SFB 649 Discussion Paper 2014-042

Beyond dimension two: A test for higher-order tail risk

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

http://sfb649.wiwi.hu-berlin.de
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
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Abstract

In practice, multivariate dependencies between extreme risks are often only assessed in a pairwise way. We propose a test to detect when tail dependence is truly high–dimensional and bivariate simplifications would produce misleading results. This occurs when a significant portion of the multivariate dependence structure in the tails is of higher dimension than two. Our test statistic is based on a decomposition of the stable tail dependence function, which is standard in extreme value theory for describing multivariate tail dependence. The asymptotic properties of the test are provided and a bootstrap based finite sample version of the test is suggested. A simulation study documents the good performance of the test for standard sample sizes. In an application to international government bonds, we detect a high tail–risk and low return situation during the last decade which can essentially be attributed to increased higher–order tail risk. We also illustrate the empirical consequences from ignoring higher-dimensional tail risk.

Keywords: decomposition of tail dependence, multivariate extreme values, stable tail dependence function, subsample bootstrap, tail correlation

JEL classification: C01, C46, C58

* M. Schienle thanks Deutsche Forschungsgemeinschaft for research support via grant SCHI-1127 and for support of various research stays at the SFB 649 "Economic Risk". J. Schaumburg thanks the European Union Seventh Framework Programme (FP7-SSH/2007-2013, grant agreement 320270 - SYRTO) for financial support.

1 Introduction

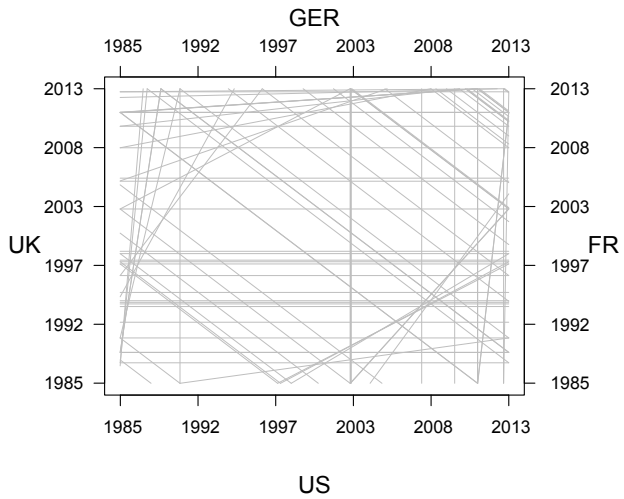
Studying extreme co-movements in multidimensional systems is a key concern in finance and insurance. However, tail dependence structures of multivariate distributions are often only treated in bivariate setups, see for instance Poon et al. (2004) and Klugman & Parsa (1999), and also de Haan & de Ronde (1998) and Ghosh (2010) for cases of extreme environmental and weather risks. This is due to the fact that in practice, bivariate models are easier tractable and computationally more appealing. But also from a theoretical point of view, statistical properties of a large group of estimators are only known up to dimension two (Coles et al. (1991), Joe et al. (1991), de Haan et al. (2008), Guillotte et al. (2011)). Though, for a variety of empirical settings, there are periods of time where a pairwise approach is too restrictive and produces misleading results. In particular, the recent financial crisis has shown that dramatic drops in the stock return of a single bank directly impact other

banks building up cascading effects which can pose a threat to the stability of the entire system. Sovereign bonds too have become increasingly interconnected with governments intervening with unprecedented bail-out measures impacting their own financial standing. In such situations, the most common bivariate measures for tail dependence, such as the tail dependence coefficient (see Straetmans et al. (2008), Poon et al. (2004), Hartmann et al. (2004)), bivariate copulas (see, e.g. Li (2013), Rodriguez (2007), and references therein), or simple product moment correlation coefficients and correlation matrices, fail to explicitly account for a large amount of the complex dependence structure among extreme risks in the system, leading to severe underestimations of the effects of extreme comovements (see also Embrechts (2009) and Mikosch (2006)). We therefore propose a test that indicates if the pairwise approach significantly underdiagnoses tail dependence in a d–dimensional random vector $\mathbb{X} = (X^{(1)}, ..., X^{(d)})'$, $d > 2$. The test is based on the stable tail dependence function (STDF) (Huang (1992), Einmahl et al. (2012), de Haan & Ferreira (2006), Resnick (1987)), which allows for a general and flexible semiparametric model of tail dependence and straightforward nonparametric estimation with a smaller risk of model misspecification than alternative parametric models, e.g. of copula type. Furthermore, its statistical properties are well understood beyond dimension two for all $\mathbb{X}$ whose distribution function lies in the domain of attraction of a multivariate extreme value distribution (Einmahl et al. (2012)). And its conservative definition of multivariate extreme events fits the needs of (financial) risk management (Segers (2012)).

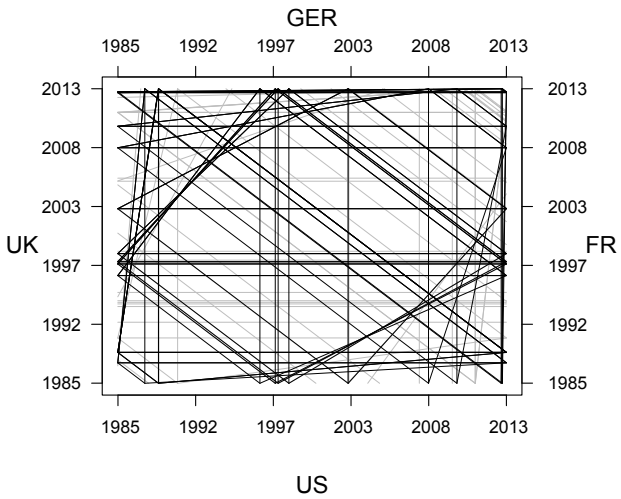
The main idea of the test is to decompose the STDF for $\mathbb{X}$ into probabilities of univariate extreme events, STDF's of all possible bivariate pairs within $\mathbb{X}$, and a remainder term capturing extreme events in dimensions three to d. We refer to the latter as higher order tail dependencies (HOTD's). If the remainder term is not significantly different from zero, we conclude that tail dependence in $\mathbb{X}$ is purely bivariate, since tail dependence in dimension d can be captured by bivariate tails. However, if we reject the null hypothesis that HOTD's have no influence, ignoring high–dimensional joint extreme events leads to an underestimation of the risks implied by joint extremes in dimension three and higher. The asymptotic properties of the test statistic are derived and a bootstrap implementation scheme for finite samples is proposed. A simulation study with standard multivariate risk structures documents the test's good finite sample properties. In a detailed empirical application, we discuss the multivariate tail properties of daily return data of government bond index prices of the United States, the United Kingdom, Germany, and France. Figure 1 shows bivariate, two– and four–dimensional joint extreme events of the four international government bond index returns.

Figure 1: Two–dimensional versus higher–dimensional extremes of international government bond index price returns. For a sample size of 2026 observations the figure displays occurrences of two–dimensional (grey lines) and higher–dimensional (black lines) extremal co–exceedances of marginal empirical 97.03% quantiles. Different extremal quantiles as a extremal threshold can be chosen as well. The sample exhibits 94 bivariate joint extremes. Within the set of bivariate extremes, there are 28 joint three– and four joint four–dimensional extremes.

(a) Joint pairwise extremes



(b) Joint three-dimensional extremes



(c) Joint four-dimensional extremes

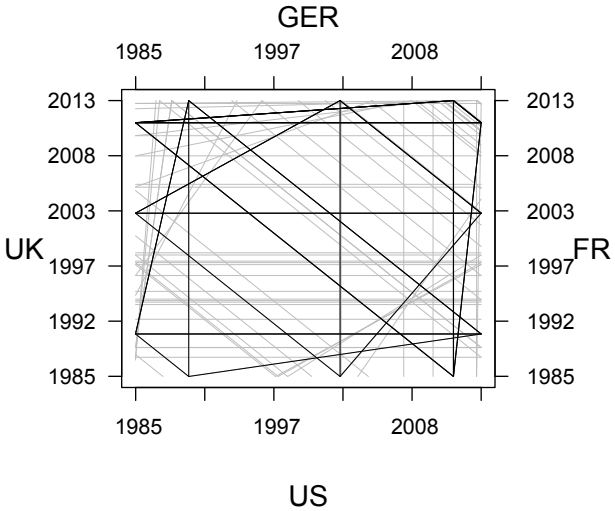


Figure 1 illustrates that a pairwise analysis of a multivariate tail can only lead to a partial assessment of the likeliness of higher–dimensional extreme events, missing out on the contribution of joint extremes of three or more marginals in 28 of 94 cases. Thus in almost 30% of the events detected as extreme by the six bivariate tails, the true level of riskiness would be underestimated if it was not accounted for the fact that the six bivariate tails can also constitute higher–dimensional joint extremes. Hence, portfolio holders that solely balance their portfolios based on bivariate tail measures would neglect the hazardous threats of potential high–dimensional extremes. We illustrate that they could therefore face substantially higher risks than intended or allowed, e.g. by solvency regulations for large bond holding insurance companies. Using the new test, we show in a rolling window analysis that HOTD's have built up during the last decade in both the left (losses) as well as the right (gains) tail of international bond index returns. Thus, a significant portion of total multivariate tail dependence is due to HOTD's. Considering the steady decrease of bond yields, we find a high tail risk and low return situation on international bond markets. Furthermore, disregarding HOTD's when estimating extreme quantile

regions produces estimates up to 60% smaller in terms of volumes compared to estimated quantile regions that do account for HOTD's. Neglecting the potential for joint extreme events in dimension three or higher can thus amount to a severe underestimation of multivariate tail behavior.

The rest of the paper is organized as follows. Section 2.1 discusses necessary concepts from multivariate extreme value theory. Section 2.2 introduces and formalizes test idea, test asymptotics and finite sample implementation. Finite sample properties are studied in section 3. Section 4 studies HOTD's in international bond indices. Section 4.3 illustrates, using an extreme quantile region approach, how severely multivariate tail risk is underestimated by a purely bivariate approach. Section 5 concludes. The appendix contains supplementary simulation and theoretical results.

2 Econometric methodology

2.1 Multivariate dependence in extreme tails

For our analysis of extreme risks, we use techniques from multivariate extreme value theory which we introduce and motivate in the following.

Denote by $\mathbb{X} := (X^{(1)}, ..., X^{(d)})'$ a d–dimensional random vector with continuous joint cumulative distribution function (cdf) $F_{\mathbb{X}}(\mathbb{x}), \mathbb{x} := (x^{(1)}, ..., x^{(d)})'$. Its univariate marginals are denoted by $F_j(x^{(j)}), j = 1, ..., d$. Suppose we observe a sample of n i.i.d. draws from the random vector $\mathbb{X}$, collected in the $(n \times d)$ sample matrix $\mathbf{X} = (\mathbf{X}_n^{(1)}, ..., \mathbf{X}_n^{(d)})$ with $\mathbf{X}_n^{(j)} = (X_1^{(j)}, ..., X_n^{(j)})'$, $j = 1, ..., d$. We write $\max(\mathbf{X}_n^{(j)}) = \max(X_1^{(j)}, ..., X_n^{(j)})$ for the sample maximum of margin j. For each marginal, we assume that there exist normalizing constants $a_n^{(j)} \in \mathbb{R}_+, b_n^{(j)} \in \mathbb{R}, j = 1, ..., d$, and a limiting distribution $G_{\mathbb{X}}(\mathbb{x})$, such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\max(\mathbf{X}_n^{(1)}) - b_n^{(1)}}{a_n^{(1)}} \leq x^{(1)}, ..., \frac{\max(\mathbf{X}_n^{(d)}) - b_n^{(d)}}{a_n^{(d)}} \leq x^{(d)}\right) = G_{\mathbb{X}}(\mathbb{x}), \tag{1}$$

for all continuity points of $G_{\mathbb{X}}(\mathbb{x})$. Then $G_{\mathbb{X}}(\mathbb{x})$ is a multivariate extreme value distribution, and $F_{\mathbb{X}}(\mathbb{x})$ is said to be in the domain of attraction of $G_{\mathbb{X}}(\mathbb{x})$, which is denoted by $F_{\mathbb{X}} \in D(G_{\mathbb{X}})$, see de Haan & Ferreira (2006) and Resnick (1987). Necessary and sufficient conditions for $F_{\mathbb{X}} \in D(G_{\mathbb{X}})$ can be found in Haan & Resnick (1977), Beirlant et al. (2004, p. 287), de Haan & Ferreira (2006), and Resnick (1987). For this theoretical section, we assume that they are fulfilled. In the application, we will illustrate that the assumptions are met. In general, closed-form expressions for $G_{\mathbb{X}}(\mathbb{x})$ do not exist. Equation 1 can be written as

$$\lim_{n\to\infty} F_{\mathbb{X}}^n(a_n^{(1)}x^{(1)} + b_n^{(1)}, ..., a_n^{(d)}x^{(d)} + b_n^{(d)}) = G_{\mathbb{X}}(\mathbb{x}), \tag{2}$$

implying that the univariate marginals converge individually to one–dimensional extreme value distributions $G_j(x^{(j)})$, which have the standard Fisher–Tippett form (Fréchet (1927), Fisher & Tippett (1928), Gnedenko (1943))

$$\lim_{n\to\infty} F_j^n(a_n^{(j)}x^{(j)} + b_n^{(j)}) = G_j(x^{(j)}) = \exp\left(-(1 + \gamma_j x^{(j)})^{-1/\gamma_j}\right), j = 1, ..., d, \tag{3}$$

where γ_j denotes the tail index of margin j. An equivalent formulation of relation 2 amounts to the concept of the stable tail dependence function (STDF) of $\mathbb{X}$, denoted by $\ell_{\mathbb{X}}(\mathbb{x})$ (Huang (1992), Einmahl et al. (2012)). Equivalent characterizations of $G_{\mathbb{X}}(\mathbb{x})$ can also be obtained via the spectral measure and the exponent measure (de Haan & Ferreira 2006, chapter 6) but are less intuitive in interpretation and decomposition. The STDF $\ell(\mathbb{x}) : \mathbb{R}^d \mapsto \mathbb{R}_+$ is defined as

$$\ell_{\mathbb{X}}(\mathbb{x}) = -\log G_{\mathbb{X}}\left(\frac{x^{(1)^{-\gamma_1}} - 1}{\gamma_1}, ..., \frac{x^{(d)^{-\gamma_d}} - 1}{\gamma_d}\right).$$

The STDF maps univariate tails to their joint limit distribution, and therefore describes the complete dependence structure of the tails of the univariate marginals. As one can express $\ell_{\mathbb{X}}(\mathbb{x})$ as

$$\ell_{\mathbb{X}}(\mathbb{x}) = \lim_{t\to 0} t^{-1}\mathbb{P}\Big(\bigcup_{i=1}^{d}\{F_i^{-1}(1-tx^{(i)}) \leq X^{(i)}\}\Big), t \in \mathbb{R}_+, \tag{4}$$

the STDF is an asymptotic measure which can be interpreted as the scaled asymptotic probability that at least one element of $\mathbb{X}$ exceeds an extreme quantile, that is, $X^{(i)}$ exceeds $F_i^{-1}(1-tx^{(i)})$, as $t \to 0$. From this representation a direct nonparametric estimate of the STDF can be derived. It is also immediate how to decompose $\ell_{\mathbb{X}}(\mathbb{x})$ into component STDF's of dimensions lower than d.

