} \otimes \beta_0^{state}\right)\right]\left(y_{j-1}^{state} \otimes I_{ne}\right) \tag{H-73}$$

$x_{j,0}$ can be written as the product of a constant matrix and a particular first order state space

$$x_{j,0} = \gamma_{20}\left(y_{j-1}^{state} \otimes I_{ne}\right) \tag{H-74}$$

$$\text{where } \gamma_{20} = \begin{bmatrix} 0 \\ {\scriptstyle ns \times (ns*ne)} \\ \beta_{20} \\ \alpha^{fwdendo}\beta_{20}^{state} + \beta_{22}^{fwdendo}\left(\alpha^{state} \otimes \beta_0^{state}\right) \\ 0 \\ {\scriptstyle ne \times (ns*ne)} \end{bmatrix} \tag{H-75}$$

The source equation takes the form

$$f_y y_{k,0,0} + f_{y^+ fwdendo} y_{k+1,1,1}^{fwdendo} + \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \begin{bmatrix} x_k \otimes x_0^{\otimes[2]} \\ x_{k,0} \otimes x_0 \\ \left(x_0 \otimes x_{k,0}\right)\left(K_{ne,ne} \otimes I_{ne}\right) \\ x_k \otimes x_{0,0} \end{bmatrix} = 0 \tag{H-76}$$

Using the constant matrices we defined and rearranging, the previous equation can be rewritten

$$f_y y_{k,0,0} + f_{y^+ fwdendo} y_{k+1,1,1}^{fwdendo} + \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \begin{bmatrix} \left(\gamma_1 \otimes x_0^{\otimes[2]}\right)\left(y_{k-1}^{state} \otimes I_{ne^2}\right) \\ \left(\gamma_{20} \otimes x_0\right)\left(y_{k-1}^{state} \otimes I_{ne^2}\right) \\ K_{nx,nx}\left(\gamma_{20} \otimes x_0\right)\left(I_{ns} \otimes K_{ne,ne}\right)\left(y_{k-1}^{state} \otimes I_{ne^2}\right) \\ \left(\gamma_1 \otimes x_{0,0}\right)\left(y_{k-1}^{state} \otimes I_{ne^2}\right) \end{bmatrix} = 0 \tag{H-77}$$

collecting terms and noting $f_{x^2} = f_{x^2} K_{nx,nx}$ yields

$$\begin{aligned} &f_y y_{k,0,0} + f_{y^+ fwdendo} y_{k+1,1,1}^{fwdendo} \\ &+ \left[f_{x^3}\left(\gamma_1 \otimes x_0^{\otimes[2]}\right) + f_{x^2}\left(\gamma_{20} \otimes x_0\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) + f_{x^2}\left(\gamma_1 \otimes x_{0,0}\right)\right]\left(y_{k-1}^{state} \otimes I_{ne^2}\right) \\ &= 0 \end{aligned} \tag{H-78}$$

Note that, from the solution (H-47)

$$y_{k+1,1,1}^{fwdendo} = \alpha^{fwdendo} y_{k,0,0}^{state} + \beta_{333}^{fwdendo} s_{k+1,1,1} \tag{H-79}$$

where

$$s_{k+1,1,1} = \begin{bmatrix} \alpha^{state} \otimes \beta_0^{state\otimes[2]} \\ \left(\beta_{20}^{state} \otimes \beta_0^{state}\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) \\ + \left(\beta_{00}^{state} \otimes \alpha^{state}\right) K_{ne^2,ns} \end{bmatrix} \left(y_{k-1}^{state} \otimes I_{ne^2}\right) \tag{H-80}$$

The solution (H-79) therefore can be written as

$$\begin{aligned} y_{k+1,1,1}^{fwdendo} =& \alpha^{fwdendo} y_{k,0,0}^{state} \\ &+ \beta_{333}^{fwdendo} \begin{bmatrix} \alpha^{state} \otimes \beta_0^{state\otimes[2]} \\ \left(\beta_{20}^{state} \otimes \beta_0^{state}\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) \\ + \left(\beta_{00}^{state} \otimes \alpha^{state}\right) K_{ne^2,ns} \end{bmatrix} \left(y_{k-1}^{state} \otimes I_{ne^2}\right) \end{aligned} \tag{H-81}$$

Inserting the previous equation in the source equation (H-78) and collecting terms yields

$$\begin{aligned} &\underbrace{\begin{bmatrix} f_{yst} & f_{ystate} + f_{y^+fwdendo}\alpha^{fwdendo} & f_{yfwd} \end{bmatrix}}_{\widetilde{A}} y_{k,0,0} \\ &= -\left\{ f_{x^3}\left(\gamma_1 \otimes x_0^{\otimes[2]}\right) + f_{x^2}\left(\gamma_{20} \otimes x_0\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) + f_{x^2}\left(\gamma_1 \otimes x_{0,0}\right) \right. \\ &\left. + f_{y^+fwdendo}\beta_{333}^{fwdendo} \begin{bmatrix} \alpha^{state} \otimes \beta_0^{state\otimes[2]} \\ \left(\beta_{20}^{state} \otimes \beta_0^{state}\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) \\ + \left(\beta_{00}^{state} \otimes \alpha^{state}\right) K_{ne^2,ns} \end{bmatrix} \right\} \left(y_{k-1}^{state} \otimes I_{ne^2}\right) \end{aligned} \tag{H-82}$$

Solving $y_{k,0,0}$ is then a standard linear problem, and it is obvious that $y_{k,0,0}$ takes the form

$$y_{k,0,0} = \beta_{300}\left(y_{k-1}^{state} \otimes I_{ne^2}\right) \tag{H-83}$$

where

$$\begin{aligned} \beta_{300} = -\widetilde{A}^{-1} &\left\{ f_{x^3}\left(\gamma_1 \otimes x_0^{\otimes[2]}\right) + f_{x^2}\left(\gamma_{20} \otimes x_0\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) + f_{x^2}\left(\gamma_1 \otimes x_{0,0}\right) \right. \\ &\left. + f_{y^+fwdendo}\beta_{333}^{fwdendo} \begin{bmatrix} \alpha^{state} \otimes \beta_0^{state\otimes[2]} \\ \left(\beta_{20}^{state} \otimes \beta_0^{state}\right)\left(I_{ns*ne^2} + I_{ns} \otimes K_{ne,ne}\right) \\ + \left(\beta_{00}^{state} \otimes \alpha^{state}\right) K_{ne^2,ns} \end{bmatrix} \right\} \end{aligned} \tag{H-84}$$

The two associated coefficients, i.e., $y_{0,j,0}$ and $y_{0,0,i}$ can be obtained by commuting $y_{k,0,0}$

$$\begin{aligned} y_{0,j,0} =& y_{k,0,0}\left(K_{ne,ne} \otimes I_{ne}\right) = \beta_{300}\left(y_{k-1}^{state} \otimes I_{ne^2}\right)\left(K_{ne,ne} \otimes I_{ne}\right) \\ =& \beta_{300}\left(K_{ns,ne} \otimes I_{ne}\right)\left(I_{ne} \otimes y_{k-1}^{state} \otimes I_{ne}\right) \end{aligned} \tag{H-85}$$

$$\begin{aligned} y_{0,0,i} =& y_{k,0,0}K_{ne,ne^2} = \beta_{300}\left(y_{k-1}^{state} \otimes I_{ne^2}\right)K_{ne,ne^2} \\ =& \beta_{300}K_{ns,ne^2}\left(I_{ne^2} \otimes y_{k-1}^{state}\right) \end{aligned} \tag{H-86}$$

therefore

(H-87) $$y_{0,j,0} = \beta_{030}\left(I_{ne} \otimes y_{j-1}^{state} \otimes I_{ne}\right)$$

(H-88) $$y_{0,0,i} = \beta_{003}\left(I_{ne^2} \otimes y_{i-1}^{state}\right)$$

where

(H-89) $$\beta_{030} = \beta_{300}\left(K_{ns,ne} \otimes I_{ne}\right)$$

(H-90) $$\beta_{003} = \beta_{300} K_{ns,ne^2}$$

H.1.4 Case 4: $k > 0$, $j > 0$ and $i = 0$

The source equation takes the form

(H-91) $$f_y y_{k,j,0} + f_{y^+ fwdendo} y_{k+1,j+1,1}^{fwdendo} + \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \begin{bmatrix} x_k \otimes x_j \otimes x_0 \\ x_{k,j} \otimes x_0 \\ \left(x_j \otimes x_{k,0}\right)\left(K_{ne,ne} \otimes I_{ne}\right) \\ x_k \otimes x_{j,0} \end{bmatrix} = 0$$

Using the constant matrices we defined and rearranging, the previous equation can be written as

(H-92)

$$f_y y_{k,j,0} + f_{y^+ fwdendo} y_{k+1,j+1,1}^{fwdendo} + \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \begin{bmatrix} \left(\gamma_1^{\otimes[2]} \otimes x_0\right)\left(y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes I_{ne}\right) \\ \left(\gamma_2 \otimes x_0\right)\left(S_{k,j} \otimes I_{ne}\right) \\ \left(\gamma_1 \otimes \gamma_{20}\right)\left(K_{ns,ns} \otimes I_{ne}\right)\left(y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes I_{ne}\right) \\ \left(\gamma_1 \otimes \gamma_{20}\right)\left(y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes I_{ne}\right) \end{bmatrix} = 0$$

collecting terms yields

(H-93)
$$\begin{aligned} & f_y y_{k,0,0} + f_{y^+ fwdendo} y_{k+1,1,1}^{fwdendo} \\ & + \begin{bmatrix} f_{x^3}\left(\gamma_1^{\otimes[2]} \otimes x_0\right) \\ + f_{x^2}\left(\gamma_1 \otimes \gamma_{20}\right)\left(I_{ns^2 * ne} + K_{ns,ns} \otimes I_{ne}\right) \end{bmatrix} \begin{bmatrix} f_{x^2}\left(\gamma_2 \otimes x_0\right) \end{bmatrix} \begin{bmatrix} y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes I_{ne} \\ S_{k,j} \otimes I_{ne} \end{bmatrix} \\ & = 0 \end{aligned}$$

