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Generalized Exogenous Processes in DSGE: A Bayesian Approach

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Generalized Exogenous Processes in DSGE: A Bayesian Approach *

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Abstract

The Reversible Jump Markov Chain Monte Carlo (RJMCMC) method can enhance Bayesian DSGE estimation by sampling from a posterior distribution spanning potentially nonnested models with parameter spaces of different dimensionality. We use the method to jointly sample from an ARMA process of unknown order along with the associated parameters. We apply the method to the technology process in a canonical neoclassical growth model using post war US GDP data and find that the posterior decisively rejects the standard AR(1) assumption in favor of higher order processes. While the posterior contains significant uncertainty regarding the exact order, it concentrates posterior density on hump-shaped impulse responses. A negative response of hours to a positive technology shock is within the posterior credible set when noninvertible MA representations are admitted.

JEL classification: C11; C32; C51; C52

Keywords: Bayesian analysis; Dynamic stochastic general equilibrium model; Model evaluation; ARMA; Reversible Jump Markov Chain Monte Carlo

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1 Introduction

Despite recent advances in improving the fit of DSGE models to the data, misspecification remains. In his Nobel Prize Lecture, Sims (2012, p. 1202) observes that "DSGEs could be made to fit better by adding parameters allowing more dynamics in the disturbances." Likewise, Del Negro and Schorfheide (2009) identify three approaches to deal with misspecification in rational expectations models: ignore it, generalize the stochastic driving forces, or relax the cross-equation restrictions. Apart from Smets and Wouters (2007) who have the price-markup disturbance follow an ARMA(1,1) process, Del Negro and Schorfheide (2009) who let government expenditures follow an AR(2) instead of an AR(1) process, or Justiniano, Primiceri, and Tambalotti (2008) who replace the ARMA(1,1) specification for the wage and price markup shocks in the Smets and Wouters (2007) model with AR(1) shocks in a robustness exercise, the DSGE literature has not yet provided a systematic framework to address the approach to misspecification of generalizing stochastic driving forces. We fill this gap by providing a Bayesian approach to estimating the order as well as the parameters of generalized ARMA representations of exogenous driving forces within DSGE models.

To accomplish the task, we adopt the Reversible Jump Markov Chain Monte Carlo (RJMCMC) methodology as pioneered by Green (1995).[1] RJMCMC provides samples from a posterior distribution spanning several, not necessarily nested, models with parameter spaces of potentially different dimensionality. In our case, each model is identified by a specific set of orders for the lag polynomials of the autoregressive and moving average components of the disturbances, each leading to a different parameter space. This approach provides a framework for the systematic exploration of the fit of DSGE models using different structures for the shock processes which provides a computationally feasible alternative to estimating all different possible combinations of shock orders individually. Additionally, it allows us to quantify posterior model uncertainty and its consequences for impulse responses and correlation structures while being agnostic regarding the order of the underlying shock processes.[2]

[1] Markov Chain Monte Carlo (MCMC) methods have become increasingly popular for the estimation of DSGE models in recent years. See Fernández-Villaverde and Rubio-Ramírez (2004), An and Schorfheide (2007) for a methodological review, and Herbst and Schorfheide (2014) for a textbook treatment.

[2] If multiple shocks are kept independent while generalizing their individual autocorrelation patterns, the resulting estimates admit a structural interpretation of the shocks that can guide the researcher in identifying those dimensions along which the model requires the most additional internal propagation. It may, furthermore, be possible to construct

The RJMCMC method rests on modifying the proposal ratios in the acceptance probability by inflating parameter vectors to common dimensionality in order to circumvent the dimensionality mismatch induced by sampling for ARMA processes of different orders. In our analysis of US post war GDP data we find that RJMCMC provides point estimates of the ARMA orders with a reliability comparable to traditional order selection criteria such as the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), and the Schwarz Criterion (SC). While the posterior mode models are AR(2) and ARMA (4,5) for first differenced and HP-filtered data respectively, RJMCMC is of primary interest for its posterior distribution over different ARMA orders and not for its point estimates of the orders. We find that the HP filtered GDP data is associated with substantial posterior model uncertainty, as testified to by the dispersed posterior over models provided by our RJMCMC analysis.

We then turn to a prototypical DSGE model, Hansen's (1985) specification of the neoclassical growth model, and relax the traditional AR(1) assumption imposed on the exogenous technology process. After confirming that RJMCMC would correctly identify the ARMA order using synthetic data generated from an AR(1) technology process, we turn to HP filtered US post war GDP data and estimate the order and parameters of the technology process. We find that the data prefers higher order exogenous processes—at the mode, ARMA(3,0), but with substantial posterior density associated with other higher order specifications, such as ARMA(2,2). The resulting posterior impulse responses are hump-shaped, reflecting common wisdom in the macroeconomics literature[3] and differing thus qualitatively from the responses to the traditional AR(1) process. From a DSGE likelihood perspective, there is, without a commensurate prior specification, no reason to prefer invertible or "fundamental" representations in the presence of MA terms; in sampling from the covariance equivalent representations for draws of the order with nonzero MA order, we find a downward shift in the amplitude of the impulse responses as well as an overall increase in the posterior uncertainty regarding the impulse responses of endogenous variables to a technology shock. Strikingly, we cannot exclude the possibility of a negative response of hours to a positive technology shock.

model selection criteria based on the comparison of the spectrum of variables of interest derived from estimates of the posterior with the spectrum using only pure white noise shocks giving a measure of how much structure has to be added to the model outside of economic theory, an idea along the lines of Watson (1993).

[3]See especially, Cogley and Nason (1995).

Our approach can be considered a Bayesian Model Averaging (BMA) method for providing impulse responses and moments under model uncertainty, in that we weigh these statistics from different models with their respective posterior probabilities. While there are certainly alternatives to our approach, for example selecting the model with the highest maximized likelihood or using model selection criteria like the Akaike Information Criterion, BMA allows us to incorporate model uncertainty into the inference of any statistic of interest. The RJMCMC algorithm allows us to explore the posterior adaptively, which would allows for a more efficient means of sampling across models than a brute force BMA approach of generating samples from the posterior of each model (for us, ARMA order combinations p and q) and then weighting according to Bayes factors. The BMA paradigm was put forth by Leamer (1978) and interest in this approach has since increased with the advent of more powerful MCMC samplers. For an overview see Hoeting, Raftery, Madigan, and Volinsky (1999) who also document an improved out-of-sample forecasting performance using BMA, which is also found by Madigan and Raftery (1994) in the context of graphical models. Kass and Raftery (1995) provide a discussion of Bayesian model selection and averaging. A recent application of RJMCMC to instrumental variable regression is presented by Koop, Leon-Gonzalez, and Strachan (2012) and Raftery, Madigan, and Hoeting (1997) discuss the merits of BMA in the context of linear regression models. In a DSGE context, Wolters (2015) uses BMA to provide meta forecasts using multiple estimated DSGE models and Strachan and Van Dijk (2013) use BMA with VARs to assess the empirical support for structural breaks and the long-run and equilibria restrictions implied by a prototypical DSGE model. Our analysis is close in spirit to theirs, yet whereas they apply BMA to estimate VARs restricted commensurate with a DSGE model or provide forecasts using estimated DSGE models, we apply BMA to estimate the DSGE model itself.