There is a rich statistical literature on general properties of the STDF and its estimators (e.g. Huang (1992), Einmahl et al. (2012), Einmahl et al. (2006), Dietrich et al. (2003), Drees et al. (2006)). It is thus a standard result that the STDF is a convex function, and homogeneous of degree one, i.e. $\ell_{\mathbb{X}}(\lambda\mathbb{x}) = \lambda\ell_{\mathbb{X}}(\mathbb{x})$ for $\lambda \in \mathbb{R}$. Importantly, $\ell_{\mathbb{X}}(\mathbb{x}) \in [\max(\mathbb{x}), \mathbb{x}\mathbf{1} := \sum_{i=1}^{d} x^{(i)}]$, where the lower (upper) bound is attained if $\mathbb{X}$ is perfectly tail dependent (independent), that is, extremes of univariate marginals always (never) occur simultaneously (de Haan & Ferreira (2006), Beirlant et al. (2004)). Tail (in)dependent is often also denoted as asymptotically (in)dependent. Numerical values of $\ell_{\mathbb{X}}(\mathbb{x})$ close to $\max(\mathbb{x})$ indicate that tails of $\mathbb{X}$ are strongly interconnected. Values of $\ell_{\mathbb{X}}(\mathbb{x})$ close to $\mathbb{x}\mathbf{1}$ mark the opposite. In practice, perfect tail dependence is rare.

It is important to see the relation but also the difference of the STDF to the so–called tail copula (TC) which is another closely related metric for tail dependence. The TC is defined as

$$R_{\mathbb{X}}(\mathbb{x}) = \lim_{t\to 0} t^{-1}\mathbb{P}\Big(\bigcap_{i=1}^{d}\{F_i^{-1}(1-tx^{(i)}) \leq X^{(i)}\}\Big),$$

which considers joint exceedances to characterize tail dependence, see Schmidt & Stadtmüller (2006). Sibuya (1960), Joe (1997) and Coles et al. (1998) introduce the concept of bivariate tail dependence in terms of the tail dependence coefficient, which corresponds to the bivariate TC at the point $\mathbb{x} = (1, 1)$. Roughly speaking, it describes the tendency of two random variables to jointly exceed a high threshold. In two dimensions, there is a one-to-one mapping between the TC and the STDF. Due to the lack of natural ordering in higher dimensions, the definition of a multivariate extreme event depends on the research objective. There are several reasons why we prefer the concept of a STDF and to the one of a TC for our purpose: Firstly, the TC captures only (the most extreme) part of the multivariate tail dependence in dimensions $d > 2$, while the STDF completely describes it (see subsection 2.2 for the relationship between the two). Secondly, a practical issue for large d is that joint d–dimensional exceedances are rarely observed in finite samples. Unless a sample contains an observation with all marginals being extreme, the TC indicates tail independence. That is, the TC only considers the most extreme event when all marginals are simultaneously extreme, and disregards more likely tail events. On the other hand, the STDF incorporates events in which a single component of $\mathbb{X}$ becomes extreme, and hence finite samples provide more relevant observations. Segers (2012) paraphrase the interpretation of $\ell_{\mathbb{X}}(\mathbb{x})$ as "trouble in the air", whereas $R_{\mathbb{X}}(\mathbb{x})$ only considers events extreme when "the sky is falling". The STDF is therefore an important ingredient for a conservative risk monitoring approach, in the sense of leading to tight univariate thresholds for a given level of multivariate joint risk in a multivariate value-at-risk setting. See subsection 4.3 for details.

2.2 A new test for higher-order extreme tail dependence

We aim to reveal the share which HOTD's contribute to overall tail dependence. Hence, we decompose the STDF for dimension d into TC's for dimensions two to d. In dimension $d = 2$, from equation 4 we have that $\ell_{\mathbb{X}}(\mathbb{x})$ is the limiting probability of a union of two events; since $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ for some events A and B. Therefore, we have $\ell_{\mathbb{X}}(x^{(1)}, x^{(2)}) = x^{(1)} + x^{(2)} - R_{\mathbb{X}}(x^{(1)}, x^{(2)})$. For similar decompositions in arbitrary dimension $2 < d < \infty$, additional notation is required. Define $\mathcal{I}_{(r)}^{(d)}$ as the combination set of all $\binom{d}{r}$ combinations

of r elements of the index set $I := \{1, 2, ..., d\}$ and let $b \in \mathcal{I}^{(d)}_{(r)}$. Furthermore, $\mathbb{x}^{(b)} := (x^{(j)}, j \in b \subset I)$, and let $R_b(\mathbb{x}^{(b)})$ and $\ell_b(\mathbb{x}^{(b)})$ be the TC and the STDF, respectively, of subvector $\mathbb{X}^{(b)}$. For example, if $d = 3$, then $\mathcal{I}^{(d=3)}_{(r=2)} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $i = \{1, 2\}, \{1, 3\}$ or $\{2, 3\}$, and if $b = \{1, 2\}$, it is $\mathbb{x}^{(b)} = (x^{(1)}, x^{(2)})$.
In $\mathbb{R}^{2<d<\infty}$, using the inclusion–exclusion principle,

$$\ell_{\mathbb{X}}(\mathbb{x}) = \sum_{i=1}^{d} x^{(i)} - \sum_{i \in \mathcal{I}^{(d)}_{(2)}} R_i(\mathbb{x}^{(i)}) + \underbrace{\sum_{i \in \mathcal{I}^{(d)}_{(3)}} R_i(\mathbb{x}^{(i)}) - ... + (-1)^{d+1} R_{\mathbb{X}}(\mathbb{x})}_{=:\mathcal{A}}, \tag{5}$$

where $\mathcal{A}$ denotes the portion HOTD's contribute to "global" tail dependence in $\mathbb{X}$. Provided global tail dependence is only caused by bivariate extreme events, i.e. by the first two terms of equation 5, $\mathcal{A}$ equals zero. In this case, higher dimensional joint extremes are irrelevant. When substituting $R(x^{(i)}, x^{(j)}) = x^{(i)} + x^{(j)} - \ell(x^{(i)}, x^{(j)}), i \neq j, i, j \in I$, equation 5 yields

$$\ell_{\mathbb{X}}(\mathbb{x}) = (2 - d) \sum_{i=1}^{d} x^{(i)} + \sum_{i \in \mathcal{I}^{(d)}_{(2)}} \ell_i(\mathbb{x}^{(i)}) + \mathcal{A}, \tag{6}$$

which decomposes global tail dependence into asymptotic probabilities for univariate extremes and STDF's for any bivariate combination and HOTD's.
With equation 6 we can test whether extreme events in dimensions larger than two have a statistically significant impact, that is, if two–dimensional tails explain tail dependence in dimension $d > 2$ sufficiently well. Formally, if $\mathcal{A} = 0$, then

$$\Delta := \ell_{\mathbb{X}}(\mathbb{x}) - (2 - d) \sum_{i=1}^{d} x^{(i)} - \sum_{i \in \mathcal{I}^{(d)}_{(2)}} \ell_i(\mathbb{x}^{(i)}) = 0. \tag{7}$$

Thus in this case, bivariate extreme relations are sufficient for capturing the full global tail dependence as the impact of higher–order tail dependencies is negligible. Hence, the null hypothesis that bivariate tails "are enough", is

$$H_0 : \Delta = 0. \tag{8}$$

For Δ substantially deviating from zero, the null is rejected. With $\mathbb{x} = \mathbf{1}$, one can show that $\Delta \in [0, \sum_{i=1}^{d-2} i], d > 2$. The following corollary clarifies that testing for $\Delta = 0$ is not equivalent to testing whether $\mathbb{X}$ is tail independent as proposed e.g. in Draisma et al. (2004). Thus, there exist multivariate distributions which are globally tail dependent, but have $\Delta = 0$. Hence their global tail dependence is exclusively caused by bivariate tails.

Corollary 2.1
If $\mathbb{X}$ is tail independent, that is if all bivariate tails of $\mathbb{X}$ are tail independent, then $\Delta = 0$. The reverse does not hold.

This can be easily shown constructively from the the family of distributions which we use in the simulation setting.

In order to apply the test, one has to estimate the STDF for $\mathbb{X}$, $\ell_{\mathbb{X}}(\mathbb{x})$, and the STDF's for all possible bivariate pairs, $\ell_b(\mathbb{x}^{(b)}), b \in \mathcal{I}^{(d)}_{(2)}$. Let $X^{(i)}_{n:m}$ denote the m–th largest order statistic of margin $X^{(i)}$, and let $1(C)$ be the indicator function for event C. In equation 4, replacing the running variable t by k/n and the extreme quantiles $F_i^{-1}(1 - tx^{(i)})$ by $X^{(i)}_{n:n+0.5-kx^{(i)}}$ we use the following nonparametric estimator for the STDF (see

Huang (1992) and Einmahl et al. (2012))

$$\hat{\ell}_{\mathbb{X}}(\mathbb{x}) = \frac{1}{k}\sum_{i=1}^{n} 1\Big\{\bigcup_{j=1}^{d}\{X_i^{(j)} \geq X_{n:n+0.5-kx^{(j)}}^{(j)}\}\Big\}, n \to \infty, k \to \infty, \frac{k}{n} \to 0. \tag{9}$$

Assuming that $\mathbb{X}$ is in the domain of attraction of a multivariate extreme value distribution, that some technical conditions on k are fulfilled, and that continuous partial derivatives of ℓ with respect to $x^{(j)} > 0, j = 1, ..., d$, exist, they show that the empirical process $\sqrt{k}(\hat{\ell}_{\mathbb{X}}(\mathbb{x}) - \ell_{\mathbb{X}}(\mathbb{x}))$ converges weakly to a sum of zero mean Brownian motions with given covariance structure. If $\mathbb{X}$ is asymptotically independent, $\hat{\ell}_{\mathbb{X}}(\mathbb{x})$ is still asymptotically normal but with zero variance (Hüsler & Li 2009).
Without loss of generality, we fix $\mathbb{x} = \mathbf{1} \in \mathbb{R}^d$, which is standard in the applied MEVT literature, see e.g. Hartmann et al. (2004). In this case for each marginal, the threshold equals $X_{n:n+1/2-k}^{(i)}$, and one observes k exceedances for each margin. Asymptotics of $\hat{\ell}(\mathbb{x} = \mathbf{1})$ simplify to

$$\sqrt{k}\left(\hat{\ell}_{\mathbb{X}}(\mathbb{x} = \mathbf{1}) - \ell_{\mathbb{X}}(\mathbb{x} = \mathbf{1})\right) \xrightarrow{d} N(0, \sigma^2_{\hat{\ell}_{\mathbb{X}}}),$$

where closed forms of $\sigma^2_{\hat{\ell}_{\mathbb{X}}}$ can be reconstructed from theorem 4.6 in Einmahl et al. (2012). Plugging $\hat{\ell}_{\mathbb{X}}$ into Δ yields the empirical test statistic

$$\widehat{\Delta} := \hat{\ell}_{\mathbb{X}}(\mathbb{x} = \mathbf{1}) - 2d + d^2 - \sum_{i \in \mathcal{I}_{(2)}^{(d)}} \hat{\ell}_i\left(\mathbb{x}^{(i)} = (1,1)\right). \tag{10}$$

These considerations lead us to the distribution of the test statistic, which is given next.

Proposition 2.2 *Let the Assumptions 1 as listed in detail in section 6.4 hold. Moreover, assume that within* $\mathbb{X}$ *at least one pair* $(X^{(i)}, X^{(j)}), i \neq j$, *is asymptotically dependent. Then,*

$$\sqrt{k}(\widehat{\Delta} - \Delta) \xrightarrow{d} N(0, \sigma^2_{\widehat{\Delta}}),$$

where $\sigma^2_{\widehat{\Delta}}$ *is the sum of all entries of the covariance matrix of*

$$\big(\hat{\ell}_{\mathbb{X}}(\mathbb{x} = \mathbf{1}), \{\hat{\ell}_i(\mathbb{x}^{(i)} = (1,1))\}_{i \in \mathcal{I}_{(2)}^{(d)}}\big).$$

The exact derivation can be found in section 6.4. Assumptions 1 consist of three components: A1(i) requires β-regular varying tails of the underlying distribution, which is a standard assumption in the extreme value literature. The parameter β can be estimated in practice e.g. via a Hill-type estimator as we illustrate in the empirical study for bonds, see section 4. A1(ii) provides a restriction on the choice of k dependent on n and β. The smoothness assumptions A1(iii) are only necessary for a simplified form of the obtained variance. In practice, it can be very restrictive in particular if there is asymptotic tail dependence. For the finite sample adjusted version of the test below, it is therefore important that A1(iii) is not necessary.

From proposition 2.2, it follows the asymptotic size of the test decreases when $\mathbb{X}$ becomes very high–dimensional. This follows from the nonparametric nature of $\hat{\ell}(\mathbb{x})$ which causes the variance of $\widehat{\Delta}$ to increase with growing dimension. Larger sample sizes can antagonize this effect. Our simulation study illustrates that the test and its finite sample version still perform well in the empirical size for dimension $d = 7$ and standard sample sizes. In contrast, if $\mathbb{X}$ exhibits tail dependence in dimension d, it necessarily exhibits tail dependence in dimensions $3 \leq g < d$. Thus, the asymptotic power of the test can increase with larger dimensions. See subsection 3.2 for details together with results on the empirical power in the simulation settings.