Using (H-36), the partition of γ_2, the previous equation can be further reduced to

(H-94)
$$\begin{aligned} & f_y y_{k,0,0} + f_{y^+ fwdendo} y_{k+1,1,1}^{fwdendo} \\ & + \begin{bmatrix} f_{x^3}\left(\gamma_1^{\otimes[2]} \otimes x_0\right) \\ + f_{x^2}\left(\gamma_1 \otimes \gamma_{20}\right)\left(I_{ns^2 * ne} + K_{ns,ns} \otimes I_{ne}\right) \\ + f_{x^2}\left(\gamma_{22} \otimes x_0\right) \end{bmatrix} \begin{bmatrix} f_{x^2}\left(\gamma_1 \otimes x_0\right) \end{bmatrix} \begin{bmatrix} y_{k-1}^{state} \otimes y_{j-1}^{state} \otimes I_{ne} \\ y_{k-1,j-1} \otimes I_{ne} \end{bmatrix} \\ & = 0 \end{aligned}$$

Note that, from the solution (H-47)

$$y^{fwdendo}_{k+1,j+1,1} = \alpha^{fwdendo} y^{state}_{k,j,0} + \beta^{fwdendo}_{333} s_{k+1,j+1,1} \tag{H-95}$$

where

$$s_{k+1,j+1,1} = \begin{bmatrix} \left(\alpha^{state\otimes[2]} \otimes \beta^{state}_0\right)\left(y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes I_{ne}\right) \\ \left[\beta^{state}_{22} \otimes \beta^{state}_0 + \left(\beta^{state}_{20} \otimes \alpha^{state}\right)\left(I_{ns} \otimes K_{ne,ns} + K_{ns*ne,ns}\right)\right]\left(y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes I_{ne}\right) \\ + \left(\alpha^{state} \otimes \beta^{state}_0\right)\left(y^{state}_{k-1,j-1} \otimes I_{ne}\right) \end{bmatrix} \tag{H-96}$$

With β_{333} conformably partitioned, the last term in the solution (H-95) takes the form

(H-97)

$$\beta^{fwdendo}_{333} s_{k+1,j+1,1} = \begin{bmatrix} \beta^{fwdendo}_{333,1}\left(\alpha^{state\otimes[2]} \otimes \beta^{state}_0\right) \\ +\beta^{fwdendo}_{22}\left[\beta^{state}_{22} \otimes \beta^{state}_0 + \left(\beta^{state}_{20} \otimes \alpha^{state}\right)\left(I_{ns} \otimes K_{ne,ns} + K_{ns*ne,ns}\right)\right] \quad \beta^{fwdendo}_{22}\left(\alpha^{state} \otimes \beta^{state}_0\right) \end{bmatrix} \begin{bmatrix} y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes I_{ne} \\ y^{state}_{k-1,j-1} \otimes I_{ne} \end{bmatrix}$$

Inserting the previous equation in the source equation (H-94) and collecting terms yields

(H-98)

$$\underbrace{\begin{bmatrix} f_{yst} & f_{ystate} + f_{y^+ fwdendo}\alpha^{fwdendo} & f_{yfwd} \end{bmatrix}}_{\widetilde{A}} y_{k,j,0}$$

$$= -\left\{ \begin{bmatrix} f_{x^3}\left(\gamma_1^{\otimes[2]} \otimes x_0\right) \\ +f_{x^2}\left(\gamma_1 \otimes \gamma_{20}\right)\left(I_{ns^2*ne} + K_{ns,ns} \otimes I_{ne}\right) \\ +f_{x^2}\left(\gamma_{22} \otimes x_0\right) \end{bmatrix} \quad f_{x^2}\left(\gamma_1 \otimes x_0\right) \right.$$

$$\left. + \begin{bmatrix} f_{y^+ fwdendo}\left[\beta^{fwdendo}_{333,1}\left(\alpha^{state\otimes[2]} \otimes \beta^{state}_0\right) \right. \\ +\beta^{fwdendo}_{22}\left(\beta^{state}_{22} \otimes \beta^{state}_0\right) \\ \left. +\beta^{fwdendo}_{22}\left(\beta^{state}_{20} \otimes \alpha^{state}\right)\left(I_{ns} \otimes K_{ne,ns} + K_{ns*ne,ns}\right)\right] \end{bmatrix} \quad f_{y^+ fwdendo}\beta^{fwdendo}_{22}\left(\alpha^{state} \otimes \beta^{state}_0\right) \right\}$$

$$\begin{bmatrix} y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes I_{ne} \\ y^{state}_{k-1,j-1} \otimes I_{ne} \end{bmatrix}$$

Solving $y_{k,j,0}$ therefore is a standard linear problem and $y_{k,j,0}$ takes the form

$$y_{k,j,0} = \beta_{330} \begin{bmatrix} y^{state}_{k-1} \otimes y^{state}_{j-1} \otimes I_{ne} \\ y^{state}_{k-1,j-1} \otimes I_{ne} \end{bmatrix} \tag{H-99}$$

where
(H-100)

$$\beta_{330} = -\widetilde{A}^{-1}\left\{\begin{bmatrix} f_{x^3}\left(\gamma_1^{\otimes[2]}\otimes x_0\right) \\ +f_{x^2}\left(\gamma_1\otimes\gamma_{20}\right)\left(I_{ns^2*ne}+K_{ns,ns}\otimes I_{ne}\right) \\ +f_{x^2}\left(\gamma_{22}\otimes x_0\right) \end{bmatrix} \quad f_{x^2}\left(\gamma_1\otimes x_0\right)\right.$$
$$\left. + \begin{bmatrix} f_{y^+fwdendo}\left[\beta_{333,1}^{fwdendo}\left(\alpha^{state\otimes[2]}\otimes\beta_0^{state}\right)\right. \\ +\beta_{22}^{fwdendo}\left(\beta_{22}^{state}\otimes\beta_0^{state}\right) \\ \left.+\beta_{22}^{fwdendo}\left(\beta_{20}^{state}\otimes\alpha^{state}\right)\left(I_{ns}\otimes K_{ne,ns}+K_{ns*ne,ns}\right)\right] \end{bmatrix} \quad f_{y^+fwdendo}\beta_{22}^{fwdendo}\left(\alpha^{state}\otimes\beta_0^{state}\right)\right\}$$

With β_{330} conformably partitioned

(H-101)
$$\begin{aligned}\beta_{330,1} = -\widetilde{A}^{-1}\Bigg[& f_{x^3}\left(\gamma_1^{\otimes[2]}\otimes x_0\right)+f_{x^2}\left(\gamma_1\otimes\gamma_{20}\right)\left(I_{ns^2*ne}+K_{ns,ns}\otimes I_{ne}\right)+f_{x^2}\left(\gamma_{22}\otimes x_0\right) \\ & +f_{y^+fwdendo}\beta_{333,1}^{fwdendo}\left(\alpha^{state\otimes[2]}\otimes\beta_0^{state}\right) \\ & +f_{y^+fwdendo}\beta_{22}^{fwdendo}\left(\beta_{22}^{state}\otimes\beta_0^{state}\right) \\ & +f_{y^+fwdendo}\beta_{22}^{fwdendo}\left(\beta_{20}^{state}\otimes\alpha^{state}\right)\left(I_{ns}\otimes K_{ne,ns}+K_{ns*ne,ns}\right)\Bigg]\end{aligned}$$

(H-102)
$$\beta_{330,2} = -\widetilde{A}^{-1}\left[f_{x^2}\left(\gamma_1\otimes x_0\right)+f_{y^+fwdendo}\beta_{22}^{fwdendo}\left(\alpha^{state}\otimes\beta_0^{state}\right)\right]$$

noting that as the right hand side of (H-102) is identical to that of (G-21), we therefore have

(H-103)
$$\beta_{330,2} = \beta_{20}$$

so that only $\beta_{330,1}$ needs to be calculated. The two associated coefficients, i.e., $y_{k,0,i}$ and $y_{0,j,i}$ can be obtained by commuting $y_{k,j,0}$.