This paper is organized as follows: We first introduce our methodology and shortly illustrate the method by constructing a sampler for a univariate autoregressive model of unknown order. Afterwards, we present the results of a small Monte Carlo study designed to gauge the power of the method for identifying univariate autoregressive moving-average models using synthetic data derived from estimated ARMA models of post war US GDP data. Lastly, we apply the method to the neoclassical growth model, using synthetic AR (1) as well as post war US data, and analyze the posterior model uncertainty and its consequences for posterior impulse responses and correlations.

2 Reversible Jump MCMC for ARMA Processes

2.1 Reversible Jump Markov Chain Monte Carlo

In this paper, we adapt and apply the Reversible Jump Markov Chain Monte Carlo (RJMCMC) methodology pioneered by Green (1995). RJMCMC generalizes the Metropolis-Hastings algorithm (Hastings 1970) to allow for moves between parameter spaces of varying dimensionality while maintaining detailed balance.[4] This transdimensionality allows for inference on a posterior distribution spanning several, not necessarily nested, models. In the following, we will illustrate the mechanics of RJMCMC starting with a short description of conventional Metropolis-Hastings samplers to fix ideas before turning to the construction of a sampler for univariate autoregressive models of unknown order using an RJMCMC approach.[5]

2.2 Conventional Metropolis-Hastings Samplers

Markov Chain Monte Carlo (MCMC) methods in general provide samples from some probability distribution of interest by constructing a Markov chain whose stationary distribution is this distribution of interest. A Markov chain with the sequence of states $\varsigma_1, \varsigma_2, \ldots$ is specified in terms of the distribution for the initial state ς_1 and the transition kernel $K(\cdot)$ that provides the conditional distribution of a state ς_{i+1} given the current state ς_i. That is, the probability that ς_{i+1} is in some set $\mathcal{A} \subseteq \mathbb{R}^d$ given that the current state of the chain is ς_i is given by

$$K(\varsigma, \mathcal{A}) = P(\varsigma_{i+1} \in \mathcal{A} | \varsigma_i = \varsigma) \tag{1}$$

A distribution π is invariant for some Markov chain if the transition kernel of the chain satisfies

$$\int K(\varsigma, \mathcal{A})\pi(\varsigma)d\varsigma = \int_{\mathcal{A}} \pi(\varsigma)d\varsigma \tag{2}$$

[4] A more extensive treatment of Metropolis-Hastings samplers can be found in Chib and Greenberg (1995). See also Tierney (1998) for a comparison of RJMCMC and conventional Metropolis-Hastings kernels. Another popular MCMC method is the Gibbs sampler which is a special case of Metropolis-Hastings samplers and ultimately RJMCMC samplers. See Gelfand and Smith (1990) for a review and comparison of Gibbs samplers as well as importance samplers and stochastic substitution and Troughton and Godsill (1998) for application to autoregressive models. Geweke (1998) provides an overview over Bayesian methods and their applications in economics.

[5] Several authors have applied RJMCMC to the problem of estimating univariate autoregressive (moving average) models, e.g., Brooks, Giudici, and Roberts (2003), Brooks and Ehlers (2004), and Ehlers and Brooks (2008).Relatedly, different approaches to statistical models of varying dimensionality have emerged; such as birth-death Markov Chain Monte Carlo, based on continuous time birth-death processes, as initiated by Stephens (2000) and applied to the analysis of autoregressive moving-average models by Philippe (2006). A summary and comparison of these methods can be found in Cappè, Robert, and Rydèn (2003).

for all subsets $\mathcal{A}$ of the state space. The task in MCMC is to construct a kernel such that the distribution of interest π is invariant with respect to the Markov chain defined by $K()$. The expression in (2), however, is not practically useful for the construction of an appropriate kernel, as verifying (2) would involve integration over the unknown distribution π being sought.

One widely used approach to overcome this hurdle are Metropolis-Hastings samplers:[6] accept-reject samplers for which proposals for a new state of the chain are drawn from some distribution γ to be chosen by the researcher and then accepted with an appropriately derived probability α. Here, the stronger condition of reversibility or detailed balance is imposed, which guarantees that π is invariant for the Markov chain. This condition holds if a sequence of two states (ς, ς') has the same distribution as the reversed subchain (ς', ς) whenever $\varsigma, \varsigma' \sim \pi$. I.e., if

$$\int_{\mathcal{A}} \pi(\varsigma) K(\varsigma, \mathcal{B}) \, d\varsigma = \int_{\mathcal{B}} \pi(\varsigma') K(\varsigma', \mathcal{A}) \, d\varsigma' \tag{3}$$

for all subsets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$. Condition (3) is more easily verified and can thus provide a starting point for the construction of a sampler.

A general Metropolis-Hastings algorithm can be written as follows: Let again ς denote a state of the Markov chain, in the case of Bayesian inference in the context of model estimation, the state is just the vector of model parameters and the distribution of interest is the posterior distribution

$$\pi(\varsigma) \propto \mathcal{L}(\varsigma)\rho(\varsigma) \tag{4}$$

where ς denotes the vector of model parameters, $\mathcal{L}$ is the likelihood of the data given the model and its parameters and ρ is the prior over the model parameters. To obtain N samples from the posterior distribution, the following algorithm is run

Metropolis-Hastings

1. Set the (arbitrary) initial state ς_0 of the Markov chain
2. For $i = 1$ to N
   (a) Set $\varsigma = \varsigma_{i-1}$
   (b) Propose a new state from some proposal distribution $\gamma(\varsigma'|\varsigma)$
   (c) Accept draw with probability
   $$\alpha(\varsigma, \varsigma') = \min(1, \chi)$$
   with
   $$\chi = \underbrace{\frac{\mathcal{L}(\varsigma')}{\mathcal{L}(\varsigma)}}_{\text{Likelihood Ratio}} \times \underbrace{\frac{\rho(\varsigma')}{\rho(\varsigma)}}_{\text{Prior Ratio}} \times \underbrace{\frac{\gamma(\varsigma|\varsigma')}{\gamma(\varsigma'|\varsigma)}}_{\text{Proposal Ratio}}$$
   (d) If the draw is accepted set $\varsigma_i = \varsigma'$. If the draw is rejected set $\varsigma_i = \varsigma$

[6] Laid out in Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller (1953) and generalized in Hastings (1970).