If is of practical interest, the test can be readily adapted to dimensions greater or equal than $d = 3$. It can thus detect if joint extremes in dimensions larger than three are of importance as the test statistic increases in d as long as joint extremes occur in dimension d.

2.3 Finite sample refinements of the test

Although it is possible to derive the explicit form and calculate empirical versions of the asymptotic variance of the test statistic, we prefer to bootstrap it. The reason is that direct estimation of $\sigma^2_{\widehat{\Delta}}$ requires the estimation of partial derivatives of the STDF and of covariances between STDF's. In principle, a weighted least squares based estimator for such partial derivatives of the STDF exists, but its statistical properties have only been established for dimension $d = 2$ so far (see Peng & Qi 2006). Moreover, in our asymptotic dependent case of interest, smoothness assumptions for the simplified partial derivative form of the variance can often not be met by construction, thus estimating from it is actually not admissible (Bücher & Dette 2013). But bootstrapping $\sigma^2_{\widehat{\Delta}}$ works under milder conditions, in particular if $\mathbb{X}$ exhibits asymptotic dependence (Bücher & Dette 2013). As we want to bootstrap extremal observations, we do not resample from the full sample, but only from a subsample (Politis & Romano (1994)). Otherwise an asymptotically vanishing bias term of $\widehat{\Delta}$, inherited by an asymptotically vanishing bias term of $\hat{\ell}_{\mathbb{X}}$ (Huang (1992)), might distort the bootstrap distribution. Peng (2010) propose a similar approach and successfully employ a subsample size of $n^{0.95}$. Qi (2008), El-Nouty & Guillou (2000), Danielsson et al. (2001), Geluk & de Haan (2002) generally document the benefits of subsampling for pointwise extreme value statistics.

We construct rejection regions for the test from the asymptotically normal distribution of $\widehat{\Delta}$ with the resampled form of the variance. In summarizing, we proceed in six steps for obtaining the test decision:

(1) Determine k^* for $\widehat{\Delta}$ from the sample of $\mathbf{X}$.

(2) Estimate $\ell_{\mathbb{X}}(k^*)$, and any $\ell_i(k^*), i \in \mathcal{I}^{(d)}_{(2)}$, to determine the full sample test statistic $\widehat{\Delta}(k^*)$ from $\mathbf{X}$.

(3) Draw at least $B = 500$ bootstrap samples with replacement from $\mathbf{X}$ with sample size $n^* = n^{0.95}$ and denote the resulting bootstrap sample by $\mathbf{X}^*_1, ..., \mathbf{X}^*_B$.

(4) For $j = 1, ..., B$, estimate $\widehat{\Delta}(k^*)$ from the bootstrap sample $\mathbf{X}^*_1, ..., \mathbf{X}^*_B$, yielding B bootstrapped estimates $\widehat{\Delta}(k^*)_1, ..., \widehat{\Delta}(k^*)_B$.

(5) Estimate $\sigma^2_{\widehat{\Delta}}$ from the bootstrapped estimates in the previous step by its empirical analogue.

(6) On a $1 - \alpha$ confidence level reject $H_0 : \Delta = 0$ if $0 < \widehat{\Delta}(k^*) + z^{\alpha}\hat{\sigma}_{\widehat{\Delta}}$, where z^{α} denotes the α quantile of the standard normal distribution.

A theoretically optimal, data-driven choice of the tuning parameter k should balance the bias–variance trade–off that is immanent to the estimation of $\ell_{\mathbb{X}}(\mathbb{x})$. Finding such a solution and deriving its optimality properties is highly non-standard even in the univariate case and is thus beyond the scope of this paper. In our simulations we choose k randomly from a suitable interval in order to minimize possible distortions from a poorly chosen k. In the application, we estimate Δ over a grid of different value for k and calculate the median over this grid of estimates. Details can be found in the respective sections. For alternative, univariate, purely data-driven procedures for determining k, we refer to Frahm et al. (2005) and Schmidt & Stadtmüller (2006).

For time-series data, serial dependence in the mean, volatility and higher moments can be addressed by implementing blocked version of the bootstrap, see, for instance, Straetmans et al. (2008). According to Hall et al. (1995), a suitable choice of block size is $n^{1/3}$. Applying a GARCH filter to the data before analyzing its tail dependence is also an alternative to handle conditionally heteroskedastic data, as it is done in Poon et al. (2004).

3 Simulation study

3.1 Size and power

In this subsection, we evaluate the empirical size and power of the test in finite samples.

We simulate from two types of distribution families with various subspecifications for which we know whether the null is true. We choose the multivariate t-distribution and (max) factor models which are commonly used in financial risk management (McNeil et al. (2006), Fama & French (1992)). Note that the multivariate t-distribution exhibits the same tail dependence structure as the t-copula (Nikoloulopoulos et al. (2009)). For testing we employ the test calibration routine introduced in subsection 2.3. Simulations are repeated $S = 1000$ times.

The exact different model specifications are listed in tables 1 and 2 in section 6.1. Model dimensions range from $d = 3$ to $d = 7$. For the test decision, considering larger dimensions is often not necessary, as higher order tail dependencies of moderate order up to seven are sufficient for concluding that bivariate methods ignore a significant portion of multivariate tail dependence. Our results indicate, however, that is crucial to consider the test not only for three or four dimensional submodels but also in moderately large dimensions up to $d = 7$ in order to check whether absent high-dimensional joint extremes affect test size and power.
A d–dimensional random vector $\mathbb{X}$ is multivariate t distributed with ν degrees of freedom, mean vector $\boldsymbol{\mu}$, and correlation matrix $\mathcal{C}$ if its probability density function is

$$f_{\mathbb{X}}(\mathbb{x}) = \frac{\Gamma\left((\nu+d)/2\right)}{(\pi\nu)^{d/2}\,\Gamma(\nu/2)\,|\mathcal{C}|^{1/2}}\left(1+\frac{1}{\nu}(\mathbb{x}-\boldsymbol{\mu})'\mathcal{C}^{-1}(\mathbb{x}-\boldsymbol{\mu})\right)^{-\frac{\nu+d}{2}},$$

e.g. Kotz & Nadarajah (2004). We set any off–diagonal elements of $\mathcal{C}$ equal to ρ, and $\boldsymbol{\mu} = (0, ..., 0)$. Exploiting results from Nikoloulopoulos et al. (2009, theorem 2.3), one can show, for all multivariate t distributions considered here, that the theoretical values of Δ are larger than zero. Δ increases both when the degrees of freedom parameter decreases, and when pairwise correlation increases. It is zero if the correlation parameter equals -1.
Generally in finance, factor models are applied when assets returns $X^{(j)}$ depend on common factors in a linear fashion. Max factor models assume $X^{(j)}$ can be modeled as the maximum of the factors times a parameter a_{mj}, the so called factor loadings. Both models have the same tail dependence structure (Einmahl et al. (2012)). Let $\mathbb{Z} := (Z^{(1)}, ..., Z^{(r)})$ be a random vector of independent Fréchet(ν :=1) random variables. A max factor model for $\mathbb{X} = (X^{(1)}, ..., X^{(d)})'$ is then defined by

$$X^{(j)} := \max(a_{1j}Z_i^{(1)}, ..., a_{rj}Z_i^{(r)}), j = 1, ..., d,$$

with $\sum_{j=1}^{d} a_{mj} = 1, a_{mj} \geq 0$. The loading matrix $A := (a_{mj})$ governs the dependence between the tails of $\mathbb{X}$. If at most two entries within a row of A are larger than zero, tail dependence is only caused by pairs, i.e. the null is true. However, $\mathbb{X}$ exhibits global tail dependence. In case of

$$A = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

the STDF for bivariate pairs are $\ell_{12}(1,1) = \ell_{13}(1,1) = \ell_{23}(1,1) = 1.5$ since

$$\mathbb{P}(X^{(1)} \text{ and } X^{(2)} \text{ extreme}) = \mathbb{P}(X^{(1)} \text{ and } X^{(3)} \text{ extreme}) = \mathbb{P}(X^{(2)} \text{ and } X^{(3)} \text{ extreme}) = 0.5$$

while $\mathbb{P}(X^{(1)} \text{ and } X^{(2)} \text{ and } X^{(3)} \text{ extreme}) = 0$. Einmahl et al. (2012) show that $\ell_{\mathbb{X}}(\mathbb{x}) = \sum_{i=1}^{r} \max_{j=1,...,d}(a_{ij}/(\sum_{i=1}^{r} a_{ij}))x^{(j)}$, and thus $\ell_{123}(1,1,1) = 1.5$ and $\Delta = 0$. If more than two elements within a row are not zero, tail dependence is also caused by higher–dimensional joint extremes, thus the null would be false. Employed calibrations of and notation for A can be found in section 6.1. For example, loading

matrix A_{3a} leads to a max–factor model with a five–dimensional $\mathbb{X}$ where global tail dependence is solely caused by pairwise tail dependence, and thus the null is true. In contrast, in case of A_{3b} global tail dependence is caused by all possible bivariate pairs and, additionally, by tail dependence between $X^{(1)}, X^{(2)}$ and $X^{(3)}$ (first row). Thus, the null is false.

In extreme value statistics, simulation results are usually sensitive to the choice of the tuning parameter k. Large values of k cause a systematic bias of $\widehat{\Delta}$, whereas small k's induce a large variance. We strive to minimize the effect of threshold choice on simulation results which is achieved by modeling k as a random variable that is uniformly distributed on the interval $[0.01n, cn^{1/2}], c \in [1, 2]$. We found the best choices for c concerning test size are 1.7 in $d = 3$, 1.5 in $d = 4$, 1.4 in $d = 5$, 1.2 in $d = 6$, 1.1 in $d = 7$. Intuitively, since $\hat{\ell}_{\mathbb{X}}(\mathbb{x})$ incorporates union–wise exceedances, a large d causes the asymptotic bias of $\widehat{\Delta}$ to increase as $\hat{\ell}_{\mathbb{X}}(\mathbb{x})$ more likely includes non–extreme data points when d is high. Thus, we let c decrease in d at the cost of a higher variance. We find that the power of the test is hardly affected by c. Without the proposed calibration of c, however, the test's size tends to be too small, i.e. the test is too conservative. The lower bound $0.01n$ prevents the variance of $\widehat{\Delta}$ to become too large. Sample sizes are given by $n_i = \exp(j), j = 5 + 0.5i, i = 1, ..., 7$. Thus for the upper limit of the domain of k_i, $\log(n_i/k_i) \in \mathcal{O}(j), j \to \infty$, and $k_i \to \infty$ as $n_i \to \infty$. Hence, $\hat{\ell}_{\mathbb{X}}(\mathbb{x})$ builds upon an intermediate sequence of the sample such that the assumptions for the asymptotics of $\hat{\ell}_{\mathbb{X}}(\mathbb{x})$ are not violated. The resulting sample sizes are of reasonable size when interest is in analyzing daily financial data ($n_1 = 149, n_2 = 245, n_3 = 404, n_4 = 666, n_5 = 1097, n_6 = 1809, n_7 = 2981$).

Figures 2 and 3 illustrate the empirical rejection rates of the test at a nominal level of 5% in each of the model classes. The exact rejection rates are reported in tables 5 and 4 in the appendix. For max-factor models, we find that the empirical power of the test is generally high independent of the dimension. For small amounts of deviations from the null, however, the test requires sample sizes greater than 1000 in order to yield satisfactory power, which appears adequate given the difficulty of the problem. Generally, empirical power converges to one for larger sample sizes. And empirical sizes appear close to the nominal level and plateaus at 5% for n sufficiently large. Depending on the exact model specification, this can occur already for the smallest sample size of 150. While empirical power is robust against the choice of k, we found that empirical sizes vary substantially with altering the domain of k. Generally, the test rejects too often if k tends to be small, thus empirical sizes are substantially smaller than nominal levels. In financial risk management, however, one would prefer a test with a larger false positive rate over a test that tends to falsely overlook prevalent HOTD's. Still, as we model k as a uniform random variable defined over an interval of reasonable possible values, reported sizes can be considered as valid.

As expected, for the multivariate t distribution, high dimension and a strong linear dependence, fatness of univariate tails cause rejection rates to be high. For all specifications empirical power monotonously converges to one as n increases. In high dimensions, the test successfully detects that low linear dependence and moderately fat tails add up to larger HOTD's; also in such cases power is close to one (e.g. $T_{1,df16}, T_{2,df16}$). For perfectly tail dependent DGP's (A_{1b}, A_{2c}, A_{4d}), and strongly linear dependent and fat–tailed DGP's (e.g $T_{4,df4}, T_{5,df4}, T_{4,df8}, T_{5,df8}$), power is nearly always one, irrespective of dimension. Conditional on the choice of k, empirical sizes oscillate around α ($A_{1a}, A_{2a}, A_{3a}, A_{4a}, A_{5a}$) independent of dimension which contradicts intuition as in higher dimension one expects nonparametric methods' performance to worsen as efficiency suffers; for small sample sizes, empirical size is slightly larger than α. Also, if Δ appears to be close to zero (e.g. $T_{1,df16}$ with $d = 3$), sample size does play a major role since the test exhibits a good power only for at least $n_5 = 1097$. We assume that test size in particular might be improved by a refined determination of k.

Thus our simulation results underline that bivariate tail measures would severely underestimate multivariate tail dependence if potential HOTD's were ignored. In case of the multivariate t distribution rejection rates are high even if the degrees of freedom are above 20, and even when linear dependence is not too far from zero.

Figure 2: Empirical rejection rates for max factor model specifications at nominal significance level of 5%. Depicted values correspond to empirical test size in figure (a) and to test power elsewhere. The dimension of $\mathbb{X}$ increases from A_1 ($d = 3$) to A_5 ($d = 7$). See Appendix 6.1 for details.