H.2 Solving the Unknown Coefficient $y_{\sigma^2,i}$

When $i > 0$, the source equation takes the form

(H-104)
$$\begin{aligned}& f_{y^-state}y_{\sigma^2,i-1}^{state}+f_y y_{\sigma^2,i}+f_{y^+fwdendo}y_{\sigma^2,i+1}^{fwdendo}+f_{x^2}\left(x_{\sigma^2}\otimes x_i\right) \\ & +\left[f_{x^3}\left(x_{\tilde{\varepsilon}}^{\otimes[2]}\otimes x_i\right)+2f_{x^2}\left(x_{\tilde{\varepsilon}}\otimes x_{\tilde{\varepsilon},i}\right)+f_{x^2}\left(x_{\tilde{\varepsilon},\tilde{\varepsilon}}\otimes x_i\right)+f_x x_{\tilde{\varepsilon}^2,i}\right]\left(E_t\varepsilon_{t+1}^{\otimes[2]}\otimes I_{ne}\right)=0\end{aligned}$$

With the following group of shifting matrices

$$x_{\tilde{\varepsilon}} = \begin{bmatrix} 0 \\ {\scriptstyle ns\times ne} \\ 0 \\ {\scriptstyle ny\times ne} \\ \beta_0^{fwdendo} \\ 0 \\ {\scriptstyle ne\times ne}\end{bmatrix},\ x_{\tilde{\varepsilon},\tilde{\varepsilon}} = \begin{bmatrix} 0 \\ {\scriptstyle ns\times ne^2} \\ 0 \\ {\scriptstyle ny\times ne^2} \\ \beta_{00}^{fwdendo} \\ 0 \\ {\scriptstyle ne\times ne^2}\end{bmatrix},\ x_{\sigma^2} = \begin{bmatrix} y_{\sigma^2}^{state} \\ y_{\sigma^2} \\ y_{\sigma^2}^{fwdendo} \\ 0 \\ {\scriptstyle ne\times 1}\end{bmatrix},\ \gamma_4 = \begin{bmatrix} 0 \\ {\scriptstyle ns\times nfwdendo} \\ 0 \\ {\scriptstyle ny\times nfwdendo} \\ I_{nfwdendo} \\ 0 \\ {\scriptstyle ne\times nfwdendo}\end{bmatrix}$$

$$x_{\tilde{\varepsilon},i} = \begin{bmatrix} 0 \\ ns\times ne^2 \\ 0 \\ ny\times ne^2 \\ \beta_{02}^{fwdendo}\left(I_{ne}\otimes y_i^{state}\right) \\ 0 \\ ne\times ne^2 \end{bmatrix} = \gamma_4\beta_{02}^{fwdendo}\left(I_{ne}\otimes\alpha^{state}\right)\left(I_{ne}\otimes y_{i-1}^{state}\right)$$

$$x_{\tilde{\varepsilon}^2,i} = \begin{bmatrix} 0 \\ ns\times ne^3 \\ 0 \\ ny\times ne^3 \\ \beta_{003}^{fwdendo}\left(I_{ne^2}\otimes y_i^{state}\right) \\ 0 \\ ne\times ne^3 \end{bmatrix} = \gamma_4\beta_{003}^{fwdendo}\left(I_{ne^2}\otimes\alpha^{state}\right)\left(I_{ne^2}\otimes y_{i-1}^{state}\right)$$

the source equation (H-104) can be written as (B-29)

The solution takes the form

$$y_{\sigma^2,i} = \alpha y_{\sigma^2,i-1}^{state} + \beta_{\sigma^2 1} y_{i-1}^{state} \tag{H-105}$$

which implies

$$y_{\sigma^2,i+1}^{fwdendo} = \alpha^{fwdendo}\alpha^{state} y_{\sigma^2,i-1}^{state} + \left(\alpha^{fwdendo}\beta_{\sigma^2 1}^{state} + \beta_{\sigma^2 1}^{fwdendo}\alpha^{state}\right) y_{i-1}^{state} \tag{H-106}$$

Inserting the previous equation in the source equation (B-29) and collecting terms yields

$$\underbrace{\left[f_{yst} \quad f_{ybwd} + f_{y^+ fwdendo}\alpha_{bwd}^{fwdendo}\right]}_{A}\beta_{\sigma^2 1}^{bwdendo} + \underbrace{\left[f_{ymix} + f_{y^+ fwdendo}\alpha_{both}^{fwdendo} \quad f_{yfwd}\right]}_{B}\beta_{\sigma^2 1}^{fwdendo}$$

$$+ \underbrace{\left[f_{y^+ mix} \quad f_{y^+}^{fwd}\right]}_{C}\beta_{\sigma^2 1}^{fwdendo}\alpha^{state} + D_{\sigma i} = 0 \tag{H-107}$$

where $D_{\sigma i}$ is a constant

$$D_{\sigma i} = f_{x^2}\left(x_{\sigma^2}\otimes\gamma_1\right) + \left[f_{x^3}\left(x_{\tilde{\varepsilon}}^{\otimes[2]}\otimes\gamma_1\right) + 2f_{x^2}\left(x_{\tilde{\varepsilon}}\otimes\left(\gamma_4\beta_{02}^{fwdendo}\right)\right)\left(I_{ne^2}\otimes\alpha^{state}\right)\right.$$

$$\left. + f_{x^2}\left(x_{\tilde{\varepsilon},\tilde{\varepsilon}}\otimes\gamma_1\right) + f_x\left(\gamma_4\beta_{003}^{fwdendo}\left(I_{ne^2}\otimes\alpha^{state}\right)\right)\right]\left(E_t\varepsilon_{t+1}^{\otimes[2]}\otimes I_{ns}\right) \tag{H-108}$$

Pre-multiplying the previous equation by the null space A^O yields the following Sylvester equation in $\beta_{\sigma^2 1}^{fwdendo}$

$$(A^O B)\beta_{\sigma^2 1}^{fwdendo} + (A^O C)\beta_{\sigma^2 1}^{fwdendo}\alpha^{state} + (A^O D_{\sigma i}) = 0 \tag{H-109}$$

With $\beta_{\sigma^2 1}^{fwdendo}$ in hand, solving $\beta_{\sigma^2 1}^{bwdendo}$ is a standard linear problem

$$\beta_{\sigma^2 1}^{bwdendo} = -pinv(A)\left[B\beta_{\sigma^2 1}^{fwdendo} + C\beta_{\sigma^2 1}^{fwdendo}\alpha^{state} + D_{\sigma i}\right] \tag{H-110}$$

We now move to the case $i = 0$. The source equation in this case takes the form

$$f_y y_{\sigma^2,0} + f_{y^+ fwdendo} y_{\sigma^2,1}^{fwdendo} + f_{x^2}\left(x_{\sigma^2}\otimes x_0\right)$$

(H-111) $$+\left[f_{x^3}\left(x_{\tilde{\varepsilon}}^{\otimes[2]}\otimes x_0\right)+f_{x^2}\left(x_{\tilde{\varepsilon},\tilde{\varepsilon}}\otimes x_0\right)+2f_{x^2}\left(x_{\tilde{\varepsilon}}\otimes x_{\tilde{\varepsilon},0}\right)+f_x x_{\tilde{\varepsilon}^2,0}\right]\left(E_t\varepsilon_{t+1}^{\otimes[2]}\otimes I_{ne}\right)=0$$

where

$$x_{\tilde{\varepsilon},0}=\begin{bmatrix}0\\ns\times ne^2\\0\\ny\times ne^2\\ \beta_{02}^{fwdendo}\left(I_{ne}\otimes\beta_0^{state}\right)\\0\\ne\times ne^2\end{bmatrix},\ x_{\tilde{\varepsilon}^2,0}=\begin{bmatrix}0\\ns\times ne^3\\0\\ny\times ne^3\\ \beta_{003}^{fwdendo}\left(I_{ne^2}\otimes\beta_0^{state}\right)\\0\\ne\times ne^3\end{bmatrix}$$

For notational consistency, we let

(H-112) $$y_{\sigma^2,0}=\beta_{\sigma^2 0}$$

and from the solution (H-105)

(H-113) $$y_{\sigma^2,1}^{fwdendo}=\alpha^{fwdendo}\beta_{\sigma^2 0}^{state}+\beta_{\sigma^2 1}^{fwdendo}\beta_0^{state}$$

inserting the last two equations in the source equation and collecting terms yields

(H-114) $$\left[f_{yst}\quad f_{ystate}+f_{y^+fwdendo}\alpha^{fwdendo}\quad f_{yfwd}\right]\beta_{\sigma^2 0}=-D_{\sigma 0}$$

where $D_{\sigma 0}$ is a constant

$$D_{\sigma 0}=\left[f_{x^3}\left(x_{\tilde{\varepsilon}}^{\otimes[2]}\otimes x_0\right)+f_{x^2}\left(x_{\tilde{\varepsilon},\tilde{\varepsilon}}\otimes x_0\right)+2f_{x^2}\left(x_{\tilde{\varepsilon}}\otimes x_{\tilde{\varepsilon},0}\right)+f_x x_{\tilde{\varepsilon}^2,0}\right]\left(E_t\varepsilon_{t+1}^{\otimes[2]}\otimes I_{ne}\right)$$

(H-115) $$+f_{y^+fwdendo}\beta_{\sigma^2 1}^{fwdendo}\beta_0^{state}$$

Solving for $\beta_{\sigma^2 0}$ therefore is a standard linear problem

(H-116) $$\beta_{\sigma^2 0}=-\left[f_{yst}\quad f_{ystate}+f_{y^+fwdendo}\alpha^{fwdendo}\quad f_{yfwd}\right]^{-1}D_{\sigma 0}$$

H.3 Solving for y_{σ^3}

The source equation takes the form

(H-117) $$y_{\sigma^3}=-(f_{y^-}+f_y+f_{y^+})^{-1}[f_{y^+3}y_0^{\otimes[3]}+3f_{y^+2}(y_{0,0}\otimes y_0)+f_{y^+}y_{0,0,0}]E_t\left(\varepsilon_{t+1}^{\otimes[3]}\right)$$

Making use of the special structure of f_{y^-}, f_y and f_{y^+} and collecting terms yields

$$y_{\sigma^3}=\left[f_{yst}\quad f_{y^-bwd}+f_{ybwd}\quad f_{y^-mix}+f_{ymix}+f_{y^+mix}\quad f_{yfwd}+f_{y^+fwd}\right]^{-1}$$

(H-118) $$[f_{y^+3}y_0^{\otimes[3]}+3f_{y^+2}(y_{0,0}\otimes y_0)+f_{y^+}y_{0,0,0}]E_t\left(\varepsilon_{t+1}^{\otimes[3]}\right)$$

Table 1: Stochastic Growth Model
Section 6.1
Baseline Parameter Values

Parameter	β	α	ρ	σ
Value	0.99	0.36	0.95	0.712%

See Hansen (1985).