This algorithm defines a transition kernel such that the Markov chain has the desired invariant distribution. The sequence of states of the chain is then a sample from this distribution of interest. The acceptance probability α corrects for differences between the proposal distribution γ and the distribution of interest.[7]

The kernel in the above is given by

$$K(\varsigma, \mathcal{B}) = \underbrace{\int_{\mathcal{B}} \gamma(\varsigma'|\varsigma)\alpha(\varsigma, \varsigma')d\varsigma'}_{\text{Probability of moving to set } \mathcal{B}} + \underbrace{\left[1 - \int_{\mathcal{B}} \gamma(\varsigma'|\varsigma)\alpha(\varsigma, \varsigma')d\varsigma'\right] \mathbb{1}_{\varsigma}}_{\text{Probability of rejecting the move and } \varsigma \in \mathcal{B}} \tag{5}$$

where $\mathbb{1}_{\varsigma} = 1$ if $\varsigma \in \mathcal{B}$ and zero otherwise giving the probability of moving to some subset $\mathcal{B}$ of the parameter space conditional on the chain currently being at ς. The crux when constructing the kernel is to define the appropriate acceptance probability α and the proposal distribution γ so as to satisfy the detailed balance condition and thereby guarantee the convergence of the Markov chain to the desired probability distribution. Indeed, plugging in the formulation of the kernel from (5) into (3) gives an expression from which, given the proposal distribution γ the appropriate acceptance probability α can be readily derived using Peskun's (1973) recipe.

2.3 Reversible Jump MCMC: AR(p) Order and Parameter Sampling

We will derive our transdimensional random walk sampler implementation of the RJMCMC with a univariate zero-mean normally distributed AR(p) model of unknown order for illustration. Our derivation follows the exposition of Waagepetersen and Sorensen (2001). Such an AR(p) model is defined as

$$y_t = P_1^p y_{t-1} + P_2^p y_{t-2} + \ldots + P_p^p y_{t-p} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}\left(0, \sigma^2\right) \tag{6}$$

P_i^p are the coefficients of the lag polynomial of order p associated with the i'th lag and ϵ_t is a zero-mean stochastic disturbance. Denote by $P^p \doteq \left\{P_1^p, P_2^p, \ldots, P_p^p\right\}$ the vector of parameters of the AR(p) model.[8] We would like to construct a posterior distribution over the orders, p, and associated parameters, P^p, given observations on y_t.

It is sensible to interpret the order of the lag polynomial p as a model indicator. We will use the terms model indicator and polynomial or lag order interchangeably. The aim is now to

[7] Note, that in the case of a standard random walk Metropolis-Hastings sampler with symmetric proposals, i.e. a Metropolis sampler, the proposal ratio reduces to one.

[8] The part of the parameter vector associated with the standard deviation of the disturbance ϵ_t, σ will be left implicit in the exposition of this section to maintain the focus on the order, p.

construct a sampler for the joint posterior distribution over the different models indexed by p and their parameters. The strategy closely resembles that for Metropolis-Hastings samplers. Indeed, Metropolis-Hastings samplers are a special case in the RJMCMC framework. It is expositionally convenient to express the state of the Markov chain as

$$\varsigma = (p, P^p) \tag{7}$$

explicitly including the order of the autoregressive polynomial p in the state.

The detailed balance condition poses the main obstacle to the transdimensional sampling's construction of a joint posterior distribution over potentially nonnested models with parameter spaces of varying dimensionality. Recall the detailed balance condition (3),

$$\int_{\mathcal{A}} \pi(\varsigma) K(\varsigma, \mathcal{B}) d\varsigma = \int_{\mathcal{B}} \pi(\varsigma') K(\varsigma', \mathcal{A}) d\varsigma' \tag{8}$$

Unlike in the foregoing section, the dimension of ς can change. I.e., the state space of the Markov chain spans parameter spaces with differing dimensionality—for a sampler for AR(p) models of unknown order, when p changes so does the number of parameters. Here, the usual strategy for the derivation of the acceptance probability will fail. Green (1995) modifies the proposals in such a way that the integrals on both sides of the detailed balance condition are over spaces of the same dimensionality by introducing an auxiliary proposal variable u together with a mapping $g_{pp'}$ that maps the auxiliary proposal u and the current state of the chain to the new proposed state. The mapping $g_{pp'}$ is chosen such that the dimensionality of the integrals on both sides of the equation is inflated to some higher common dimensionality.

In order to be able to easily verify adherence to detailed balance for a move from a state (p, P^p) to $(p', P^{p'})$ the vectors of Markov chain states and the random auxiliary proposal variables (P^p, u) and $(P^{p'}, u')$ must be of equal dimension. This dimension matching condition ensures that $\pi(P^p|p)\gamma_{pp'}(P^p, u)$ and $\pi(P^{p'}|p')\gamma_{p'p}(P^{p'}, u')$ are "joint densities on spaces of equal dimension," (Waagepetersen and Sorensen 2001, p. 54) allowing an application of a change of variables in the detailed balance equation to facilitate the construction of the transition kernel of the Markov chain. Here, $\gamma_{pp'}(P^p, u)$ is the proposal density for the auxiliary variable u going from an AR model of order p to one with order p' which may also depend on the current parameter vector P^p. The proposed new order p' is drawn from some $\gamma_p(p'|p)$ and the joint proposal density is $\gamma(\varsigma) = \gamma_{pp'}(P^p, u)\gamma_p(p'|p)$.

In our implementation of the method, we use the following differentiable bijection for $g_{pp'}$

$$\begin{bmatrix} P^{p'} \\ u' \end{bmatrix} = g_{pp'}(P^p, u) = \begin{bmatrix} A(p,p')_{p'\times p} & I_{p'\times p'} \\ I_{p\times p} & 0_{p\times p'} \end{bmatrix} \begin{bmatrix} P^p \\ u \end{bmatrix} \tag{9}$$

where

$$A(p,p') = \begin{cases} \begin{bmatrix} I_{p\times p} \\ 0_{(p'-p)\times p} \end{bmatrix} & \text{if } p' > p \\ \begin{bmatrix} I_{p'\times p'} 0_{p'\times(p-p')} \end{bmatrix} & \text{if } p' < p \\ \begin{bmatrix} I_{p'\times p'} \end{bmatrix} & \text{if } p' = p \end{cases} \tag{10}$$

This mapping leads to the transdimensional analog of a full-site updating random walk sampler. Proposals for "newly born" parameters, i.e., those $P_i^{p'}$ for $i = p+1, \ldots, p'$, are centered around zero. If $p' < p$ the parameter vector is truncated and proposals for these parameters are centered around their previous values. For $p' = p$ this mapping gives a standard random walk sampler.