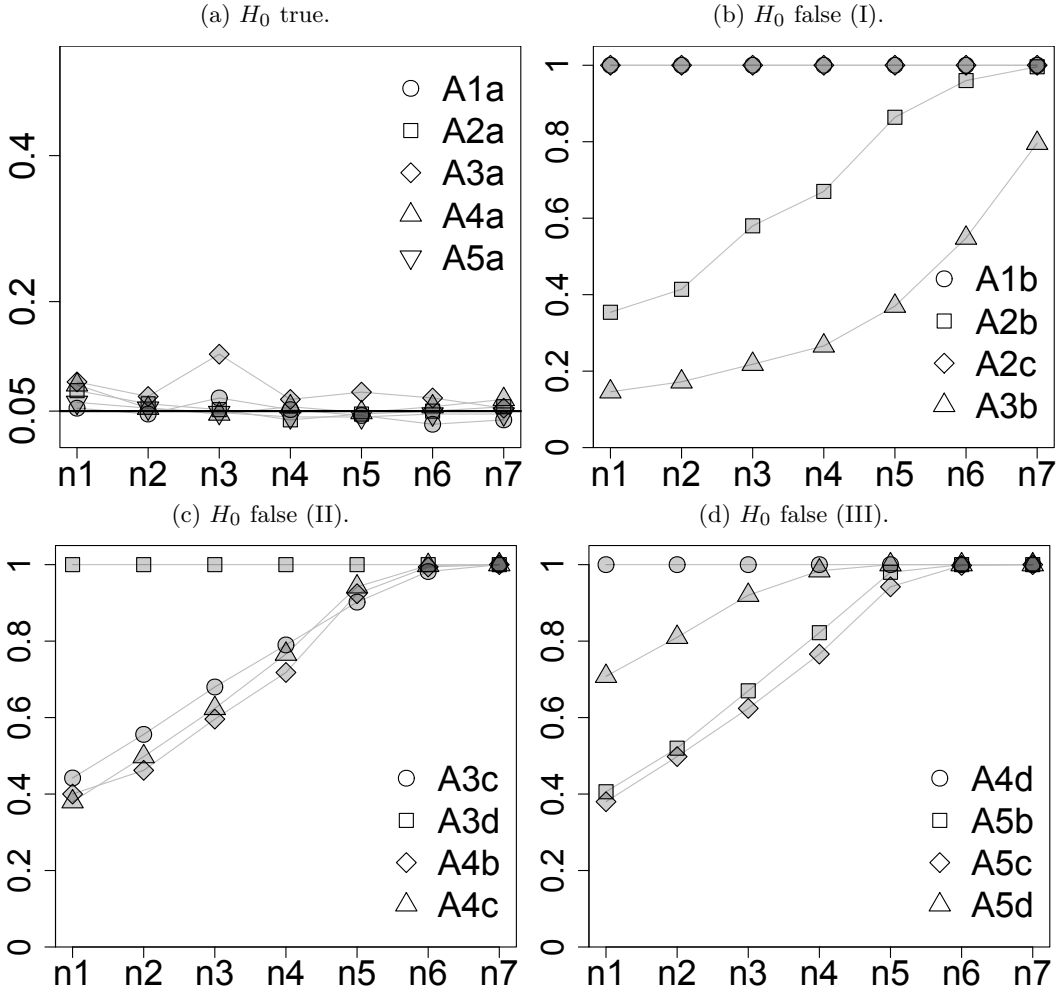
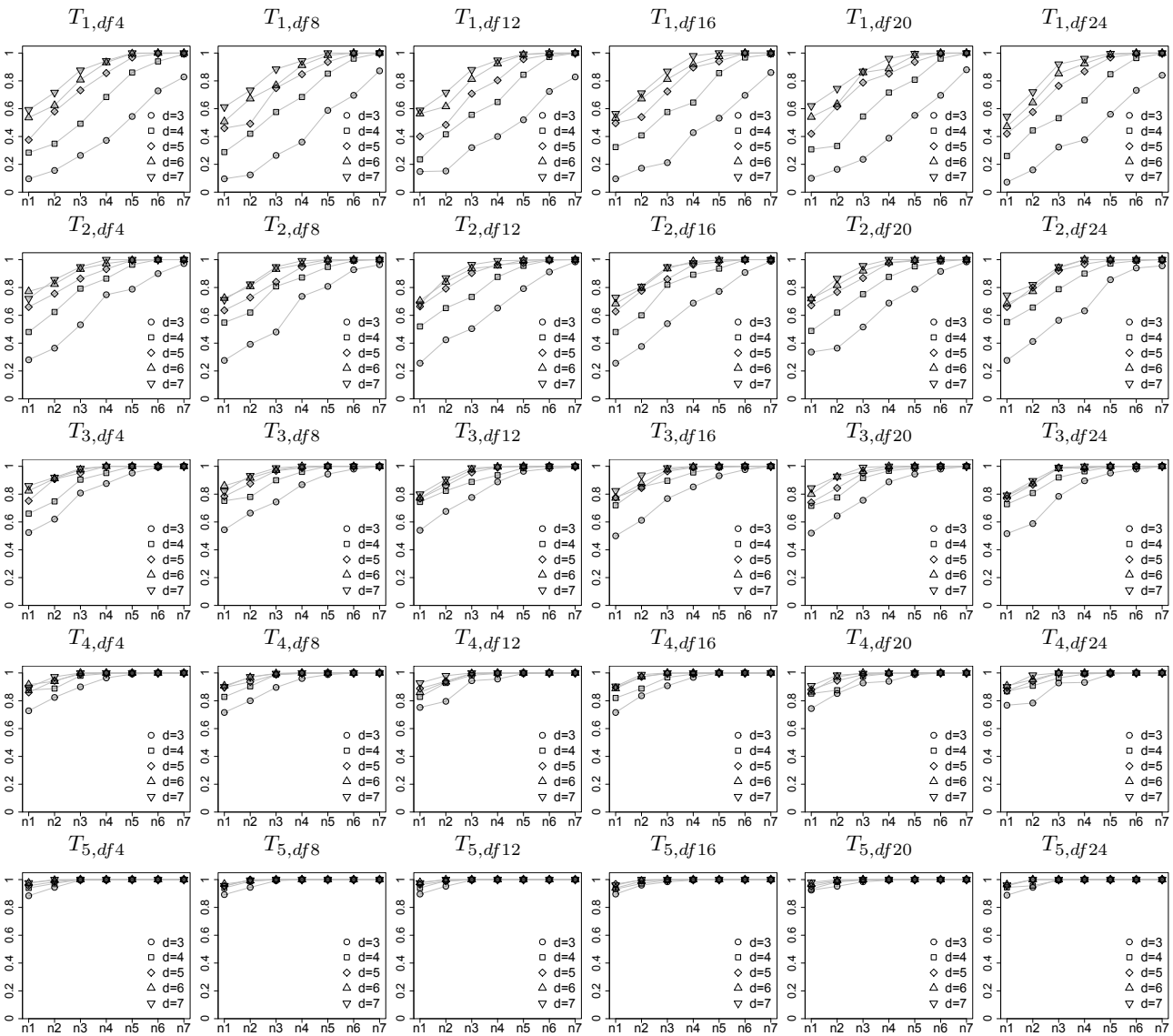


Figure 3: Rejection rates of the test at nominal level of 5% for model specifications from multivariate t distributions. Linear dependence increases from top to bottom. Fatness of univariate tails decreases from left to right. Sample size increases within each figure from left to right. For each sample size dimension ranges from $d = 3$ to $d = 7$. See Appendix 6.1 for details.



3.2 Local power analysis

In this subsection, we study the performance of the test under a series of local alternatives from the null. In contrast to the fixed alternatives of the subsection before, alternatives here are very close to the null and their distance to the null can shrink with increasing sample size revealing the power optimality properties of the consistent test. Thus, we evaluate the ability of the test to detect a violation of the null if the nature of the underlying distribution of $\mathbb{X}$ is such that only a few joint extremes in dimension ≥ 3 occur in finite samples. Following Berg & Quessy (2009) and Kojadinovic & Yan (2010), such distributions are generated by mixing distributions that violate the null, denoted by $F_{\mathbb{X},H_1}$, with distributions that comply with the null, denoted by $F_{\mathbb{X},H_0}$. We define the mixture distribution by

$$\mathcal{F}_{\mathbb{X},\lambda_n}(\mathbb{x}) := (1-\lambda_n)F_{\mathbb{X},H_0}(\mathbb{x}) + \lambda_n F_{\mathbb{X},H_1}(\mathbb{x}), \tag{11}$$

where λ_n decreases to zero for increasing sample size n and $F_{\mathbb{X},H_0}(\mathbb{x})$ satisfying $\Delta = 0$, $F_{\mathbb{X},H_1}(\mathbb{x})$ satisfying $\Delta > 0$, and $F_{\mathbb{X},H_0}(\mathbb{x}) \leq F_{\mathbb{X},H_1}(\mathbb{x}), \forall \mathbb{x}$, ensuring realizations from $F_{\mathbb{X},H_1}$ enter the extreme part of the sample. Denote the test statistic resulting from the mixture distribution $\mathcal{F}_{\mathbb{X},\lambda_n}(\mathbb{x})$ by Δ_n. For $\lambda_n = O((\sqrt{k(n)})^{-1})$, we can show that, asymptotically,

$$\sqrt{k}(\widehat{\Delta}_n - \Delta_n) \xrightarrow{d} N(0, \sigma^2_{\widehat{\Delta}_n,\lambda_n}),$$

where the asymptotic variance can again be obtained analytically from theorem 4.3 in Einmahl et al. (2012). Thus the test has power against any local alternatives if and only if they are at least of order $(\sqrt{k(n)})^{-1}$ apart from the null.

In the following simulations, we illustrate this result. Hence, we are interested in rejection rates of $\Delta = 0$ from mixture distributions defined in equation 11 for $\lambda_n := \lambda k(n)^{-1/2}$, with $0 < \lambda \leq k(n)^{-1/2}$. We determine k in the same way as in the simulations before. In order to calculate local powers p_n, we generate $S = 1000$ times a DGP of mixture distribution kind with fixed sample size and increasing λ. Local power is estimated by $\hat{p}_n = 1/S \sum_{i=1}^{S} 1\{\widehat{\Delta}_n > z^{\alpha}\hat{\sigma}_{\widehat{\Delta}_n,\lambda_n} k^{-1/2}\}$ for every λ_n. The asymptotic variance $\sigma^2_{\widehat{\Delta}_n,\lambda_n}$ is estimated by the bootstrap procedure presented in section 2.3. Berg & Quessy (2009), Kojadinovic & Yan (2010) carry out similar analyses for goodness–of–fit tests of parametric (extreme value) copulas.

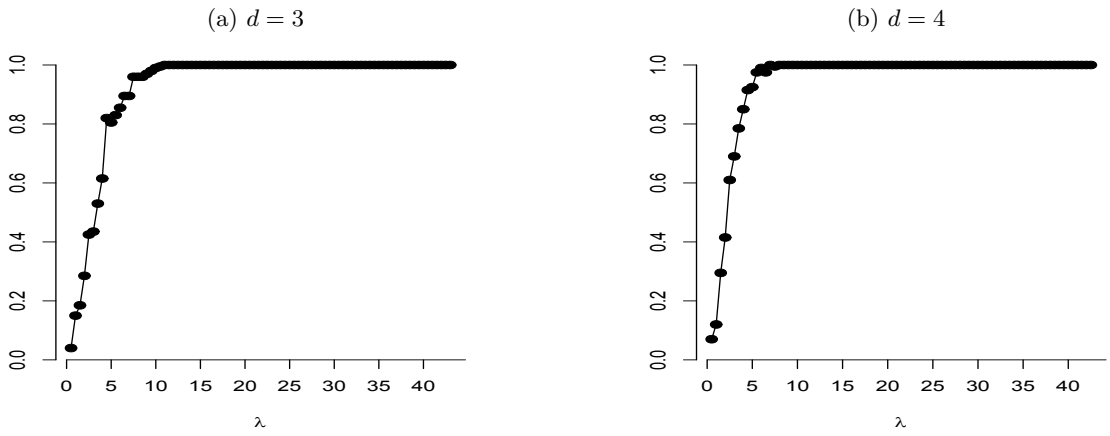
For the sake of brevity, we concentrate on dimensions $d = 3, 4$, sample size $n = 2000$, and we let λ increase. For $d = 3$,

$$\mathcal{F}_{\mathbb{X},\lambda_n}(\mathbb{x}) = (1-\lambda_n)F_{\mathbb{Y}}(\mathbb{y}) + \lambda_n F_{\mathbb{W}}(\mathbb{w}), \tag{12}$$

where $F_{\mathbb{Y}}(\mathbb{y})$ and $F_{\mathbb{W}}(\mathbb{w})$ are the cdf's of the max factor model A_{1a} and A_{1b}, respectively. To ensure realizations of A_{1b} actually enter the extremal part of the sample, factors Z are first used to generate data from $F_{\mathbb{Y}}(\mathbb{y})$ and then multiplied by a constant larger than one when generating data from $F_{\mathbb{W}}(\mathbb{w})$. For $d = 4$, we mix the cdf of A_{2a} and A_{2c}. If, for example, $\lambda = 2$, it holds $\lambda_n \approx 0.045$. Thus 4.5% of the extreme part of the sample is generated by the $F_{\mathbb{W}}$ which violates the null. This share increases in λ.

Figure 4 shows estimated local powers with $\alpha = 0.05$. The test successfully detects minor violations from the null. Even for small λ, when the impact of the perturbating DGP is small, rejection rates quickly converge to one. Also, an increasing dimension d appears not to have a significant effect on the convergence speed of empirical power.

Figure 4: Empirical test power for the mixture distributions defined in equation 12 with sample size $n = 2000$ at level 5%.