Table 2: E_1 Performance of the Different Algorithms
Model of Section 6.1

	Baseline	$\chi = 3$	$\chi = 10$	$\chi = 25$	$\chi = 50$
First	5.90E-04	5.02E-03	4.55E-02	1.73E-01	2.50E-01
Second-Perturbation	1.13E-05	2.86E-04	8.07E-03	6.36E-02	NaN
Second-Kim et al	1.09E-05	2.76E-04	8.11E-03	7.28E-02	1.82E-01
Second-Den Haan and De Wind	1.09E-05	2.76E-04	8.11E-03	7.28E-02	1.82E-01
Second-NLMA	1.09E-05	2.76E-04	8.11E-03	7.28E-02	1.82E-01
Third-Perturbation	5.72E-08	4.99E-06	NaN	NaN	NaN
Third-Andreasen	1.79E-07	1.35E-05	1.29E-03	2.79E-02	1.26E-01
Third-Den Haan and De Wind	5.80E-04	4.88E-03	4.74E-02	2.43E-01	5.39E-01
Third-Fernandez-Villaverde et al	1.62E-06	4.35E-05	1.85E-03	3.18E-02	1.34E-01
Third-Juillard	1.33E-06	3.58E-05	1.60E-03	2.93E-02	1.27E-01
Third-NLMA	1.79E-07	1.35E-05	1.29E-03	2.79E-02	1.26E-01

Table 3: E_2 Performance of the Different Algorithms
Model of Section 6.1

	Baseline	$\chi = 3$	$\chi = 10$	$\chi = 25$	$\chi = 50$
First	4.43E-08	3.64E-06	5.24E-04	4.90E-02	2.68E+01
Second-Perturbation	3.02E-11	2.26E-08	3.94E-05	2.25E-02	NaN
Second-Kim et al	3.04E-11	2.24E-08	3.52E-05	1.83E-02	2.48E+01
Second-Den Haan and De Wind	3.04E-11	2.24E-08	3.52E-05	1.83E-02	2.48E+01
Second-NLMA	3.04E-11	2.24E-08	3.52E-05	1.83E-02	2.48E+01
Third-Perturbation	1.40E-15	2.27E-11	NaN	NaN	NaN
Third-Andreasen	1.66E-14	1.11E-10	1.92E-06	5.77E-03	2.19E+01
Third-Den Haan and De Wind	4.77E-08	3.54E-06	5.21E-04	5.58E-02	2.94E+01
Third-Fernandez-Villaverde et al	5.06E-13	4.70E-10	2.46E-06	6.00E-03	2.19E+01
Third-Juillard	3.54E-13	3.67E-10	2.41E-06	6.12E-03	2.20E+01
Third-NLMA	1.66E-14	1.11E-10	1.92E-06	5.77E-03	2.19E+01

Table 4: E_∞ Performance of the Different Algorithms
Model of Section 6.1

	Baseline	$\chi = 3$	$\chi = 10$	$\chi = 25$	$\chi = 50$
First	1.50E-02	1.06E-01	7.59E-01	2.34E+00	2.32E+00
Second-Perturbation	8.96E-04	1.99E-02	3.30E-01	1.14E+00	Inf
Second-Kim et al	9.07E-04	1.87E-02	4.55E-01	3.55E+00	7.63E+00
Second-Den Haan and De Wind	9.07E-04	1.87E-02	4.55E-01	3.55E+00	7.63E+00
Second-NLMA	9.07E-04	1.87E-02	4.55E-01	3.55E+00	7.63E+00
Third-Perturbation	1.60E-05	2.45E-03	Inf	Inf	Inf
Third-Andreasen	4.14E-05	2.50E-03	1.96E-01	3.94E+00	1.70E+01
Third-Den Haan and De Wind	2.98E-02	1.19E-01	1.63E+00	1.66E+01	7.00E+01
Third-Fernandez-Villaverde et al	1.29E-04	4.14E-03	1.83E-01	3.69E+00	1.57E+01
Third-Juillard	1.28E-04	4.12E-03	1.64E-01	3.56E+00	1.53E+01
Third-NLMA	4.14E-05	2.50E-03	1.96E-01	3.94E+00	1.70E+01

Table 5: Asset Pricing Model
Section 6.2
Baseline Parameter Values

Parameter	α	β	μ	ρ	σ
Value	-1.5	0.95	0.0179	-0.139	0.0348

See Burnside (1998) and Collard and Juillard (2001).

Table 6: E_1 Performance of the Different Algorithms
Model of Section 6.2

	Baseline	$\sigma = 1E-04$	$\sigma = 0.1$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
First	1.42E-02	1.18E-07	1.16E-01	1.85E-02	4.94E-02	1.91E-01
Second-Perturbation	1.92E-04	9.74E-11	1.29E-02	3.29E-04	2.96E-03	6.74E-02
Second-Kim et al	1.92E-04	9.74E-11	1.29E-02	3.29E-04	2.96E-03	6.74E-02
Second-Den Haan and De Wind	1.92E-04	9.74E-11	1.29E-02	3.29E-04	2.96E-03	6.74E-02
Second-NLMA	1.92E-04	9.74E-11	1.29E-02	3.29E-04	2.96E-03	6.74E-02
Third-Perturbation	1.91E-04	9.74E-11	1.29E-02	3.29E-04	2.62E-03	5.82E-02
Third-Andreasen	1.91E-04	9.74E-11	1.29E-02	3.29E-04	2.62E-03	5.82E-02
Third-Den Haan and De Wind	1.92E-04	1.93E-09	1.29E-02	3.29E-04	2.91E-03	8.29E-02
Third-Fernandez-Villaverde et al	1.91E-04	9.74E-11	1.29E-02	3.29E-04	2.62E-03	5.82E-02
Third-Juillard	1.91E-04	9.74E-11	1.29E-02	3.29E-04	2.62E-03	5.82E-02
Third-NLMA	1.91E-04	9.74E-11	1.29E-02	3.29E-04	2.62E-03	5.82E-02

	$\beta = 0.5$	$\beta = 0.99$	$\alpha = -10$	$\alpha = -5$	$\alpha = 0$	$\alpha = 0.5$
First	2.36E-03	2.92E-02	2.28E-01	9.06E-02	9.95E-11	2.85E-03
Second-Perturbation	1.28E-05	8.30E-04	4.65E-02	7.52E-03	9.95E-11	8.41E-06
Second-Kim et al	1.28E-05	8.30E-04	4.65E-02	7.52E-03	9.95E-11	8.41E-06
Second-Den Haan and De Wind	1.28E-05	8.30E-04	4.65E-02	7.52E-03	9.95E-11	8.41E-06
Second-NLMA	1.28E-05	8.30E-04	4.65E-02	7.52E-03	9.95E-11	8.41E-06
Third-Perturbation	3.78E-06	8.31E-04	4.66E-02	7.54E-03	9.95E-11	7.84E-06
Third-Andreasen	3.78E-06	8.31E-04	4.66E-02	7.54E-03	9.95E-11	7.84E-06
Third-Den Haan and De Wind	6.49E-06	8.32E-04	4.67E-02	7.54E-03	9.95E-11	8.10E-06
Third-Fernandez-Villaverde et al	3.78E-06	8.31E-04	4.66E-02	7.54E-03	9.95E-11	7.84E-06
Third-Juillard	3.78E-06	8.31E-04	4.66E-02	7.54E-03	9.95E-11	7.84E-06
Third-NLMA	3.78E-06	8.31E-04	4.66E-02	7.54E-03	9.95E-11	7.84E-06

See Burnside (1998) and Collard and Juillard (2001).

Table 7: E_2 Performance of the Different Algorithms
Model of Section 6.2

	Baseline	$\sigma = 1E-04$	$\sigma = 0.1$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
First	3.17E-02	2.10E-12	2.66E+00	5.36E-02	4.29E-01	1.73E+01
Second-Perturbation	7.05E-06	1.44E-18	3.37E-02	1.70E-05	2.38E-03	4.00E+00
Second-Kim et al	7.05E-06	1.44E-18	3.37E-02	1.70E-05	2.38E-03	4.00E+00
Second-Den Haan and De Wind	7.05E-06	1.44E-18	3.37E-02	1.70E-05	2.38E-03	4.00E+00
Second-NLMA	7.05E-06	1.44E-18	3.37E-02	1.70E-05	2.38E-03	4.00E+00
Third-Perturbation	5.74E-06	1.44E-18	3.29E-02	1.70E-05	1.21E-03	1.75E+00
Third-Andreasen	5.74E-06	1.44E-18	3.29E-02	1.70E-05	1.21E-03	1.75E+00
Third-Den Haan and De Wind	6.50E-06	5.92E-12	3.29E-02	1.70E-05	1.75E-03	4.58E+00
Third-Fernandez-Villaverde et al	5.74E-06	1.44E-18	3.29E-02	1.70E-05	1.21E-03	1.75E+00
Third-Juillard	5.74E-06	1.44E-18	3.29E-02	1.70E-05	1.21E-03	1.75E+00
Third-NLMA	5.74E-06	1.44E-18	3.29E-02	1.70E-05	1.21E-03	1.75E+00

	$\beta = 0.5$	$\beta = 0.99$	$\alpha = -10$	$\alpha = -5$	$\alpha = 0$	$\alpha = 0.5$
First	5.04E-06	6.43E-01	1.37E+00	4.43E-01	3.58E-18	4.36E-03
Second-Perturbation	2.38E-10	5.48E-04	5.95E-02	3.27E-03	3.58E-18	5.33E-08
Second-Kim et al	2.38E-10	5.48E-04	5.95E-02	3.27E-03	3.58E-18	5.33E-08
Second-Den Haan and De Wind	2.38E-10	5.48E-04	5.95E-02	3.27E-03	3.58E-18	5.33E-08
Second-NLMA	2.38E-10	5.48E-04	5.95E-02	3.27E-03	3.58E-18	5.33E-08
Third-Perturbation	1.30E-11	5.21E-04	5.71E-02	3.07E-03	3.58E-18	3.31E-08
Third-Andreasen	1.30E-11	5.21E-04	5.71E-02	3.07E-03	3.58E-18	3.31E-08
Third-Den Haan and De Wind	4.84E-09	5.25E-04	5.72E-02	3.07E-03	3.58E-18	3.15E-07
Third-Fernandez-Villaverde et al	1.30E-11	5.21E-04	5.71E-02	3.07E-03	3.58E-18	3.31E-08
Third-Juillard	1.30E-11	5.21E-04	5.71E-02	3.07E-03	3.58E-18	3.31E-08
Third-NLMA	1.30E-11	5.21E-04	5.71E-02	3.07E-03	3.58E-18	3.31E-08

See Burnside (1998) and Collard and Juillard (2001).