The detailed balance condition holds if[9]

$$\int_{\mathcal{A}_p} \pi(\varsigma)\, Q\left(\varsigma, \mathcal{B}_{p'}\right) dP^p = \int_{\mathcal{B}_{p'}} \pi(\varsigma')\, Q\left(\varsigma', \mathcal{A}_p\right) dP^{p'} \tag{11}$$

for all subsets $\mathcal{A}_p$ and $\mathcal{B}_{p'}$ of the parameter spaces associated with autoregressive polynomials of order p and p' respectively and where

$$Q\left(\varsigma, \mathcal{B}_{p'}\right) = \int_{\mathcal{B}_{p'}} \gamma(\varsigma'|p, P^p)\alpha_{pp'}(\varsigma, \varsigma')d\varsigma'$$

is the first part of the kernel in (5), i.e. the part of the conditional distribution of ς' associated with acceptance of the proposal.

Implementing the change of variables with the mapping $g_{pp'}$, the detailed balance condition is satisfied if

$$\pi(\varsigma)\,\gamma_p(p'|p)\alpha_{pp'}\gamma_{pp'}(P^p, u) = \pi(\varsigma')\,\gamma_p(p|p')\alpha_{p'p}\gamma_{p'p}(g_{pp'}(P^p, u)) \tag{12}$$

where the details of the derivation can be found in the appendix.

Following Peskun (1973), we set the acceptance probability, $\alpha_{pp'}$, as large as possible,[10]

$$\alpha_{pp'} = \min\left(1, \chi_{pp'}(\varsigma, \varsigma')\right) \tag{13}$$

with

$$\chi_{pp'}(\varsigma, \varsigma') = \underbrace{\frac{\mathcal{L}(\varsigma')}{\mathcal{L}(\varsigma)}}_{\text{Likelihood Ratio}} \underbrace{\frac{\rho(\varsigma')}{\rho(\varsigma)}}_{\text{Prior Ratio}} \underbrace{\frac{\gamma_p(p|p')\gamma_{p'p}(g_{pp'}(P^p, u))}{\gamma_p(p'|p)\gamma_{pp'}(P^p, u)}}_{\text{Proposal Ratio}} \tag{14}$$

Having chosen an appropriate acceptance probability to maintain detailed balanced, we can now implement the procedure. The resulting sequence of states approximates the joint posterior over

[9] See also Waagepetersen and Sorensen (2001).

[10] Which, as noted by Green (1995), is "optimal in the sense of reducing the autocorrelation of the chain."

all models indexed by their order p and the corresponding parameter vectors.

RJMCMC Algorithm

1. Set the initial state ς_0 of the Markov chain
2. For $i = 1$ to N
   (a) set $\varsigma = \varsigma_{i-1}$
   (b) Propose a visit to model p' with probability $\gamma_p(p'|p)$
   (c) Sample u from $\gamma_{pp'}(P^p, u)$
   (d) Set $(P', u') = g_{pp'}(P^p, u)$
   (e) Accept draw with probability
   $$\alpha = \min\left(1, \chi_{pp'}(\varsigma, \varsigma')\right)$$
   $\chi_{pp'}$ is defined as in (14)
   (f) If the draw is accepted set $\varsigma_i = \varsigma'$. If the draw is rejected set $\varsigma_i = \varsigma$

The application to moving average models follows by analogy and the extension to autoregressive moving average (ARMA) models is straightforward. One simply defines the model indicator as a two-element vector, proposing not only visits to some model with autoregressive order p' but also for a new order for the MA-polynomial q'.

For many applications, it is desirable to restrict the parameter spaces of ARMA processes to ensure stationarity and/or invertibility.[11] To constrain sampling to these invertible and stationary regions of the parameters spaces of each model, we reparametrize the AR (and MA) polynomial in terms of its (inverse) partial autocorrelations (PACs). Details are in the appendix.

3 RJMCMC ARMA Order and Parameter Estimation: Monte Carlo Evidence

We examine the performance of the RJMCMC method for ARMA processes of unknown order introduced in the foregoing section by carrying out two Monte Carlo experiments. For both experiments, we compare the model chosen by the posterior mode of our RJMCMC algorithm with the choices that follow from using the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), and the Schwarz Criterion (SC). We orient the Monte Carlo exper-

[11] For the DSGE application in sections 4 and 5, we will require stationarity of the exogenous driving forces. In section 5, we will examine the consequences of imposing or not imposing invertibility on MA components, should they exist, on impulse responses.

iments around the same post war US per capita real GDP data[12] that will inform our DSGE model in the following section by applying our RJMCMC algorith to obtain 3,000,000 draws from the posterior distribution of first-differenced and demeaned quarterly observations of the logarithm of US per capita real GDP as well as 7,000,000 draws from the posterior distribution of the cyclical component of US GDP extracted using a Hodrick-Prescott filter with the smoothing parameter set to 1600 for the period from 1947:1 - 2013:3. The first Monte Carlo is carried out by taking every 30,000th draw from the posterior for first differences and the second with every 70,000th draw from the posterior for HP-filtered data, giving 100 different models each, and then for each generating 250 observations using the corresponding model and parameter values.