4 Higher order tail dependencies in bond markets

4.1 Data

We test for HOTD's in a multivariate sample of government bond index price returns, consisting of 10–year government bonds of the United States (US), Germany (GER), United Kingdom (UK) and France (FRA). Government bonds are issued by governments in order to finance public spending. Buyers are private investors, hedge funds, private banks, and central banks. Government bonds of developed countries enjoy a reputation of a safe investment with moderate returns. Hence, government bonds can be used as hedging tools in order to narrow overall portfolio risk. Diminishing bond prices indicate rising bond yields and decreasing demand in the bond market. Governments have interest in a high demand for bonds in order to plan and maintain the financing of public spending. Private investors prefer high yields, ceteris paribus. Thus, for both governments and investors it is critical to know about the interdependencies of extreme return drops between different government bonds. As analyzing the left tail, i.e. losses, of bond price returns is analogue to analyzing the right tail, i.e. gains, of bond yields (and vice versa), we assess (tail) dependence in both tails of price returns separately.
Data stem from Datastream and run under the mnomics BMUS10Y, BMBD10Y, BMUK10Y and BMFR10Y. The sample period ranges from 31/01/1985 to 09/10/2013. See figure 9 for plots of the price-level data. We transform all marginal daily return series $y_t, t = 1, ..., T$, to a filtered series

$$i = 1, ..., T/3 : \quad \tilde{y}_t = 1/3(y_t + y_{t-1} + y_{t-2}), t = 3i, \tag{13}$$

which levels out intercontinental time zone effects. Before applying the test, we check if extremes are independent over time. Clusters of extremes are not in line with the assumption of a random sample, which is needed for the limit law in equation 1 to hold. The degree of autocorrelation in extremes can be measured by the extremal index (Embrechts et al. (1997)), which describes the tendency of extremes to cluster. Values close to one indicate that the series can be treated as if it originated from a white noise sequence. We estimate the extremal indices for each filtered bond return series with the runs estimator, which is standard in the literature (see, e.g., Embrechts et al. (1997), Chapter 8). Figure 10 shows that the estimates for all four marginals are close to one. Therefore, we proceed by analyzing the filtered series without further accounting for time dependences.

4.2 Empirical results

We test for HOTD's between filtered daily return gains and losses by repeatedly estimating 95% confidence intervals for Δ using a rolling window scheme. Sample size corresponds to a window length of roughly 10 years

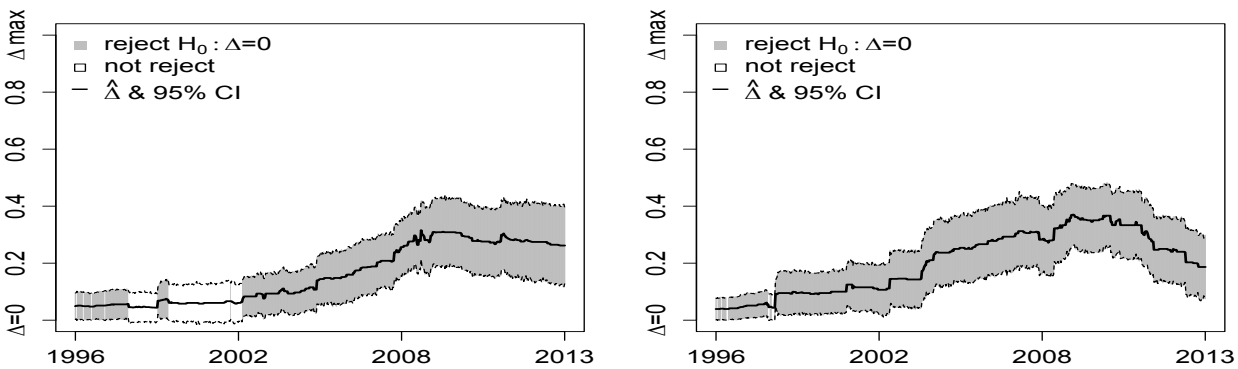
with $n = 710$ observations per window. Results for smaller windows are similar and available upon request. Sample days with missing data or constant bond prices are discarded. We use a grid of threshold parameters k for $\widehat{\Delta}$: $\widehat{\Delta}$ is calculated for every $k \in [0.01n, 1.5n^{1/2}]$, where n denotes the length of the rolling window. The final $\hat{\Delta}$ is then determined as the median thereof. In contrast to automatic data–driven procedures for finding k, in every iteration the same extremal part of the sample tail is evaluated as k is fixed. Variances are estimated by the block bootstrap routine outlined in section 2.2, with $B = 500$ and block size $n^{1/3}$.
A univariate tail analysis would solely involve estimating tail indices γ_j for each margin j, see equation 3. A widely used estimator for the tail index is the Hill estimator,

$$\hat{\gamma} = \frac{1}{k}\sum_{i=0}^{k-1}(\log X_{(n-i):n} - \log X_{(n-k):n})^j. \tag{14}$$

Figure 11 shows plots of the tail index estimates $\hat{\gamma}$ for the four government bond return series. All marginal distributions appear to be fat–tailed, since the estimates are decisively larger than zero. Thus, the marginals lie in the domain of max–attraction of the Fréchet distribution. Interestingly, the estimated tail indices do not alter substantially over time. The univariate tails therefore indicate that tail risk remains constant, despite the presence of economic shocks such as the dot–com crisis, 9\11\01, the subprime crisis and the Euro crisis.
Figure 5 displays the dynamics of the estimated test statistic Δ and according confidence intervals ($\alpha = 0.05$) for return gains and return losses. For gains, we only observe no HOTD's before 2002. During 2002–2013, HOTD's have significantly inflated for both gains and losses. Plateaus of $\widehat{\Delta}$ far away from zero for both tails indicate HOTD's persist until today. Thus, the likelihood for three or four price returns to become jointly extreme cannot not be dismissed without a loss of information about the joint tail of all four bond returns. Notably, after 2010, $\widehat{\Delta}$ has slightly decreased for losses. In both cases, the portion of global tail dependence HOTD's rises during or peaks at financial crises on the stock markets. Thus, the tail dependence structure within bond markets is not immune against extreme events on stock markets.

Figure 5: Dynamics of the test statistic $\widehat{\Delta}$ (see equation 10), together with 95% confidence intervals, for 10–years government bond returns of US, Germany, United Kingdom and France, using a rolling window of 10 years. Figure (a) shows test decisions of a rolling window with length 10 years for return gains while figure (b) displays test decisions for return losses. Gray confidence intervals correspond to rejections of the null that tail dependence is purely bivariate.

(a) Gains (b) Losses



As figure 9 indicates bond yields have gradually dwindled during the past fifteen years. In addition, as seen during the Euro crisis and the government shutdowns in the US in 2013 and 2013, the default probability of government bond issuers is most likely not zero – which might have been the case before the 2000's. Thus for government bonds we identify a period of high tail risk and low returns: A pronounced market risk due

to an increase in HOTD's, an increased credit risk and diminishing bond yields can be interpreted as general mispricing on international bond markets. Consequently, government bonds of highly developed countries should not be considered safe haven per se.
Keeping in mind that, despite the crash in 2008 on stock and real estate markets, Western stock and real estate markets reached record levels after 2010, this situation resembles the prelude of the bubble burst in 1987. During the 1980's Japan suffered from weak economic growth, a bond market with low yields and high risk, causing investors seek high yields investments other than bonds. This excess liquidity flowed to stock and real estate markets, and increased demand escalated both markets. The central bank's sudden announcement to increase interest rates thus amounted to the stock market bubble to burst.

4.3 Impact of disregarding higher order tail dependencies

The impact of disregarding HOTD's may be measured by comparing quantile regions (QR's). In a risk monitoring context, QR's give rise to a conservative multivariate extension of the widely used Value-at-Risk (VaR), which is directly linked to the STDF. We define a QR according to Einmahl et al. (2009) as vector of univariate thresholds $\tilde{x}^{(j)}_{1-p}$, $j = 1, ..., d$, such that for any small fixed probability p,

$$\mathbb{P}(X^{(1)} > \tilde{x}^{(j)}_{1-p_1} \cup \ldots \cup X^{(d)} > \tilde{x}^{(d)}_{1-p_4}) = p, \tag{15}$$

with returns $X^{(j)}, j = 1, ..., 4$. Thus the probability that at least one return exceeds an extreme quantile is p. The vector $Q := \{(-\infty, \tilde{x}^{(j)}_{1-p_j}), j = 1, ..., d\}$ spans a subspace in $\mathbb{R}^d$ satisfying $\mathbb{P}(\mathbb{X} \in Q) = 1 - p$.
In order to illustrate how neglecting HOTD's leads to substantial underestimation of overall risk in the case of the four government bonds, we first estimate a QR that copes for HOTD's. Second, we estimate a QR that only copes for bivariate tail dependencies. According to its definition in 15, the $(1-p)$–QR in our case is the vector of marginal quantiles $\tilde{x}^{(i)}_{1-p}, i = 1, ..., 4$, fulfilling

$$Q_{1-p} := \{(\tilde{x}^{(1)}_{1-p^*}, ..., \tilde{x}^{(4)}_{1-p^*}) : \mathbb{P}(\cup_{i=1}^{4}\{X^{(i)} > \tilde{x}^{(i)}_{1-p^*}\}) = p\}.$$

Note, the individual exceedance probability (p^*) is the same for all margins such that the thresholds are uniquely identified. A QR is termed extreme if $p < 1/n$. Extreme QR's can be consistently estimated as $\widehat{Q}^{\Delta>0}_{1-p} = (\widehat{\tilde{x}}^{(1)}_{1-p^{\Delta>0}}, ..., \widehat{\tilde{x}}^{(4)}_{1-p^{\Delta>0}})$ with estimated thresholds

$$\begin{aligned}
\widehat{\tilde{x}}^{(i)}_{1-p^{\Delta>0}} &= \frac{k/(np^{\Delta>0})^{\hat{\gamma}} - 1}{\hat{\gamma}}\hat{a} + \hat{b}, \\
&\text{with} \\
M^{[j]} &= 1/k \sum_{i=0}^{k-1} \big(\log \frac{X_{n-i:n}}{X_{n-k:n}}\big)^j, \\
\hat{\xi}^+ &= M^{[1]}, \quad \hat{\xi}^- = 1 - 0.5\big(1 - \frac{(M^{[1]})^2}{M^{[2]}}\big)^{-1}, \\
\hat{\gamma} &= \hat{\gamma}^+ + \hat{\gamma}^-, \\
\hat{a} &= X_{n-k:n} M^{[1]}(1 - \hat{\gamma}^-), \quad \hat{b} = X_{n-k:n}, \\
p^{\Delta>0} &= \frac{p}{\ell(1,1,1,1)}, \; i = 1, ..., 4.
\end{aligned}$$

Rewriting the exceedance probability p in terms of the STDF for the entire vector respects bivariate, three–dimensional and four–dimensional tail dependencies within the return vector. To mimic a purely bivariate approach we rewrite the exceedance probability only in terms of bivariate STDF's. If tail dependence was solely

bivariate, p must only be expressed in terms of bivariate tail dependencies, i.e.

$$p^{\Delta=0} = \frac{p}{4 - \sum_{i \neq j}(2 - \ell_{ij}(1,1))}.$$

The purely bivariate approach constitutes $\widehat{Q}_{1-p}^{\Delta=0} = (\widehat{\tilde{x}}_{1-p^{\Delta=0}}^{(1)}, ..., \widehat{\tilde{x}}_{1-p^{\Delta=0}}^{(4)})$. A comparison between $\widehat{Q}_{1-p}^{\Delta>0}$ and $\widehat{Q}_{1-p}^{\Delta=0}$, for example in terms of volumes $V(\widehat{Q}_{1-p}) = \prod_{i=1}^{4} \widehat{\tilde{x}}_{1-p}^{(i)}$, then quantifies the impact of disregarding HOTD's.

Figure 6 maps the time index against the estimates of all four quantile thresholds for both types of QR's and an extremal exceedance probability of $p = 0.0014$. Figure 7 compares both QR's for both tails of all marginals in terms of the volume ratio between both QR's, i.e. $V(\widehat{Q}_{1-p}^{\Delta=0})/V(\widehat{Q}_{1-p}^{\Delta>0})$, and of ratios between the marginal thresholds, i.e. $\widehat{\tilde{x}}_{1-p^{\Delta=0}}^{(i)}/\widehat{\tilde{x}}_{1-p^{\Delta>0}}^{(i)}, i = 1, ..., 4$. Alternatively, in order to compare both QR's, we also contrast the realized exceedance probability of a $(1-p)$–QR accounting for HOTD's against a fixed exceedance probability of a same sized $(1-p)$–QR that ignores HOTD's. This corresponds to the estimation of a portfolio VaR with a given admissible risk level $\overline{p}$ based on bivariate tail-risks, though the truly underlying risk level p in fact differs. Here we fix $\overline{p}$ for $Q_{1-\overline{p}}^{\Delta=0}$ and subsequently determine the realized exceedance probability p for $Q_{1-p}^{\Delta>0}$ such that univariate thresholds of both QR's are identical, i.e. both QR's are identical by construction. Analytically, we get

$$p^{\Delta>0} \stackrel{!}{=} p^{\Delta=0} \quad \Leftrightarrow \quad \frac{\dot{p}}{\ell(1,1,1,1)} = \frac{\overline{p}}{4 - \sum_{i \neq j}(2 - \ell_{ij}(1,1))}$$

$$\Leftrightarrow$$

$$\dot{p} = \frac{\ell(1,1,1,1)\overline{p}}{4 - \sum_{i \neq j}(2 - \ell_{ij}(1,1))}.$$

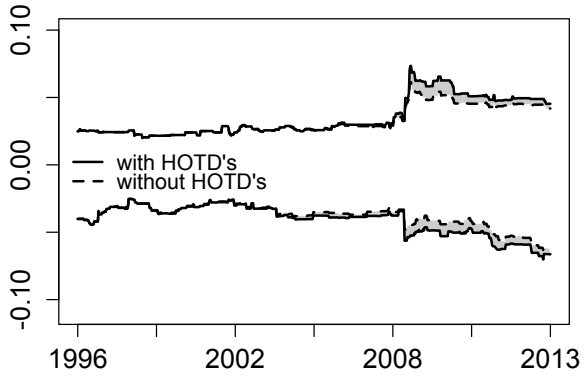
Figure 8 shows the dynamics of the realized exceedance probability of the HOTD approach with respect to the bivariate approach.