Table 8: E_∞ Performance of the Different Algorithms
Model of Section 6.2

	Baseline	$\sigma = 1E-04$	$\sigma = 0.1$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
First	1.48E-02	1.22E-07	1.21E-01	1.85E-02	7.10E-02	6.61E-01
Second-Perturbation	6.63E-04	1.09E-10	2.27E-02	3.29E-04	1.43E-02	5.15E-01
Second-Kim et al	6.63E-04	1.09E-10	2.27E-02	3.29E-04	1.43E-02	5.15E-01
Second-Den Haan and De Wind	6.63E-04	1.09E-10	2.27E-02	3.29E-04	1.43E-02	5.15E-01
Second-NLMA	6.63E-04	1.09E-10	2.27E-02	3.29E-04	1.43E-02	5.15E-01
Third-Perturbation	1.99E-04	9.74E-11	1.34E-02	3.29E-04	3.90E-03	3.89E-01
Third-Andreasen	1.99E-04	9.74E-11	1.34E-02	3.29E-04	3.90E-03	3.89E-01
Third-Den Haan and De Wind	1.96E-02	5.68E-05	5.40E-02	3.29E-04	1.18E-01	1.36E+00
Third-Fernandez-Villaverde et al	1.99E-04	9.74E-11	1.34E-02	3.29E-04	3.90E-03	3.89E-01
Third-Juillard	1.99E-04	9.74E-11	1.34E-02	3.29E-04	3.90E-03	3.89E-01
Third-NLMA	1.99E-04	9.74E-11	1.34E-02	3.29E-04	3.90E-03	3.89E-01

	$\beta = 0.5$	$\beta = 0.99$	$\alpha = -10$	$\alpha = -5$	$\alpha = 0$	$\alpha = 0.5$
First	2.97E-03	2.98E-02	2.48E-01	9.65E-02	9.95E-11	2.91E-03
Second-Perturbation	9.13E-05	1.78E-03	8.55E-02	1.67E-02	9.95E-11	3.94E-05
Second-Kim et al	9.13E-05	1.78E-03	8.55E-02	1.67E-02	9.95E-11	3.94E-05
Second-Den Haan and De Wind	9.13E-05	1.78E-03	8.55E-02	1.67E-02	9.95E-11	3.94E-05
Second-NLMA	9.13E-05	1.78E-03	8.55E-02	1.67E-02	9.95E-11	3.94E-05
Third-Perturbation	5.19E-06	8.48E-04	5.12E-02	8.07E-03	9.95E-11	8.02E-06
Third-Andreasen	5.19E-06	8.48E-04	5.12E-02	8.07E-03	9.95E-11	8.02E-06
Third-Den Haan and De Wind	2.09E-02	1.88E-02	1.36E-01	5.85E-02	9.95E-11	6.53E-03
Third-Fernandez-Villaverde et al	5.19E-06	8.48E-04	5.12E-02	8.07E-03	9.95E-11	8.02E-06
Third-Juillard	5.19E-06	8.48E-04	5.12E-02	8.07E-03	9.95E-11	8.02E-06
Third-NLMA	5.19E-06	8.48E-04	5.12E-02	8.07E-03	9.95E-11	8.02E-06

See Burnside (1998) and Collard and Juillard (2001).

Table 9: Recursive Utility and Stochastic Volatility
Section 6.3
Constant Parameter Values

Parameter	β	ν	ξ	δ	λ	ρ
Value	0.99	0.36218	0.3	0.0196	0.95	0.9

See Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012).

Table 10: Recursive Utility and
Stochastic Volatility
Section 6.3
Values for Different
Parameterizations

Parameter	γ	σ_z	η
Baseline Value	5	0.007	0.06
Extreme Value	40	0.021	0.1

See Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012).

Table 11: E_1 Performance of the Different Algorithms
Model of Section 6.3
Baseline Parameterization

	k_t	c_t	l_t	i_t	y_t	R_t^f	R_t
First	6.25E-03	3.81E-03	1.61E-03	1.32E-02	5.31E-03	1.10E-04	1.20E-04
Second-Perturbation	1.79E-03	3.06E-03	3.57E-03	2.78E-02	8.16E-03	2.20E-04	2.44E-04
Second-Kim et al	1.12E-03	6.16E-04	3.83E-04	2.64E-03	7.94E-04	2.21E-05	2.22E-05
Second-Den Haan and De Wind	1.18E-03	6.38E-04	3.90E-04	2.67E-03	7.95E-04	2.40E-05	2.43E-05
Second-NLMA	1.12E-03	6.16E-04	3.83E-04	2.64E-03	7.94E-04	2.21E-05	2.23E-05
Third-Perturbation	1.56E-03	3.01E-03	3.58E-03	2.79E-02	8.11E-03	2.21E-04	2.46E-04
Third-Andreasen	7.19E-04	3.42E-04	3.27E-04	2.03E-03	3.60E-04	1.63E-05	1.50E-05
Third-Den Haan and De Wind	5.58E-03	3.40E-03	1.47E-03	1.20E-02	4.74E-03	9.91E-05	1.08E-04
Third-Fernandez-Villaverde et al	7.37E-04	3.53E-04	3.29E-04	2.10E-03	3.83E-04	1.65E-05	1.52E-05
Third-Juillard	7.19E-04	3.42E-04	3.27E-04	2.04E-03	3.62E-04	1.64E-05	1.50E-05
Third-NLMA	7.20E-04	3.42E-04	3.27E-04	2.03E-03	3.60E-04	1.63E-05	1.50E-05

Table 12: E_2 Performance of the Different Algorithms
Model of Section 6.3
Baseline Parameterization

	k_t	c_t	l_t	i_t	y_t	R_t^f	R_t
First	6.68E-03	1.39E-05	6.28E-07	1.48E-05	4.65E-05	2.15E-08	2.60E-08
Second-Perturbation	5.57E-04	8.14E-06	2.45E-06	5.08E-05	9.78E-05	8.24E-08	1.02E-07
Second-Kim et al	3.02E-04	4.49E-07	4.94E-08	9.33E-07	1.54E-06	1.06E-09	1.10E-09
Second-Den Haan and De Wind	3.27E-04	4.75E-07	4.99E-08	9.33E-07	1.55E-06	1.16E-09	1.20E-09
Second-NLMA	3.02E-04	4.49E-07	4.94E-08	9.33E-07	1.54E-06	1.06E-09	1.10E-09
Third-Perturbation	3.97E-04	7.97E-06	2.47E-06	5.13E-05	9.83E-05	8.29E-08	1.03E-07
Third-Andreasen	1.12E-04	1.29E-07	3.21E-08	4.76E-07	3.10E-07	5.92E-10	4.76E-10
Third-Den Haan and De Wind	5.19E-03	1.09E-05	5.02E-07	1.17E-05	3.64E-05	1.72E-08	2.08E-08
Third-Fernandez-Villaverde et al	1.34E-04	1.47E-07	3.25E-08	5.35E-07	4.25E-07	5.85E-10	4.78E-10
Third-Juillard	1.12E-04	1.29E-07	3.21E-08	4.79E-07	3.15E-07	5.93E-10	4.76E-10
Third-NLMA	1.12E-04	1.29E-07	3.21E-08	4.75E-07	3.09E-07	5.91E-10	4.75E-10

Table 13: E_∞ Performance of the Different Algorithms
Model of Section 6.3
Baseline Parameterization

	k_t	c_t	l_t	i_t	y_t	R_t^f	R_t
First	5.61E-02	3.49E-02	2.80E-02	1.79E-01	5.66E-02	1.19E-03	1.40E-03
Second-Perturbation	1.87E-02	2.45E-02	3.42E-02	2.40E-01	7.36E-02	1.91E-03	2.26E-03
Second-Kim et al	2.00E-02	1.05E-02	1.08E-02	6.29E-02	1.84E-02	3.42E-04	4.15E-04
Second-Den Haan and De Wind	2.03E-02	1.07E-02	1.09E-02	6.27E-02	1.85E-02	3.33E-04	4.06E-04
Second-NLMA	2.00E-02	1.05E-02	1.08E-02	6.29E-02	1.84E-02	3.42E-04	4.15E-04
Third-Perturbation	1.43E-02	2.39E-02	3.39E-02	2.38E-01	7.17E-02	1.86E-03	2.20E-03
Third-Andreasen	1.09E-02	5.10E-03	7.07E-03	3.86E-02	8.49E-03	2.52E-04	2.30E-04
Third-Den Haan and De Wind	5.01E-02	3.25E-02	2.31E-02	1.45E-01	5.01E-02	9.07E-04	1.15E-03
Third-Fernandez-Villaverde et al	1.38E-02	6.03E-03	7.58E-03	4.63E-02	1.22E-02	2.56E-04	2.36E-04
Third-Juillard	1.10E-02	5.07E-03	7.11E-03	3.86E-02	8.65E-03	2.56E-04	2.29E-04
Third-NLMA	1.08E-02	5.12E-03	7.08E-03	3.88E-02	8.46E-03	2.52E-04	2.31E-04