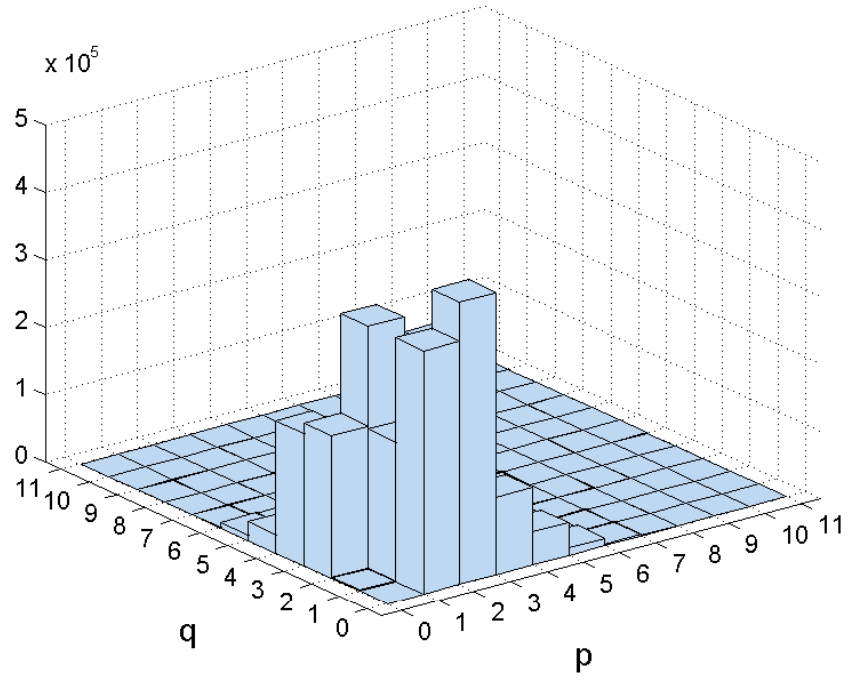

Figure 1: Posterior over the orders p, q for first differenced data

For first-differenced data, the model at the mode is an AR(2), with the posterior mean parameters conditional on the AR(2) model being

$$y_t = 0.3184 y_{t-1} + 0.1297 y_{t-2} + \epsilon_t; \epsilon_t \sim N(0, 0.9025)$$

[12] We take 1947:1-2013:3 real GDP from the NIPA tables, expressed on a per capita basis using the BLS series on the civilian noninstitutional population. Both data sets were downloaded from the St. Louis Federal Reserve's FRED database.

The posterior over models can be found in figure (1). Note that there is a substantial amount of posterior uncertainty regarding the model with textbook representations such as Blanchard and Fischer's (1989, p. 9) ARMA(2,2) estimated on first differenced log GNP comfortably in the posterior distribution over models.

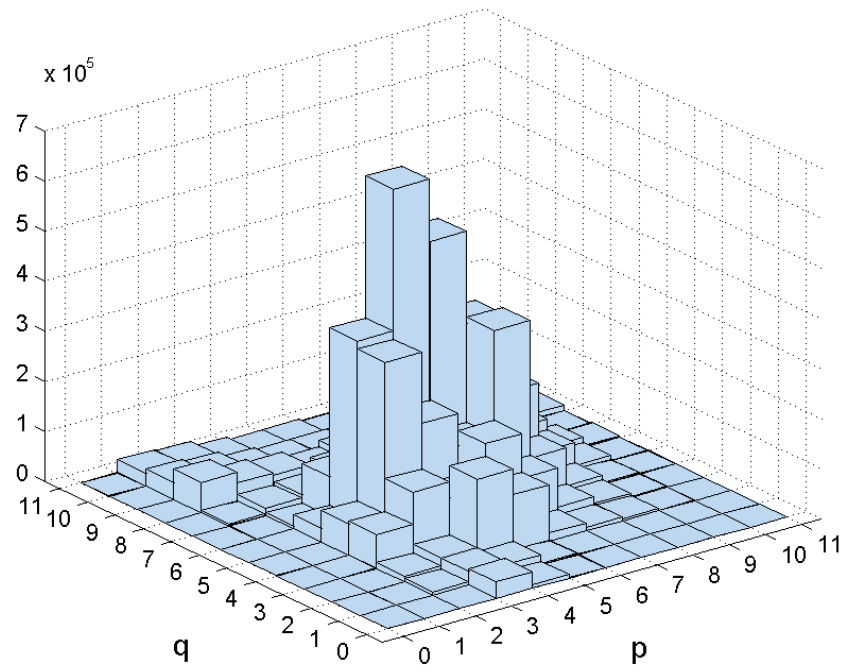

Figure 2: Posterior over the orders p, q for two-sided HP-filtered data

With HP-filtered data, the model at the mode is an ARMA(4,5), with the posterior mean parameters conditional on posterior mode model given by

$$\begin{aligned} y_t =& 0.6027 y_{t-1} + 0.5304 y_{t-2} + 0.0861 y_{t-3} - 0.4196 y_{t-4} + \dots \\ &+ \epsilon_t + 0.3786 \epsilon_{t-1} - 0.2556 \epsilon_{t-2} - 0.5812 \epsilon_{t-3} - 0.2706 \epsilon_{t-4} - 0.2154 \epsilon_{t-5} \\ \epsilon_t &\sim N(0, 0.7551) \end{aligned}$$

Figure (2) shows the posterior distribution over the orders p, q for the HP-filtered data. Clearly, there is significant posterior uncertainty regarding the model reflected in the dispersion of posterior density spread over many more models than was the case with first differenced data. This is consistent with relatively high orders for the lag polynomials preferred at the posterior mode with

many neighboring models mimicking the covariance structure of the model model.

We implement RJMCMC by generating 1,500,000 draws from the posterior, discarding the first 1,000,000 as burn-in, and identifying the model at the mode in (p, q). The first state of the chain was set to white noise with unit standard deviation, i.e. $p = q = 0$ where p denotes the autoregressive order, q the moving average order, and $\sigma = 1$. Our metric for model choice is in accordance with a $1 - 0$ loss function, selecting the model at the mode of the posterior distribution over (p, q). It should be noted that one of the strengths of our method is the ability to quantify posterior uncertainty over models directly, such that model uncertainty can be incorporated in the calculation of posterior credible sets over impulse responses, correlations structures, or the like, providing more than just a point estimate of the model order.

We compare the model choice of our method with the choices that follow from minimizing the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), and the Schwarz Criterion (SC).[13] These are defined as

$$\text{AIC} = 2k - 2\ln(\hat{\mathcal{L}}), \quad \text{AICC} = AIC + \frac{2k(k+1)}{n-k-1}, \quad \text{SC} = -2\ln(\hat{\mathcal{L}}) + k\ln(n)$$

with k being the number of model parameters and n the number of observations. $\hat{\mathcal{L}}$ denotes the maximized likelihood value of a model, i.e., for given ARMA orders p and q.