Until 2002, the difference between both QR's is very small as both volumes are roughly the same, and also no evident difference between marginal thresholds is found. Note, the test for HOTD's indicates HOTD's also exist before 2002 but they have only minor impact on global tail dependence. Thus, as seen in section 4, HOTD's kick in after 2002, and consequently the differential between both QR's steadily escalates over time for all bonds and for both tails. By the end of 2008, the volume of the bivariate tail based QR for return losses collapses to 47% of the QR accounting for HOTD's, but recovers to 73% in 2013. Underestimation is in general more severe for return losses, while also pronounced for return gains. Interestingly, figure 6 allows to track the driver of the increase in HOTD's. While thresholds for UK, GER and FRA remain constant or even decrease until 2008, the US thresholds bounce in the beginning of 2008. Although HOTD's have relinquished importance since 2008, they still cause a major underestimation of extreme QR's.

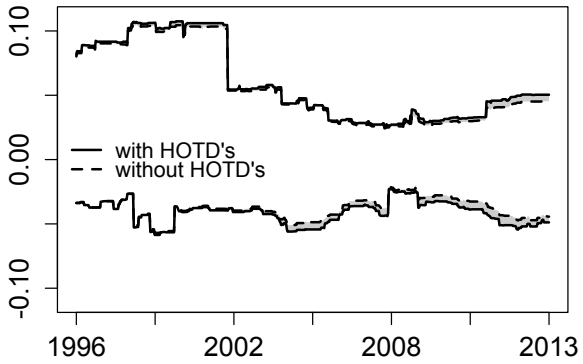
Comparing the nominal exceedance probability $\overline{p}$ of the bivariate approach with the realized exceedance probability $\dot{p}$ of the HOTD approach in figure 8, reveals that $\dot{p} > \overline{p}$. Thus the bivariate approach generally underestimates the true QR given HOTD's are present. Put differently, the bivariate QR approach consistently undervalues the true exceedance probability p, which we approximate by the realized exceedance probability of the HOTD QR. Especially in 2009, the exceedance probability of the bivariate QR is less than one half of the true p, both for gains and losses. Therefore, if one intends to estimate a, say, multivariate 99% VaR, the estimated QR without HOTD's has in fact a chance of 2% to be exceeded. Although a bivariate approach is a noticeable advancement from a univariate analysis, it still underestimates a large portion of multivariate tail risk already in this simplistic illustrative setting, and might lull financial institutions into a false sense of security.

Figure 6: Dynamics of 99.86%–quantile regions accounting for HOTD's and 99.86%–quantile regions not accounting for HOTD's.

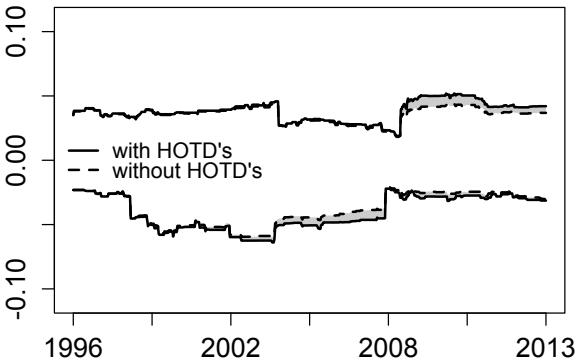
(a) Marginal threshold of US bonds



(b) Marginal threshold of UK bonds



(c) Marginal threshold of German bonds



(d) Marginal threshold of French bonds

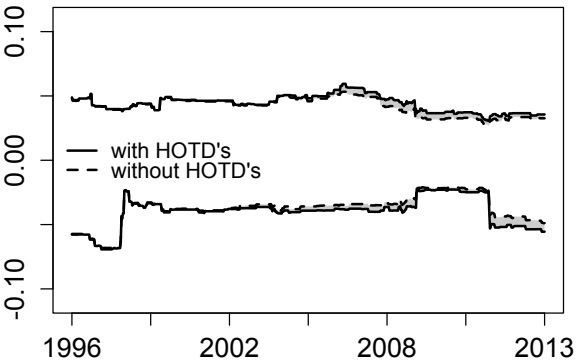
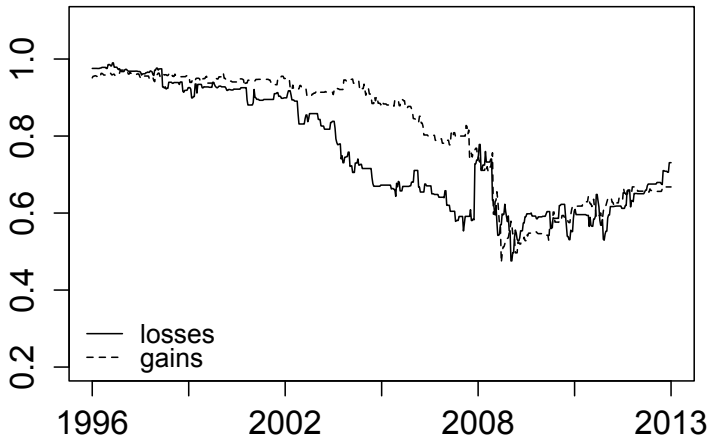
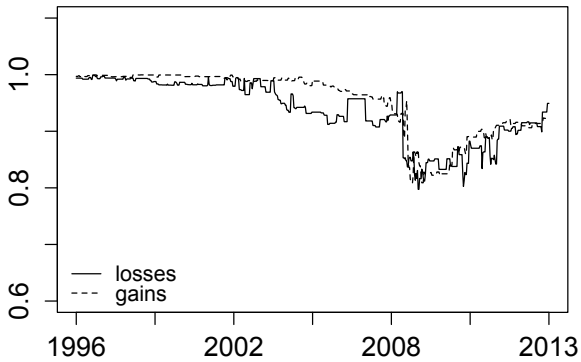


Figure 7: Dynamics of the differential between 99.86%–quantile regions accounting for HOTD's and 99.86%–quantile regions disregarding HOTD's.

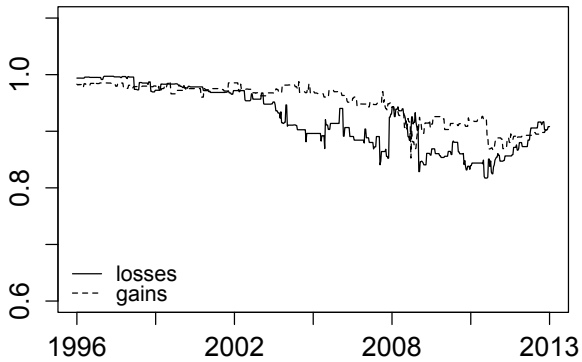
(a) Volume ratio



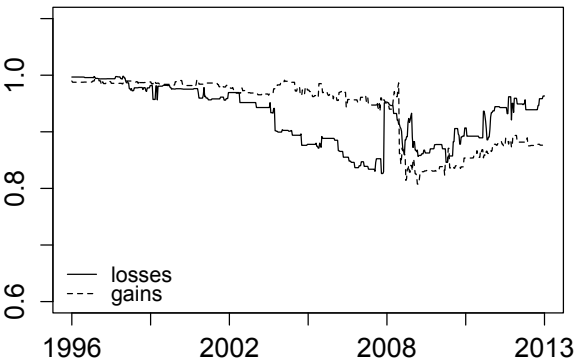
(b) Ratio of marginal thresholds of US bonds



(c) Ratio of marginal thresholds of UK bonds



(d) Ratio of marginal thresholds of German bonds



(e) Ratio of marginal thresholds of French bonds

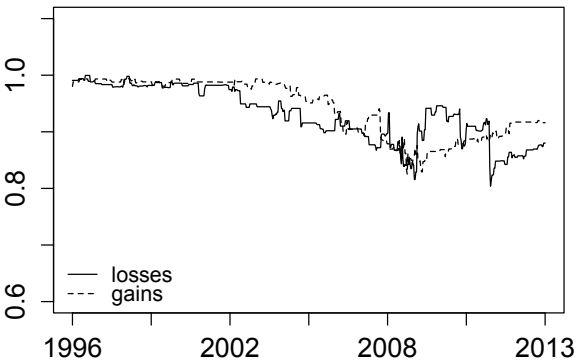
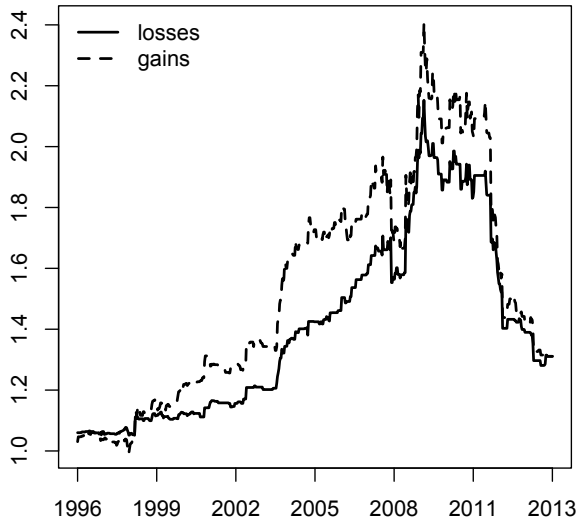


Figure 8: Dynamics of the ratio of realized exceedance probability for $\widehat{Q}_{1-p}^{\Delta>0}$ and nominal exceedance probability for $\widehat{Q}_{1-p}^{\Delta=0}$: $\dot{p}/\overline{p}$.



5 Summary

This paper proposes a test that indicates situations in which common bivariate measures for tail dependence underdiagnose the potential for higher-dimensional extreme events. Test asymptotics are derived and simulations show the bootstrap implementation routine features attractive finite sample properties although the problematic threshold choice, immanent to EVT, affects test size distinctly. Simulations show that even for standard models for multivariate risk management traditional pairwise methods to measure tail dependence tend to ignore dependence between a multiple of extremes. In higher dimensions, higher order tail dependencies critically add up even if linear dependence and fatness of univariate tails are moderate.

On international bond markets, we identify a high tail risk and low return situation. Furthermore, we find that dismissing HOTD's in a quantile region framework can lead to an underestimation of the tail dependence between international bond returns by up to 40%.

6 Appendix

6.1 Model specifications

Table 1: Specifications for multivariate t distributions.

	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
degrees of freedom					
$df = 4$	$\boldsymbol{T_{1,df4}}$	$\boldsymbol{T_{2,df4}}$	$\boldsymbol{T_{3,df4}}$	$\boldsymbol{T_{4,df4}}$	$\boldsymbol{T_{5,df4}}$
$df = 8$	$\boldsymbol{T_{1,df8}}$	$\boldsymbol{T_{2,df8}}$	$\boldsymbol{T_{3,df8}}$	$\boldsymbol{T_{4,df8}}$	$\boldsymbol{T_{5,df8}}$
$df = 12$	$\boldsymbol{T_{1,df12}}$	$\boldsymbol{T_{2,df12}}$	$\boldsymbol{T_{3,df12}}$	$\boldsymbol{T_{4,df12}}$	$\boldsymbol{T_{5,df12}}$
$df = 16$	$\boldsymbol{T_{1,df16}}$	$\boldsymbol{T_{2,df16}}$	$\boldsymbol{T_{3,df16}}$	$\boldsymbol{T_{4,df16}}$	$\boldsymbol{T_{5,df16}}$
$df = 20$	$\boldsymbol{T_{1,df20}}$	$\boldsymbol{T_{2,df4}}$	$\boldsymbol{T_{3,df20}}$	$\boldsymbol{T_{4,df20}}$	$\boldsymbol{T_{5,df20}}$
$df = 24$	$\boldsymbol{T_{1,df24}}$	$\boldsymbol{T_{2,df4}}$	$\boldsymbol{T_{3,df24}}$	$\boldsymbol{T_{4,df24}}$	$\boldsymbol{T_{5,df24}}$

Table 2: Notation for max factor models.

dimension	H_0 true	H_0 false
$d = 3$	$\boldsymbol{A_{1a}}$	$\boldsymbol{A_{1b}}$
$d = 4$	$\boldsymbol{A_{2a}}$	$\boldsymbol{A_{2b}}, \boldsymbol{A_{2c}}$
$d = 5$	$\boldsymbol{A_{3a}}$	$\boldsymbol{A_{3b}}, \boldsymbol{A_{3c}}, \boldsymbol{A_{3,d}}$
$d = 6$	$\boldsymbol{A_{4a}}$	$\boldsymbol{A_{4b}}, \boldsymbol{A_{4c}}, \boldsymbol{A_{4,d}}$
$d = 7$	$\boldsymbol{A_{5a}}$	$\boldsymbol{A_{5b}}, \boldsymbol{A_{5c}}, \boldsymbol{A_{5,d}}$

Table 3: Calibration of the max factor models.

$$\boldsymbol{A_{1a}} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{1b}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} \end{pmatrix}$$

$$\boldsymbol{A_{2a}} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{2b}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{2c}} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

$$\boldsymbol{A_{3a}} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{3b}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{pmatrix}$$

$$\boldsymbol{A_{3c}} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{3d}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

$$\boldsymbol{A_{4a}} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}, \boldsymbol{A_{4b}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.\overline{3} & 0.\overline{3} & 0.\overline{3} \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

$$\boldsymbol{A_{4c}} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{4d}} = \begin{pmatrix} 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} \end{pmatrix}$$

$$\boldsymbol{A_{5a}} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}, \boldsymbol{A_{5b}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

$$\boldsymbol{A_{5c}} = \begin{pmatrix} 0.\overline{3} & 0.\overline{3} & 0.\overline{3} & 0 & 0 & 0 & 0 \\ 0.\overline{3} & 0 & 0 & 0.\overline{3} & 0.\overline{3} & 0 & 0 \\ 0 & 0.\overline{3} & 0 & 0 & 0 & 0.\overline{3} & 0.\overline{3} \end{pmatrix}, \boldsymbol{A_{5d}} = \begin{pmatrix} 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0.1\overline{6} & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \end{pmatrix}$$

6.2 Simulation results

Table 4: Rejection rates: Max factor models.