Table 14: E_1 Performance of the Different Algorithms
Model of Section 6.3
Extreme Parameterization

	k_t	c_t	l_t	i_t	y_t	R_t^f	R_t
First	4.88E-02	2.19E-02	8.81E-03	6.93E-02	3.06E-02	8.55E-04	8.52E-04
Second-Perturbation	1.08E-02	1.05E-02	1.16E-02	8.11E-02	2.58E-02	7.28E-04	8.05E-04
Second-Kim et al	1.10E-02	5.06E-03	2.62E-03	1.93E-02	7.11E-03	1.66E-04	1.71E-04
Second-Den Haan and De Wind	3.41E-02	1.46E-02	5.33E-03	2.79E-02	9.83E-03	7.74E-04	7.91E-04
Second-NLMA	1.10E-02	5.08E-03	2.63E-03	1.94E-02	7.11E-03	1.67E-04	1.72E-04
Third-Perturbation	7.24E-03	9.37E-03	1.15E-02	8.15E-02	2.52E-02	6.94E-04	7.70E-04
Third-Andreasen	8.06E-03	2.66E-03	1.97E-03	1.24E-02	3.58E-03	1.22E-04	1.14E-04
Third-Den Haan and De Wind	4.58E-02	2.30E-02	8.99E-03	5.86E-02	2.55E-02	8.95E-04	9.30E-04
Third-Fernandez-Villaverde et al	8.78E-03	3.13E-03	2.01E-03	1.40E-02	4.38E-03	1.60E-04	1.53E-04
Third-Juillard	8.01E-03	2.63E-03	1.97E-03	1.25E-02	3.57E-03	1.23E-04	1.15E-04
Third-NLMA	8.15E-03	2.60E-03	1.84E-03	1.25E-02	3.59E-03	1.19E-04	1.10E-04

Table 15: E_2 Performance of the Different Algorithms
Model of Section 6.3
Extreme Parameterization

	k_t	c_t	l_t	i_t	y_t	R_t^f	R_t
First	4.66E-01	5.24E-04	1.90E-05	6.21E-04	1.91E-03	1.10E-06	1.14E-06
Second-Perturbation	3.22E-02	1.11E-04	2.80E-05	6.24E-04	1.18E-03	9.40E-07	1.16E-06
Second-Kim et al	4.27E-02	3.89E-05	2.41E-06	9.62E-05	1.86E-04	5.04E-08	5.72E-08
Second-Den Haan and De Wind	1.69E-01	1.61E-04	4.75E-06	1.00E-04	2.46E-04	6.50E-07	6.81E-07
Second-NLMA	4.28E-02	3.90E-05	2.42E-06	9.63E-05	1.86E-04	5.09E-08	5.78E-08
Third-Perturbation	1.09E-02	8.62E-05	2.77E-05	6.16E-04	1.13E-03	8.63E-07	1.08E-06
Third-Andreasen	1.74E-02	9.63E-06	1.13E-06	3.57E-05	4.50E-05	2.66E-08	2.28E-08
Third-Den Haan and De Wind	3.52E-01	4.91E-04	1.68E-05	3.90E-04	1.25E-03	1.15E-06	1.26E-06
Third-Fernandez-Villaverde et al	2.56E-02	1.47E-05	1.27E-06	5.63E-05	8.22E-05	4.42E-08	4.20E-08
Third-Juillard	1.74E-02	9.51E-06	1.14E-06	3.68E-05	4.64E-05	2.73E-08	2.35E-08
Third-NLMA	1.52E-02	8.57E-06	1.04E-06	3.07E-05	4.02E-05	2.29E-08	1.95E-08

Table 16: E_∞ Performance of the Different Algorithms
Model of Section 6.3
Extreme Parameterization

	k_t	c_t	l_t	i_t	y_t	R_t^f	R_t
First	3.31E-01	1.94E-01	1.44E-01	6.58E-01	3.26E-01	8.18E-03	1.15E-02
Second-Perturbation	1.49E-01	1.01E-01	1.47E-01	1.01E+00	2.48E-01	9.84E-03	1.04E-02
Second-Kim et al	1.93E-01	8.81E-02	6.92E-02	3.36E-01	1.83E-01	3.00E-03	5.48E-03
Second-Den Haan and De Wind	2.08E-01	9.68E-02	7.42E-02	3.53E-01	1.86E-01	2.33E-03	4.79E-03
Second-NLMA	1.93E-01	8.81E-02	6.92E-02	3.36E-01	1.83E-01	3.00E-03	5.48E-03
Third-Perturbation	8.53E-02	8.92E-02	1.47E-01	1.11E+00	3.03E-01	9.32E-03	9.96E-03
Third-Andreasen	1.25E-01	4.67E-02	3.93E-02	2.18E-01	1.06E-01	1.89E-03	2.50E-03
Third-Den Haan and De Wind	2.71E-01	1.83E-01	1.19E-01	8.03E-01	2.97E-01	5.90E-03	6.99E-03
Third-Fernandez-Villaverde et al	1.59E-01	5.74E-02	4.30E-02	2.80E-01	1.41E-01	1.90E-03	3.07E-03
Third-Juillard	1.26E-01	4.52E-02	3.99E-02	2.25E-01	1.10E-01	1.97E-03	2.67E-03
Third-NLMA	1.19E-01	4.42E-02	4.17E-02	2.21E-01	1.04E-01	1.82E-03	2.42E-03