3.1 Priors and Proposals

Table 1 summarizes the priors and proposals used in the Monte Carlo study. We choose a uniform

Variable	Prior	Proposal
p	U(0,10)	LaplaceD(p,2)
q	U(0,10)	LaplaceD(q,2)
AR PAC	TN(0,0.25)	TN(PAC,0.0025)
MA inverse PAC	TN(0,0.25)	TN(PAC,0.0025)
σ: Standard Deviation ϵ_t	IG(1,1)	TN(σ,0.0025)

Table 1: Prior and Proposal Distribution for Monte Carlo Experiment

prior over the AR and MA orders, restricting the highest allowed order to 10 for both the AR and MA polynomials. Proposals for the AR and MA orders are taken to follow a discretized Laplace

[13] Calculations for the three standard measures were carried out using the R package auto.arima.

distribution, LaplaceD(μ, b), with location parameter, μ, and shape parameter, b, such that

$$\gamma_p(p'|p) \propto exp(-b|p - p'|) \text{ with } p', p \in [0, 1, \ldots, 10] \tag{15}$$

$$\gamma_q(q'|q) \propto exp(-b|q - q'|) \text{ with } q', q \in [0, 1, \ldots, 10] \tag{16}$$

For the (inverse) partial autocorrelations, our prior is a truncated normal distribution, TN($\mu, \sigma, -1, 1$), with location parameter, μ, and dispersion σ, and truncations at 1 and -1, imposing invertibility and stationarity. With these proposal distributions, we center the (inverse) partial autocorrelations around their previous values and new (inverse) partial autocorrelations are centered around zero.

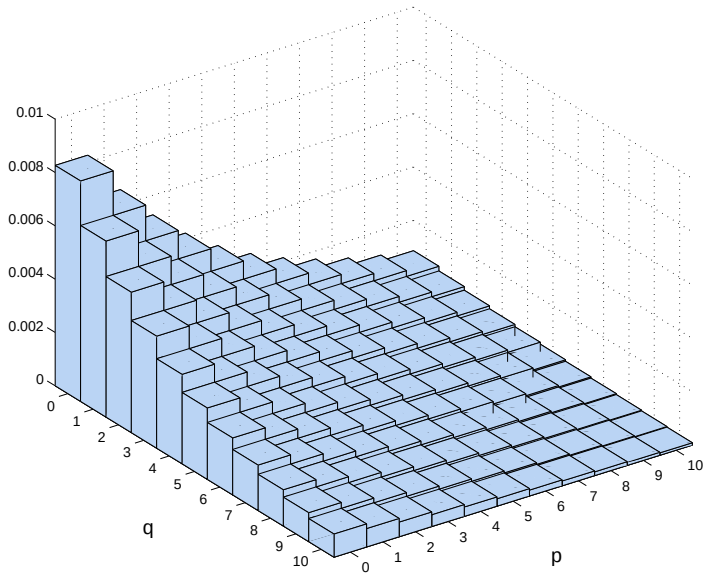

Figure 3: Implied prior over the orders p, q

All three standard information criteria penalize for the number of parameters in the model. This feature is also present in the posterior of our RJMCMC method with proper priors over the (inverse) partial autocorrelations. Increasing the order of, say, an autoregressive model and setting the new parameter to zero gives a model identical to the previous one with lower order; hence, does not change the likelihood. Yet, the posterior with the additional parameter is penalized as the prior probability assigned to the value of the new parameter is smaller than one, yielding a posterior probability lower than with the original, lower order. Even though the prior on the orders is uniform the prior resulting from the combination of the prior over the orders and the prior over

the parameters can be thought of as behaving implicitly like a prior of exponential form as shown in figure (3).

3.2 Likelihood

For the ARMA (p, q) model introduced in (A-12), we employ the Kalman filter to evaluated the log likelihood, $\ln \mathcal{L}\left(\{y_t\}_{t=1}^{T}; \varsigma\right)$, as a sequence of conditional log likelihoods

$$\ln \mathcal{L}\left(\{y_t\}_{t=1}^{T}; \varsigma\right) = \sum_{t=1}^{T} \ln \mathcal{L}\left(y_t | \left\{y_j\right\}_{j=1}^{t-1}; \varsigma\right) = -\frac{1}{2} \sum_{t=1}^{T} \left[\ln \omega_t + \frac{\upsilon_t^2}{\omega_t} + \ln(2\pi)\right] \tag{17}$$

where the last equality follows from the assumption of normality; the sample size is $T = 100$; υ_t is the innovation in the current observation, $\upsilon_t \doteq y_t - E\left[y_t | \left\{y_j\right\}_{j=1}^{t-1}\right]$; and ω_t the conditional variance of this innovation, $\omega_t \doteq E\left[\upsilon_t^2 | \left\{y_j\right\}_{j=1}^{t-1}\right]$.

The innovation and its conditional variance are recovered from the Kalman filter recursion[14] where we follow Harvey (1993, p. 96) in setting up the recursion for ARMA(p,q) processes.[15] The state equation is

$$w_{t+1} = Aw_t + R\epsilon_t, \ \epsilon_t \sim N(0, \sigma^2) \tag{18}$$

and the observation equation is given by

$$y_t = Zw_t \tag{19}$$

where

(20)

$$Z = \begin{bmatrix} 1 & 0_{1\times m-1} \end{bmatrix}, \quad A = \begin{bmatrix} \mathbf{P}_{m-1}^{p,q} & I_{m-1} \\ P_m^{p,q} & 0_{1\times m-1} \end{bmatrix}, \quad \mathbf{P}_{m-1}^{p,q} = \begin{bmatrix} P_1^{p,q} & \dots & P_{m-1}^{p,q} \end{bmatrix}', \quad R = \begin{bmatrix} 1 & Q_1^{p,q} & \dots & Q_m^{p,q} \end{bmatrix}'$$

for $m = \max(p, q+1)$.

3.3 Results

We report the proportion of correctly identified models in table 2. The RJMCMC method outperforms the set of traditional information criteria in all cases except for the model at the posterior mode of HP-filtered data. An increase in the number of the draws from the posterior could further improve the performance of our implementation.

With the generally higher order processes obtained from the posterior obtained using HP-filtered data all methods identify the correct model only in very few cases. This is not surprising

[14] See, e.g., Anderson and Moore (1979).

[15] See de Jong and Penzer (2004) for an overview of alternate state space formulations of ARMA models.

Method	First Differences	HP-Filter
RJMCMC	0.23	0.05
AIC	0.08	0.03
AICC	0.09	0.02
SC	0.18	0.01

Table 2: Proportion of Correctly Identified Models

as the autocorrelation structure of ARMA models of higher orders may be very close even if the orders of the lag polynomials differ and the likelihood is therefore rather flat across models. This was reflected likewise in the posterior distribution over models in the estimation using post war US in figure (2). However, RJMCMC enables the characterization of the resulting uncertainty regarding model selection choices and the posterior therefore provides the researcher with a tool to gauge the extent of model uncertainty.

Of course, the ability of the method to estimate the parameters of the model along with the order of the model is of importance. Figure 4 reports the recursive means of the parameter draws of the model parameters conditional on $p = 2$ and $q = 0$ from a chain from experiment 1 where the model was correctly identified. These values clearly converge close to the values underlying the data generating process.