Model	n_1	n_2	n_3	n_4	n_5	n_6	n_7
A_{1a}	5.4	4.6	6.8	5.2	4.4	4.2	4.8
A_{2a}	7.8	6.0	5.2	3.8	4.6	5.0	5.6
A_{3a}	9.0	7.0	12.8	6.6	7.6	6.8	5.4
A_{4a}	8.6	5.4	4.6	5.6	4.8	5.6	6.6
A_{5a}	6.2	5.4	4.8	4.2	4.2	4.6	4.7
A_{1b}	100	100	100	100	100	100	100
A_{2b}	35.4	41.4	58.0	67.0	86.4	96.0	99.6
A_{2c}	100	100	100	100	100	100	100
A_{3b}	14.6	17.2	21.8	26.6	37.0	54.8	79.6
A_{3c}	44.2	55.6	68.0	79.0	90.2	98.2	100
A_{3d}	100	100	100	100	100	100	100
A_{4b}	40.4	46.2	59.6	71.8	92.4	99.4	100
A_{4c}	38.0	49.8	62.4	76.6	94.2	98.8	100
A_{4d}	100	100	100	100	100	100	100
A_{5b}	40.6	52.9	67.3	82.2	98.0	100	100
A_{5c}	38.5	49.8	62.4	76.6	94.2	99.8	100
A_{5d}	70.8	81.0	92.0	98.4	100	100	100

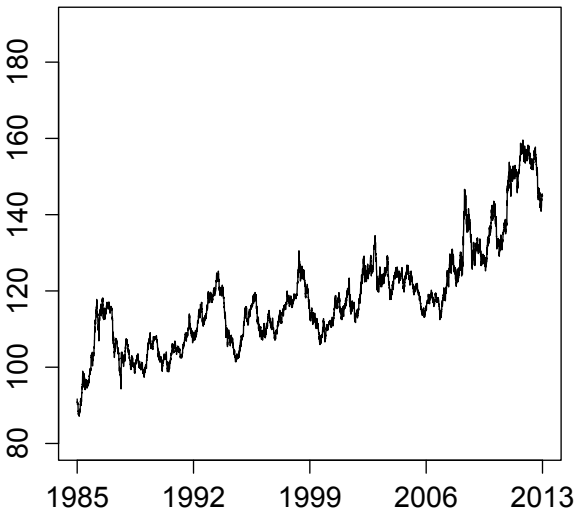
Table 5: Rejection rates: Multivariate t distributions.

		n_1					n_2					n_3					n_4					n_5					n_6					n_7				
	d	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7
T_1	df4	9.6	28.4	37.6	53.6	59.2	15.6	34.8	58.0	62.4	71.6	26.4	49.2	73.2	80.8	87.6	37.2	68.4	85.6	93.2	94.4	54.4	86.0	96.8	99.6	100	72.8	94.0	99.6	100	100	82.8	99.2	100	100	100
	df8	9.6	28.8	46.0	50.8	61.2	12.4	42.0	49.2	67.2	73.2	26.4	57.6	74.8	76.8	88.4	36.0	68.4	84.8	91.2	94.4	58.8	85.2	93.6	98.4	100	69.6	96.0	99.6	100	100	87.2	100	100	100	100
	df12	14.8	23.6	40.0	56.4	58.8	15.2	41.6	48.4	61.6	71.6	32.0	55.6	70.8	81.2	88.0	40.0	64.8	80.4	92.4	94.8	52.0	84.4	95.6	98.8	99.2	72.4	97.2	98.8	100	100	82.8	100	100	100	100
	df16	9.6	32.4	49.6	53.2	56.4	17.2	40.8	54.0	67.2	71.2	21.2	57.6	72.4	81.2	86.8	42.8	64.4	89.6	92.0	98.0	53.2	85.6	94.0	97.6	100	69.6	96.8	100	100	100	86.0	99.2	100	100	100
	df20	10.0	30.8	42.0	54.0	62.0	16.4	33.2	61.6	63.2	74.4	23.6	54.4	78.8	86.0	86.8	38.8	71.6	85.2	88.8	96.0	55.2	80.8	93.6	98.4	99.6	69.6	96.0	99.6	100	100	88.0	100	100	100	100
	df24	7.2	26.0	42.0	47.2	54.4	16.0	44.4	57.6	64.4	72.0	32.4	53.2	76.4	85.2	92.0	37.6	66.0	86.8	92.4	96.0	56.0	84.8	96.8	98.8	99.6	73.2	96.4	99.6	100	100	84.0	99.6	100	100	100
T_2	df4	28.0	48.0	66.0	77.2	72.0	36.4	62.4	75.6	82.4	85.6	53.2	79.2	86.4	93.2	94.8	74.8	86.4	93.2	97.2	100	78.8	96.4	99.6	99.6	100	90.0	100	100	100	100	97.2	100	100	100	100
	df8	27.6	54.8	63.6	72.4	71.2	39.2	62.0	72.8	80.8	82.4	48.0	80.8	84.0	93.2	94.8	73.6	87.2	94.8	96.8	99.2	80.8	94.8	98.8	100	100	92.8	99.2	100	100	100	96.4	100	100	100	100
	df12	25.6	52.0	66.4	70.4	68.0	42.4	65.2	79.2	84.0	86.8	50.4	73.2	90.4	93.6	96.4	65.2	87.6	96.4	95.6	99.2	79.2	95.6	97.6	99.6	99.6	91.2	99.6	100	100	100	98.4	100	100	100	100
	df16	25.6	48.0	62.8	68.4	72.8	37.6	60.0	77.6	79.6	80.8	54.0	82.0	86.0	93.6	94.8	68.8	89.2	96.4	98.8	97.6	77.2	93.6	98.4	99.6	100	90.8	100	100	100	100	98.8	100	100	100	100
	df20	33.6	48.8	67.2	72.0	71.6	36.4	62.0	76.8	81.6	86.4	51.6	75.2	86.8	92.0	95.6	68.8	87.6	97.6	98.0	100	78.8	95.2	99.2	99.6	100	91.6	98.8	100	100	100	98.4	100	100	100	100
	df24	27.6	55.2	66.0	68.4	74.4	41.2	65.6	80.0	77.2	82.0	56.4	78.8	92.0	94.0	94.8	63.2	90.0	96.8	100	100	85.6	97.2	99.6	100	100	94.0	99.2	100	100	100	95.6	100	100	100	100
T_3	df4	52.4	66.0	75.2	82.4	86.0	62.0	74.8	90.8	91.2	92.0	80.8	90.4	95.2	97.6	98.4	87.6	95.2	100	100	100	95.2	100	100	100	100	99.2	100	100	100	100	100	100	100	100	100
	df8	54.4	75.2	78.4	86.0	82.0	66.4	78.0	87.6	91.6	93.2	74.4	90.0	96.8	97.2	98.8	86.8	96.0	98.8	100	100	94.4	100	100	100	100	98.0	100	100	100	100	100	100	100	100	100
	df12	54.0	74.4	76.8	77.2	80.0	67.6	82.4	86.0	88.4	90.8	77.6	88.8	95.6	97.6	98.8	88.8	93.6	99.2	99.6	99.6	96.4	98.8	100	100	100	98.4	100	100	100	100	99.6	100	100	100	100
	df16	50.0	72.0	77.6	77.2	82.4	61.2	85.2	84.4	87.6	93.6	76.8	89.6	96.4	97.6	98.8	85.2	95.6	99.2	99.6	100	93.2	99.2	100	100	100	97.6	100	100	100	100	100	100	100	100	100
	df20	52.0	71.6	74.0	80.0	84.4	64.4	77.6	84.4	92.4	92.8	75.6	91.6	95.2	96.4	99.2	88.8	96.8	98.4	100	100	94.4	98.8	100	100	100	98.0	100	100	100	100	100	100	100	100	100
	df24	51.6	72.8	76.8	78.8	79.2	58.8	80.8	86.8	87.6	89.6	78.4	92.0	99.2	98.4	99.2	89.6	96.4	98.4	99.2	100	95.2	99.6	100	100	100	98.0	100	100	100	100	100	100	100	100	100
T_4	df4	72.8	87.6	86.0	91.6	89.6	82.4	88.8	94.4	94.0	97.2	90.0	98.0	99.2	100	100	96.4	99.6	100	100	100	99.2	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df8	71.6	82.8	89.6	90.8	90.0	80.0	90.4	93.6	96.8	97.2	89.6	98.8	98.8	99.6	99.6	96.0	99.6	100	100	100	98.8	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df12	75.2	82.8	88.8	86.0	92.8	79.6	92.8	94.0	93.6	98.0	94.4	98.4	99.6	99.6	100	95.6	99.6	100	100	100	99.6	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df16	71.6	82.0	89.6	88.8	90.4	83.6	88.8	97.2	97.2	98.8	90.8	96.8	99.6	100	100	96.8	99.2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df20	74.4	85.2	88.0	86.4	90.8	85.2	87.6	94.4	97.2	98.4	92.8	97.6	98.8	100	99.6	94.0	99.6	99.6	100	100	98.8	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df24	76.8	86.8	87.2	90.8	90.0	78.4	90.8	94.0	95.2	98.4	92.8	96.4	100	100	100	93.2	99.2	99.6	100	100	99.2	100	100	100	100	100	100	100	100	100	100	100	100	100	100
T_5	df4	88.4	94.0	95.6	97.6	97.6	94.4	97.2	98.4	99.6	100	99.6	100	100	100	100	99.6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df8	89.2	94.0	95.2	96.8	95.6	94.4	98.8	100	99.2	99.6	99.2	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df12	89.6	94.0	96.4	98.0	97.6	95.2	99.2	98.8	100	100	99.6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df16	89.6	93.2	96.8	94.0	96.0	96.0	97.2	99.6	98.8	100	98.4	99.6	100	100	100	99.6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df20	92.4	93.2	95.2	96.8	98.0	95.2	98.4	98.8	99.2	100	98.4	99.6	100	100	100	99.6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	df24	88.8	94.4	95.6	96.4	96.0	94.4	95.6	100	100	100	100	99.6	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

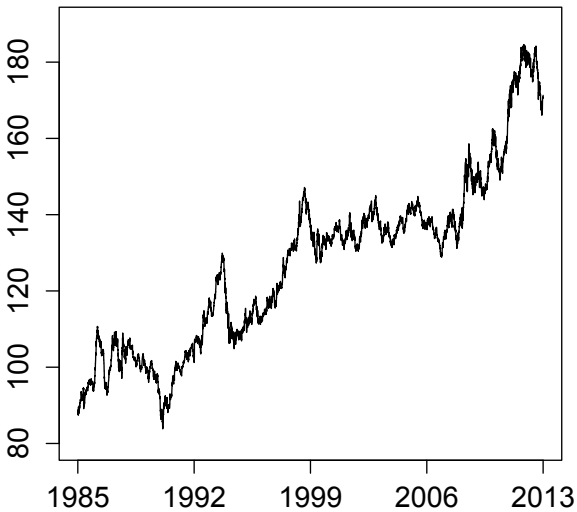
6.3 Auxiliary results for government bonds

Figure 9: Dynamics of government bond indices in price levels.

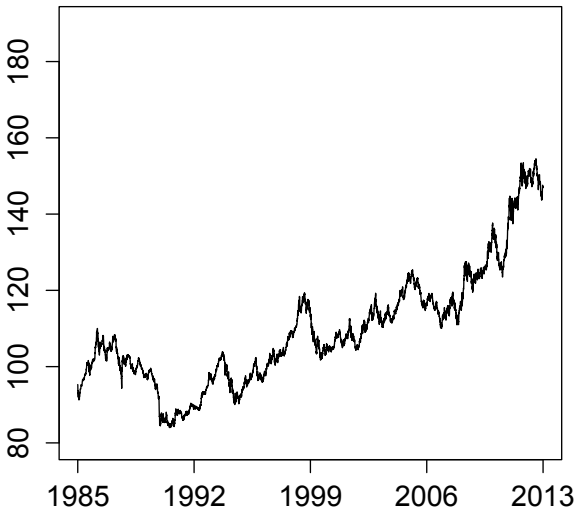
(a) United States



(b) United Kingdom



(c) Germany



(d) France

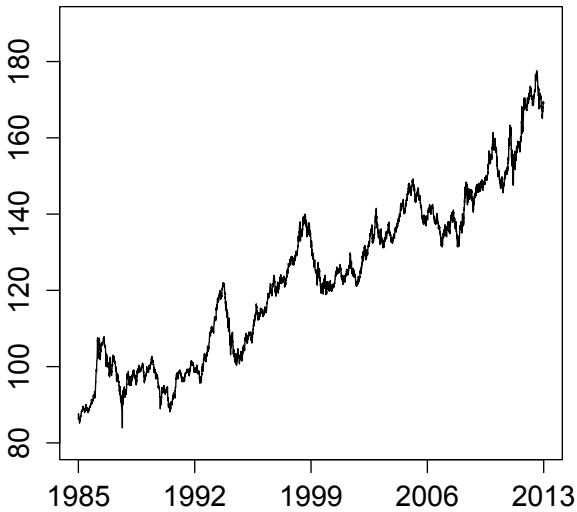
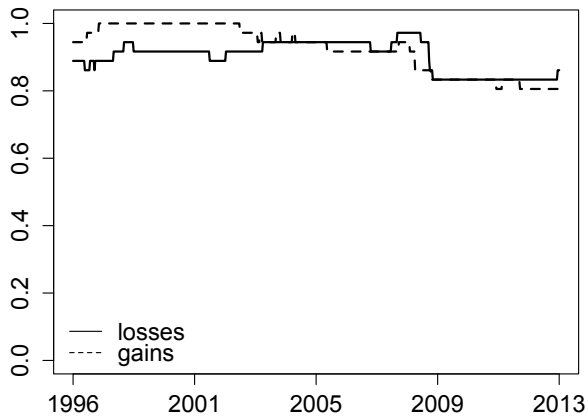
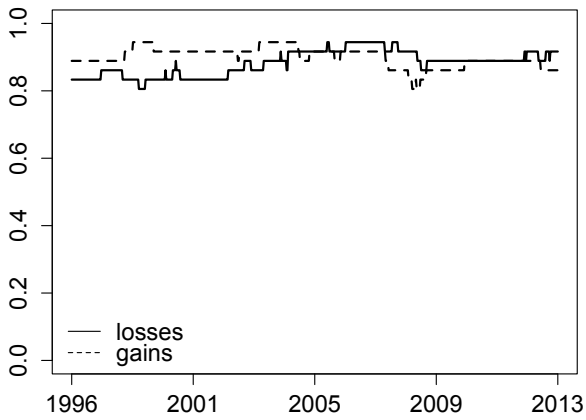


Figure 10: Dynamics of the runs estimator for the extremal indices of government bond returns, using a rolling window scheme.