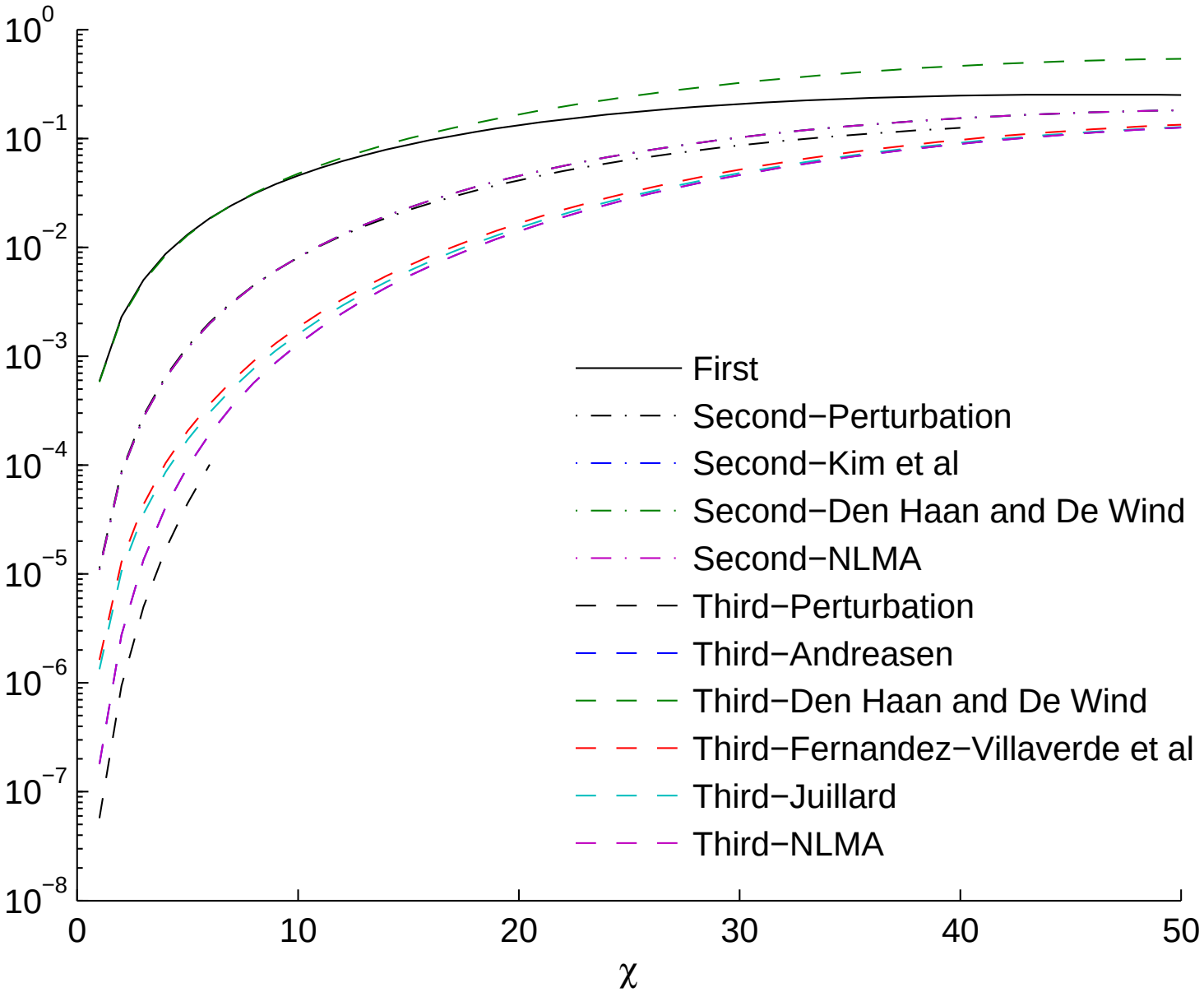

Figure 1: E_1 Performance of the Different Algorithms, Model of Section 6.1

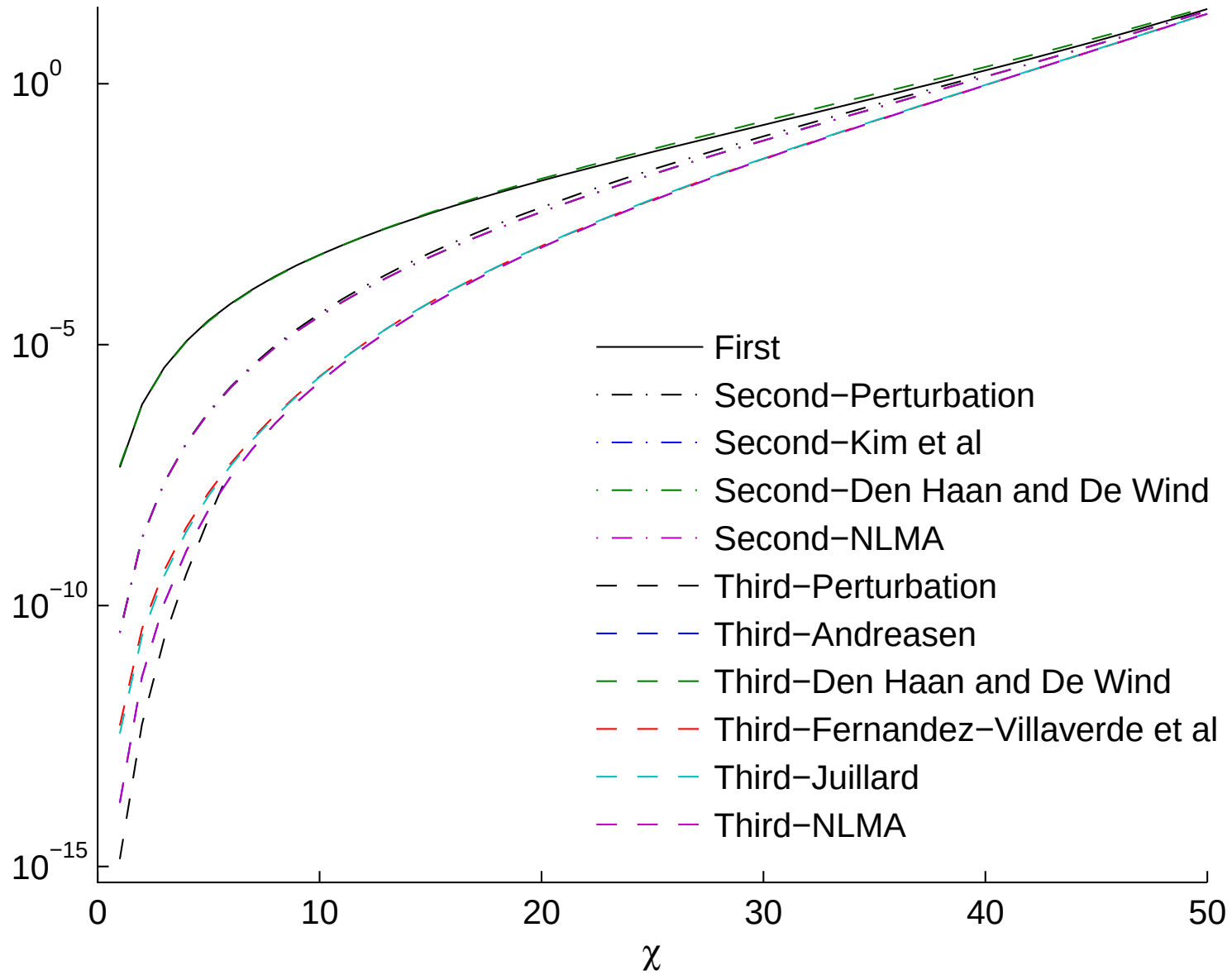

Figure 2: E_2 Performance of the Different Algorithms, Model of Section 6.1

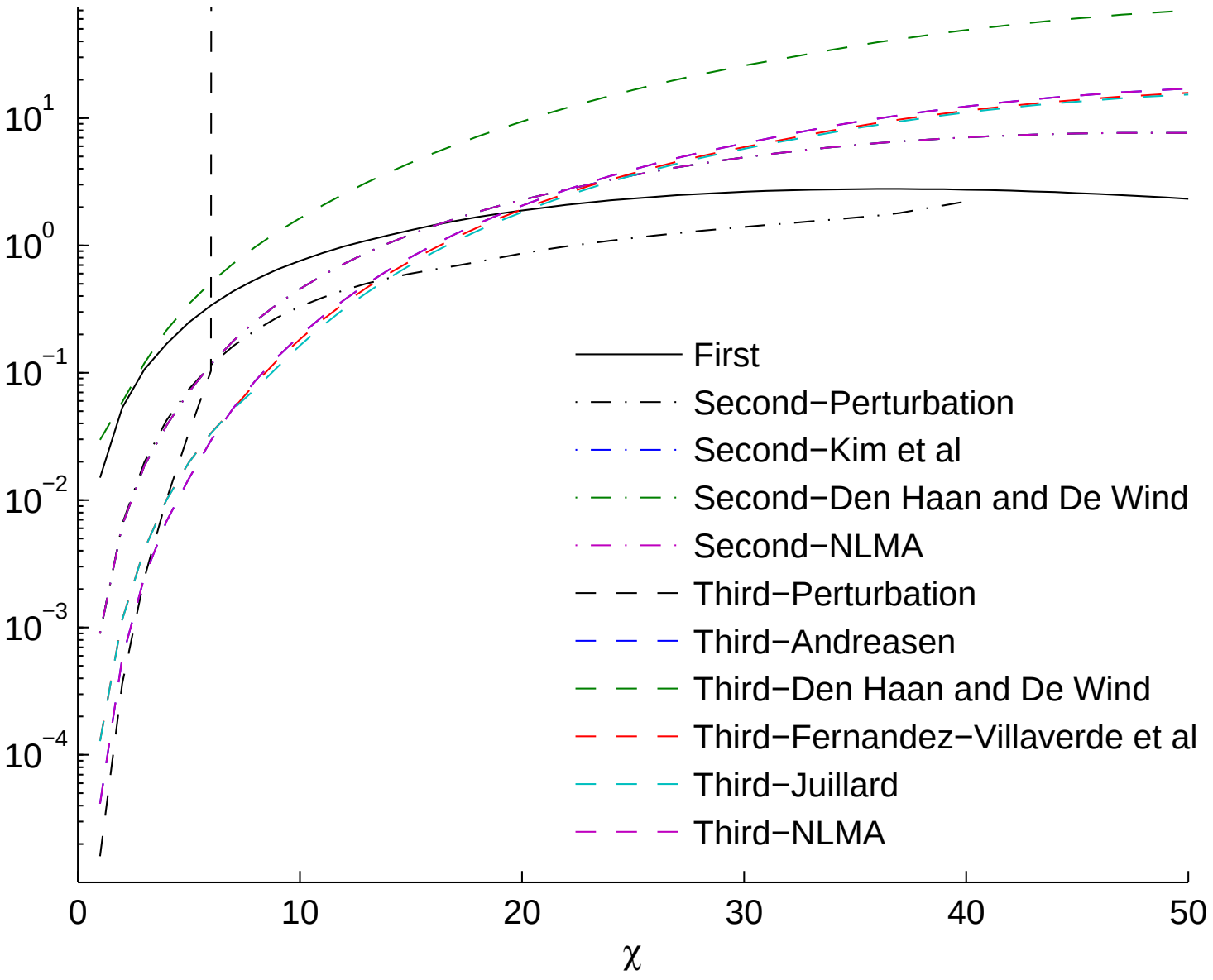

Figure 3: E_∞ Performance of the Different Algorithms, Model of Section 6.1