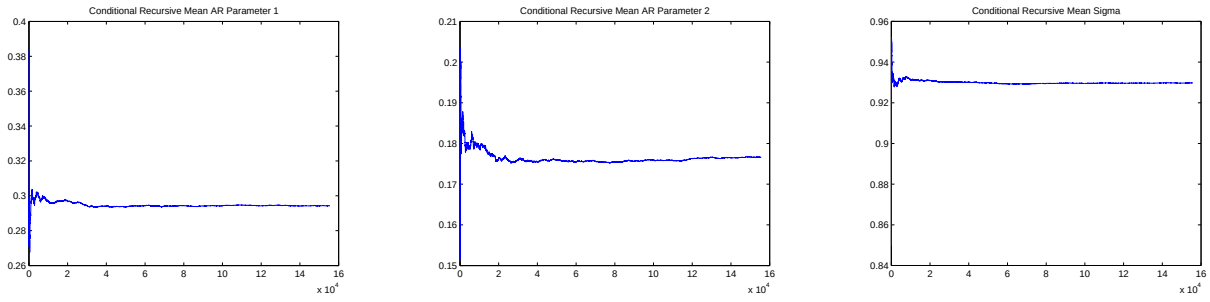

Figure 4: Recursive Parameter Means from the Conditional Posterior

In conclusion, our method exhibits roughly the same or better performance as classical methods concerning order identification while providing a complete posterior distribution over parameters and model orders that can be used for the posterior analysis of statistics of interest. We are interested in posterior statistics of DSGE models such as impulse responses and correlation structures and will now turn to a DSGE setting and apply the RJMCMC method there.

4 Neoclassical Growth Model

As a baseline model to examine how the RJMCMC model can be applied to a DSGE model, we consider Hansen (1985) specification of the neoclassical growth model. In this simple model, the social planner's problem is to maximize the discounted lifetime expected utility of a representative household given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ln\left(c_t\right) + \psi ln\left(1 - l_t\right) \right], \ 0 < \beta < 1 \tag{21}$$

with c_t representing consumption and l_t hours; $\beta \in (0, 1)$ is the subjective discount factor of the household and ψ weights the utility of leisure, $1 - l_t$, in the household's utility function. The social planner faces the resource constraint

$$c_t + i_t = y_t \tag{22}$$

where investment, i_t, contributes to the accumulation of capital, k_t, through

$$k_t = \left(1 - \delta\right) k_{t-1} + i_t \tag{23}$$

with the depreciation rate, δ, and where production, y_t is neoclassical and given by

$$y_t = e^{z_t} k_{t-1}^{\alpha} l_t^{1-\alpha} \tag{24}$$

with z_t being stationary stochastic productivity. Hansen (1985) assumed a highly autocorrelated AR(1) process—with the autoregressive parameter set to 0.95— following Kydland and Prescott (1982). Relaxing this assumption will be the focus of our investigation.

The first order conditions of the social planner's problem are given by

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left(1 - \delta + \alpha e^{z_{t+1}} \left(\frac{l_{t+1}}{k_t} \right)^{1-\alpha} \right) \right] \tag{25}$$

$$\frac{\psi}{1 - l_t} = \frac{1}{c_t} \left(1 - \alpha\right) e^{z_t} \left(\frac{k_{t-1}}{l_t} \right)^{\alpha} \tag{26}$$

An equilibrium is defined by the equations (22) through (26) along with a specification for the stochastic productivity process, z_t.

$\overline{L}$	$\frac{1}{3}$	Steady state employment 1/3 of total time endowment
α	0.36	Capital share
δ	0.025	Depreciation rate for capital
$\overline{R}$	1.01	One percent real interest rate per quarter

Table 3: Model Calibration

In this exercise, we will take the parameters of Hansen's (1985) calibration of all parameters outside the specification of the stochastic productivity process, z_t, as given. This will allow us to concentrate on the contribution of the RJMCMC algorithm in estimating the order and parameters of the exogenous process. The calibrated parameters reported in table 3 deliver standard values for parameters, imposing , e.g., that about one third of agents' time endowment is spent in employment activities, capital contributes a little more than one third to production. As we will consider arbitrary ARMA processes for z_t, the model does not fit canonical DSGE linear problem statements, e.g., Klein (2000), which allow for straightforward calculation of the likelihood function. While we could redefine the model to include the entire state vector induced by the ARMA exogenous process as endogenous variables to bring the model into the canonical form, doing so would significantly increase the computation costs involved in the QZ decomposition for the state transition and the Sylvester equation for the impact matrix of shocks. In the appendix, we provide an extension of multivariate DSGE linear solution methods to arbitrary vector ARMA exogenous driving forces.

5 Estimation Results for the Neoclassical Growth Model Model

We carry out two exercises using the neoclassical growth model model as presented above. First, in order to check whether the method could pick up the correct underlying process for a technology shock in this model, we generated 250 observations of synthetic data using the AR(1) process as reported by Hansen (1985) in his original study. Second, we estimate the order and parameters of the technology shock process for the model using US GDP data, treated with the HP filter as in Hansen's (1985) original study.

5.1 Priors and Proposals

The priors and proposals for the shock process orders and parameters are reported in table 4.

The priors remain the same as in the Monte Carlo study, while the dispersion parameters of the proposals were tuned using short pilot runs to increase the efficiency of the RJMCMC algorithm.

Variable	Prior	Proposal
p	U(0,10)	LaplaceD(p,2.2)
q	U(0,10)	LaplaceD(q,2.2)
AR PAC	TN(0,0.25)	TN(PAC,0.0016)
MA PAC	TN(0,0.25)	TN(PAC,0.0016)
σ	IG(1,1)	TN(σ,0.0025)

Table 4: Priors and Proposals for RBC Model Estimation

5.2 Synthetic AR(1) Data

For this exercise we generated 250 realizations for the technology shock according to the AR(1) specification and calibration in Hansen (1985)

$$z_t = 0.95 z_{t-1} + \epsilon_t \tag{27}$$

We then fed the resulting series for z_t into the linearized RBC model and applied our method to the resulting synthetic data on output, y_t, generating 650.000 draws discarding the first 100.000 draws as burn in. Standard visual measures over the chains indicated convergence. Figure 5 shows the posterior distribution over the orders for the disturbance. The method places an overwhelming majority of the posterior weight on the AR(1) model—obviously correctly identifying the AR(1) data generating process for the productivity process with observations on output, y_t.

This result gives us further confidence that, if the real world process for the productivity shock were AR(1), it would be correctly identified by the RJMCMC method we propose.