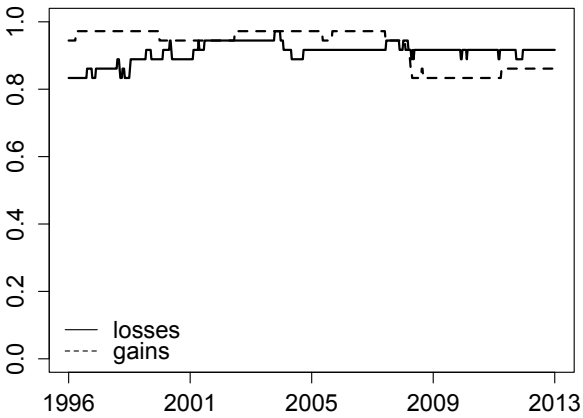
(a) USA



(b) United Kingdom



(c) France



(d) Germany

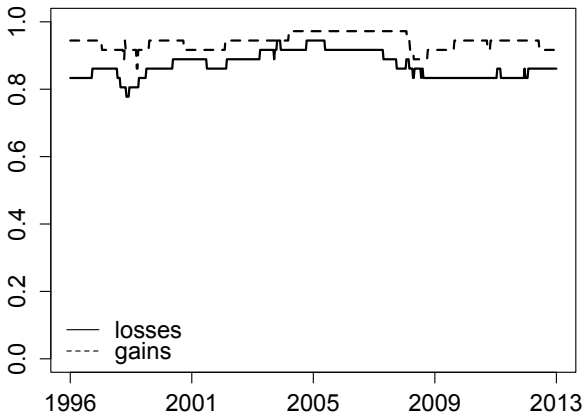
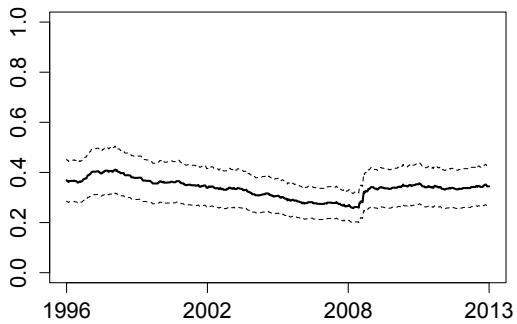
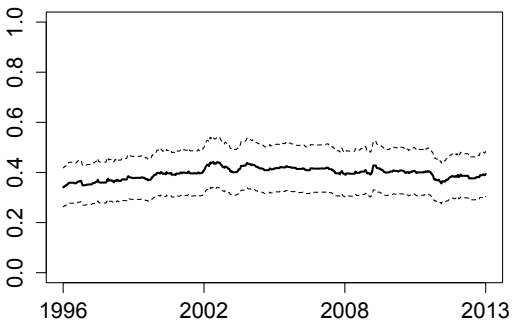


Figure 11: Dynamics of rolling-window estimates of the tail indices of four government bond index returns (see equation 14), together with 95% confidence intervals.

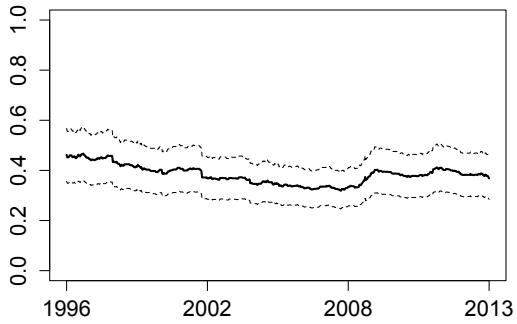
(a) US bond gains.



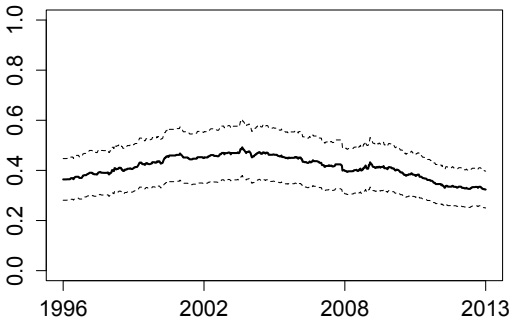
(b) US bond losses.



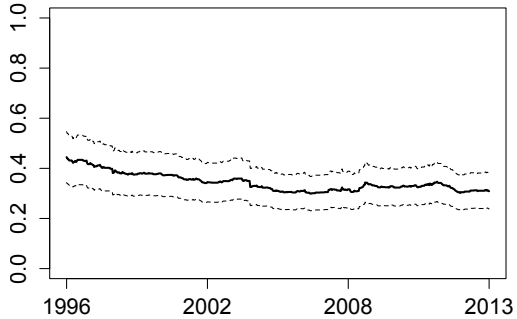
(c) UK bond gains.



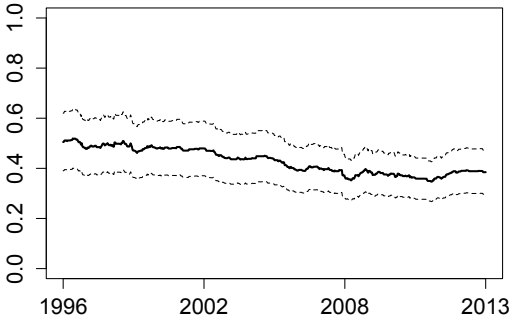
(d) UK bond losses.



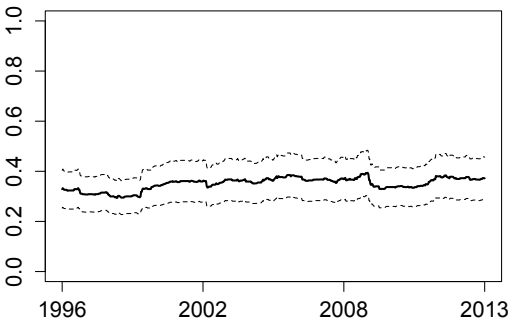
(e) German bond gains.



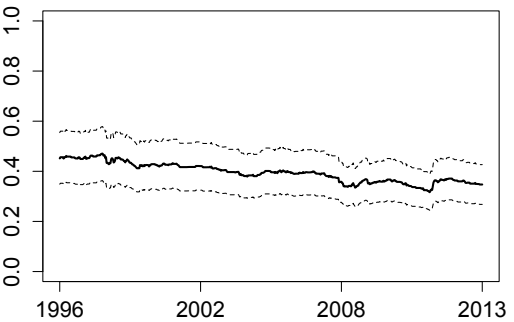
(f) German bond losses.



(g) French bond gains.



(h) French bond losses.



6.4 Proofs and Assumptions

Assumption 1 *Let*

(i) $t^{-1}\mathbb{P}(\bigcup_{i=1}^{d} F_i^{-1}(1-tx^{(i)}) \leq X^{(i)}) = \ell_{\mathbb{X}}(\mathbb{x}) + \mathcal{O}(t^{\beta})$ *uniformly on the unit simplex in* $\mathbb{R}^d$*, and* $t \downarrow 0$*, with a constant* $\beta > 0$*,*

(ii) $k = o(n^{2\beta}/(1+2\beta)), k \to \infty, n \to \infty,$

(iii) The first order derivatives of $\ell_{\mathbb{X}}(\mathbb{x})$ *with respect to* $x^{(j)}, j = 1, ..., d$*, exist and are continuous.*

Proof of proposition 2.2 According to Einmahl et al. (2012), theorem 4.6, $\sqrt{k}(\hat{\ell}_{\mathbb{X}}(\mathbb{x}) - \ell_{\mathbb{X}}(\mathbb{x}), \mathbb{x} \in [0,1]^d$, is normal with zero mean and covariance matrix equal to a sum of partial derivatives of $\ell_{\mathbb{X}}(\mathbb{x})$ and Brownian bridges. It is assumed that $\ell_{\mathbb{X}}(\mathbb{x}) < \mathbb{x}'\mathbf{1}$ to ensure the asymptotic variance of $\hat{\ell}_{\mathbb{X}}(\mathbb{x})$ is non–zero. This holds if at least one bivariate pair $(X^{(i)}, X^{(j)})$ is asymptotic dependent. In $\mathbb{R}^2$, where $\mathbb{x} = (x^{(1)}, x^{(2)})$, it holds that

$$\sqrt{k}\hat{\ell}_{\mathbb{X}}(x^{(1)}, x^{(2)}) \xrightarrow{d} N(\ell_{\mathbb{X}}(x^{(1)}, x^{(2)}), \sigma_{\ell}^2), x^{(1)}, x^{(2)} > 0,$$

where

$$\begin{aligned}\sigma_{\ell}^2 = {} & \ell_{\mathbb{X}}(x^{(1)}, x^{(2)}) - 2x^{(1)}\ell_{\partial 1}(x^{(1)}, x^{(2)}) - 2x^{(2)}\ell_{\partial 2}(x^{(1)}, x^{(2)}) + x^{(1)}\ell_{\partial 1}^2(x^{(1)}, x^{(2)}) \\ & + x^{(2)}\ell_2(x^{(1)}, x^{(2)}) + 2\ell_{\partial 1}(x^{(1)}, x^{(2)})\ell_{\partial 2}(x^{(1)}, x^{(2)})(x^{(1)} + x^{(2)} - \ell_{\mathbb{X}}(x^{(1)}, x^{(2)}),\end{aligned}$$

with $\ell_{\partial j}(\mathbb{x}) := (\partial \ell_{\mathbb{X}}/\partial x^{(j)})(\mathbb{x})$ denoting the partial derivative of the STDF with respect to argument $x^{(j)}$. According to equations 5 and 6, and setting $\mathbb{x} = \mathbf{1}$, $\widehat{R}_{\mathbb{X}}(\mathbb{x})$ is also asymptotic normal. Asymptotic normality of $\widehat{\Delta}$ directly follows from equation 7. Thus,

$$\widehat{\Delta} \xrightarrow{d} N(\Delta, \sigma_{\Delta}^2),$$

with

$$\sigma_{\Delta}^2 = k^{-1}\sigma_{\hat{\ell}_{\mathbb{X}}}^2 + k^{-1}\sum_{i \in \mathcal{I}_{(2)}^{(d)}} \sigma_{\hat{\ell}_i}^2 + 2\Big(\sum_{i \in \mathcal{I}_{(2)}^{(d)}} Cov(\hat{\ell}_i, \hat{\ell}_{\mathbb{X}}) + \sum_{i,j \in \mathcal{I}_{(2)}^{(d)}, i \neq j} Cov(\hat{\ell}_i, \hat{\ell}_j)\Big) \in (0, \infty).$$

Proof of corollary 2.1
If $\mathbb{X}$ is tail independent, $\ell_{\mathbb{X}}(\mathbb{x}) = \mathbb{x}\mathbf{1} \Leftrightarrow \ell_i(\mathbb{x}^{(i)}) = \mathbb{x}^{(i)}\mathbf{1}$, for all possible bivariate combinations i. Plugging this into the general form of Δ, and realizing that in this case $\sum_{i \in \mathcal{I}_{(2)}^{(d)}} \ell_i(\mathbb{x}^{(i)}) = (d-1)\sum_{i=1}^{d} x^{(i)}$, it follows that

$$\begin{aligned}\Delta &= \ell_{\mathbb{X}}(\mathbb{x}) - 2\sum_{i=1}^{d} x^{(i)} + d\sum_{i=1}^{d} x^{(i)} - \sum_{i \in \mathcal{I}_{(2)}^{(d)}} \ell_i(\mathbb{x}^{(i)}) \\ &= \sum_{i=1}^{d} x^{(i)} - 2\sum_{i=1}^{d} x^{(i)} + d\sum_{i=1}^{d} x^{(i)} - \sum_{i \in \mathcal{I}_{(2)}^{(d)}} \ell_i(\mathbb{x}^{(i)}) \\ &= \sum_{i=1}^{d} x^{(i)} - 2\sum_{i=1}^{d} x^{(i)} + d\sum_{i=1}^{d} x^{(i)} - (d-1)\sum_{i=1}^{d} x^{(i)} \\ &= 0.\end{aligned}$$

Furthermore, $\sigma_{\hat{\ell}_{\mathbb{X}}}^2 = 0$. The test would not reject the null, although the distribution is degenerated.
The reverse does not hold true. E.g. let $\mathbb{X} := (X^{(1)}, X^{(2)}, X^{(3)})$, with $\mathbb{X}^{(3)}$ being independent of $X^{(1)}$, and

let $X^{(1)} \overset{a.s.}{=} X^{(2)}$, i.e. $X^{(1)}$ and $X^{(2)}$ are perfectly tail dependent. Thus, $\ell_{12}(x^{(1)}, x^{(2)}) \equiv \ell_{11}(x^{(1)}, x^{(1)}) = x^{(1)}, \ell_{13}(x^{(1)}, x^{(3)}) = x^{(1)} + x^{(3)}$, and

$$\begin{aligned}
\ell_{123}(x^{(1)}, x^{(1)}, x^{(3)}) &= \lim_{t \downarrow 0} t\mathbb{P}\Big(\bigcup_{i \in \{1,2,3\}} \{X^{(i)} \geq F_i^{-1}(1 - tx^{(i)})\}\Big) \\
&= \lim_{t \downarrow 0} t\mathbb{P}\left(\{X^{(1)} \geq F_1^{-1}(1 - tx^{(1)})\} \cup \{X^{(3)} \geq F_3^{-1}(1 - tx^{(3)})\}\right) \\
&= x^{(1)} + x^{(3)}.
\end{aligned}$$

Rewriting Δ yields

$$\begin{aligned}
\Delta &= \ell_{123}(x^{(1)}, x^{(1)}, x^{(3)}) - 2(2x^{(1)} + x^{(3)}) + 3(2x^{(1)} + x^{(3)}) \\
&\qquad - 2\ell_{11}(x^{(1)}, x^{(1)}) - \ell_{13}(x^{(1)}, x^{(3)}) \\
&= x^{(1)} + x^{(3)} - 2(2x^{(1)} + x^{(3)}) + 3(2x^{(1)} + x^{(3)}) - x^{(1)} - 2(x^{(1)} + x^{(3)}) \\
&= 0.
\end{aligned}$$

Hence, we have tail dependence in $\mathbb{X}$ and Δ is zero as extreme events in dimension three do not matter.

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

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This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

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