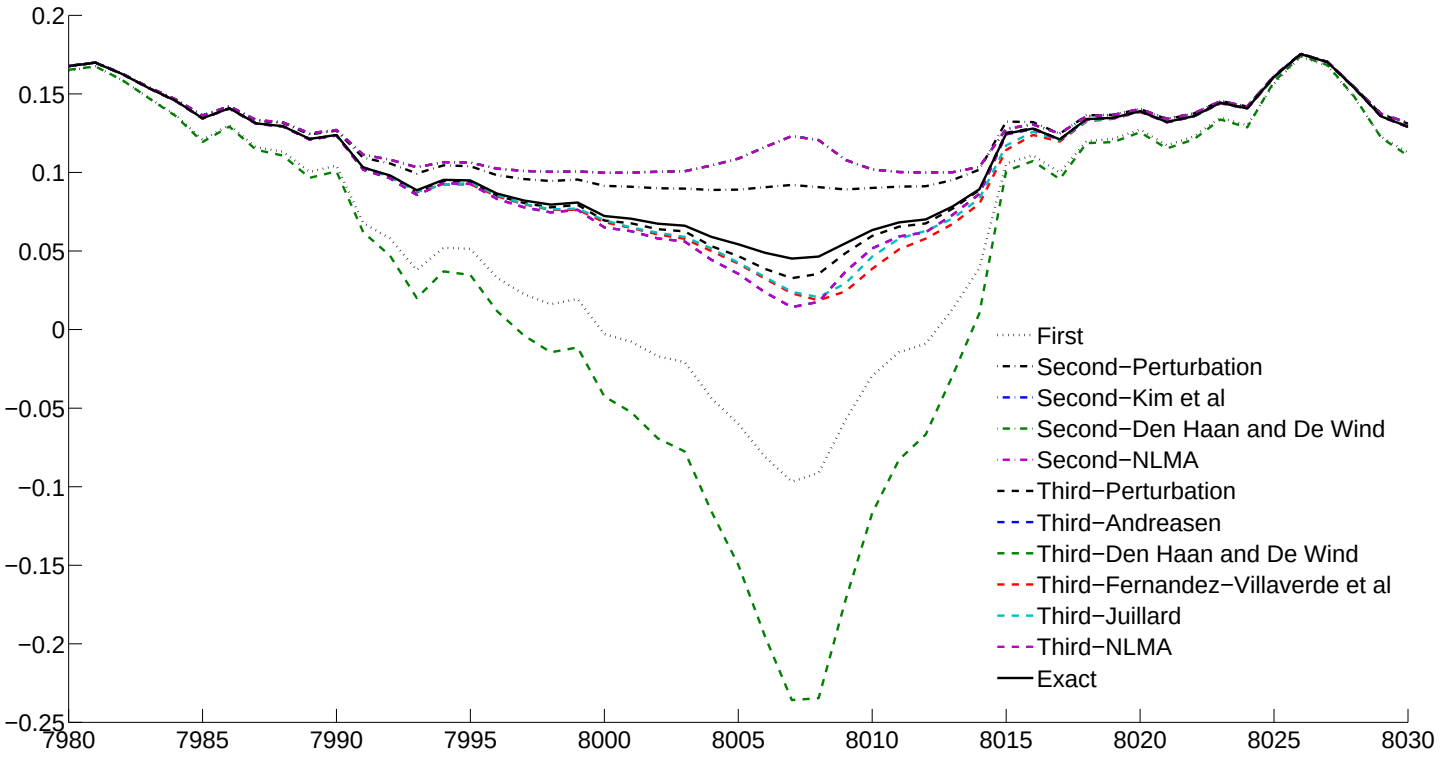

Figure 4: Simulation, Model of Section 6.1, $\chi = 10$

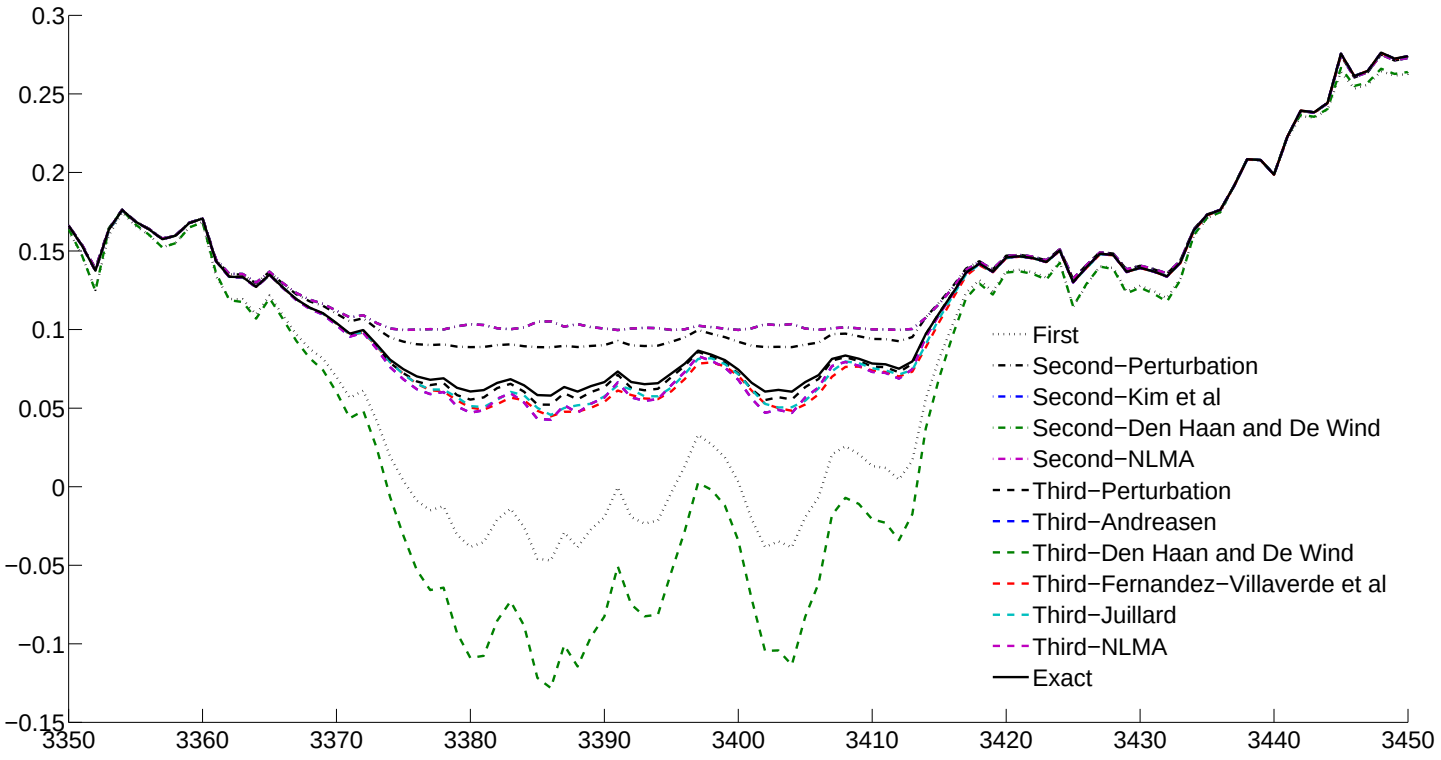

Figure 5: Simulation, Model of Section 6.1, $\chi = 10$

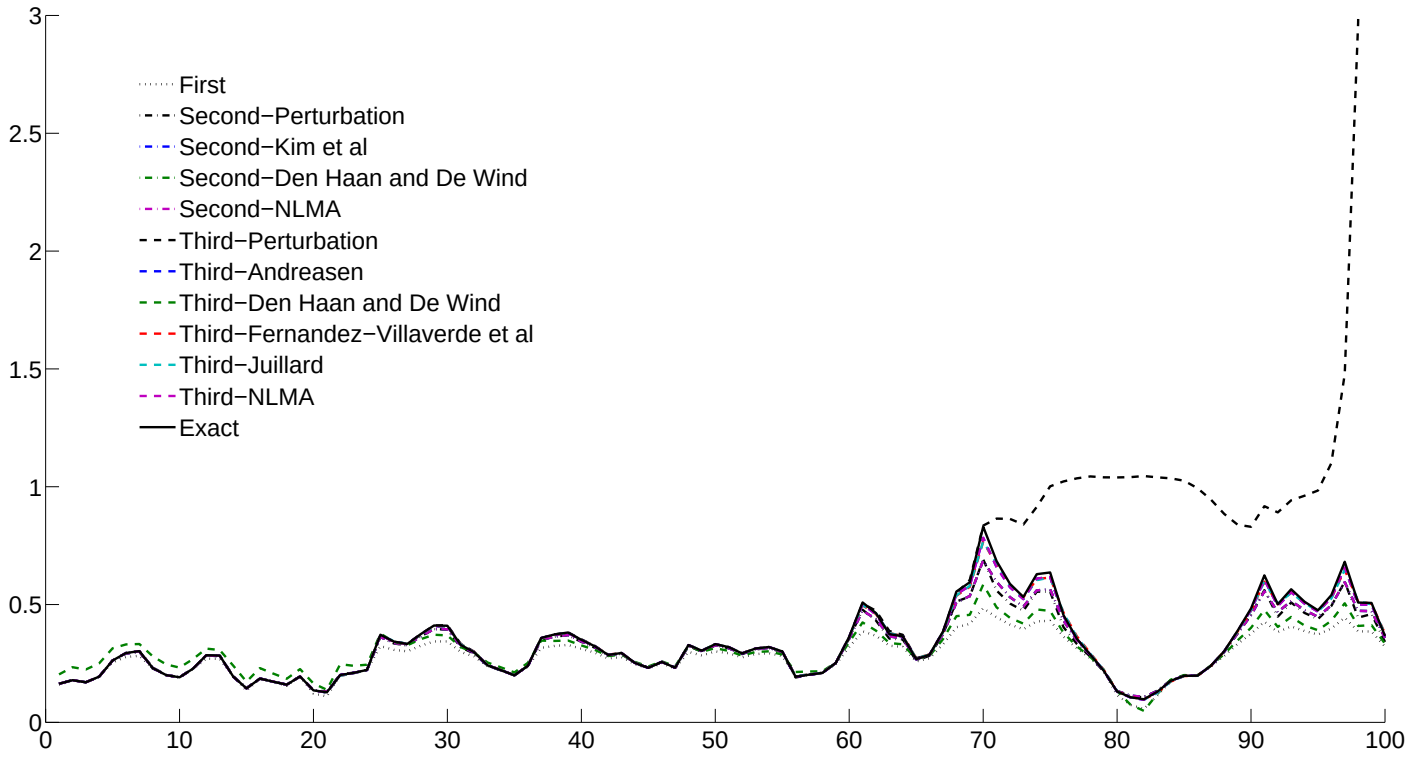

Figure 6: Simulation, Model of Section 6.1, $\chi = 31.2294$

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