5.3 US GDP Data: Estimates

We now address what US postwar GDP data can reveal about the productivity shock in Hansen's (1985) model. We estimated the productivity shock process using HP-filtered quarterly US GDP per capita as in Hansen (1985) taking his original calibration and value of 1600 for the smoothing parameter in the HP filter as given.[16] In applying the RJMCMC method introduced in section 2, we generated 4.000.000 draws discarding the first 1.000.000 draws as burn in. The HP filter was applied to the DSGE model when evaluating the likelihood, thus treating the data and the model with the same filter.[17]

[16] See footnote 12 for details on the data series.

[17] See section A.6 for details.

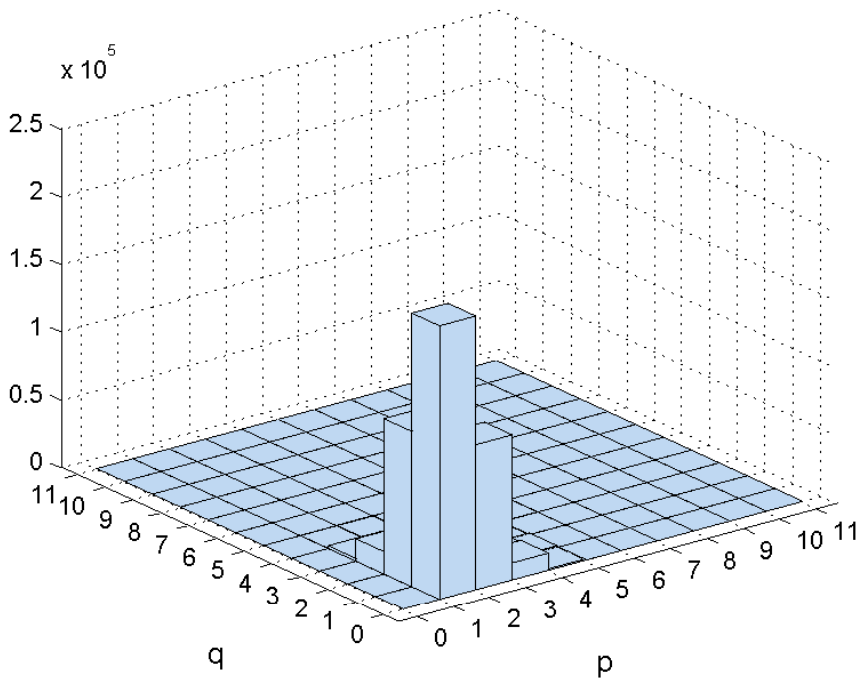

Figure 5: Posterior over the Orders for the Shock Process, Synthetic AR(1) Data from (27)

Figure 6 shows the posterior over (p, q) for this exercise. The model at the mode is ARMA(3,0) and the baseline AR(1) specification of Hansen (1985) is clearly rejected. There is much more substantial uncertainty regarding the correct shock process than in the Monte Carlo exercises above. The prior posterior plots in figure (7) are indicative that our results are not being overly driven by our choice of priors, likewise confirmed by comparing the posteriors over orders in figure 6 to the implied priors in figure 3.

Figure 8 reports recursive means of the first AR parameter for three chains with differing initial states for the orders of the ARMA polynomial for the technology shock, calculated both conditional on the model at the mode of the posterior as well as unconditional means. Inspection suggests that all three chains have converged. It is not clear, however, whether these standard graphical or other formal measures of convergence, e.g., Brooks and Gelman (1998), apply without adaptation in transdimensional analyses, see e.g., Fan and Sisson (2011). In any case, the posterior statistics, such as impulse responses, that we will examine are not indicative of a lack of convergence.

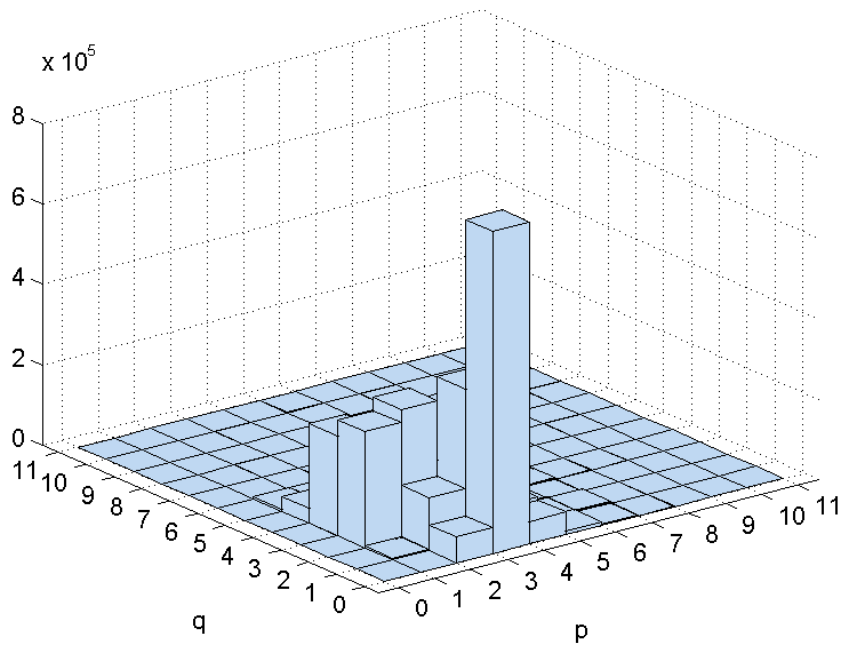

Figure 6: Posterior over the Orders for the Shock Process

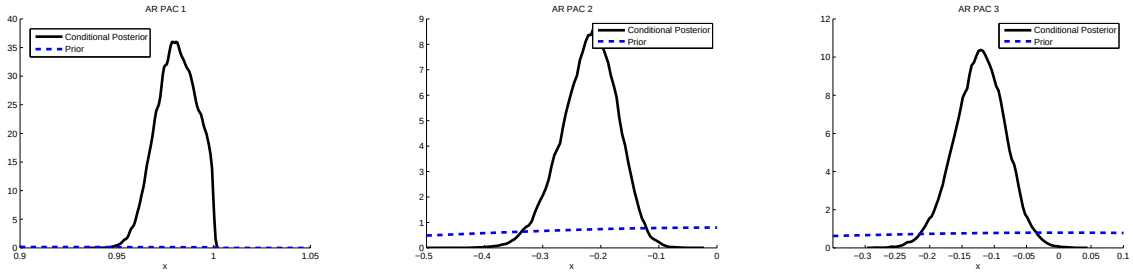

Figure 7: Priors and Posteriors for Partial Autocorrelations

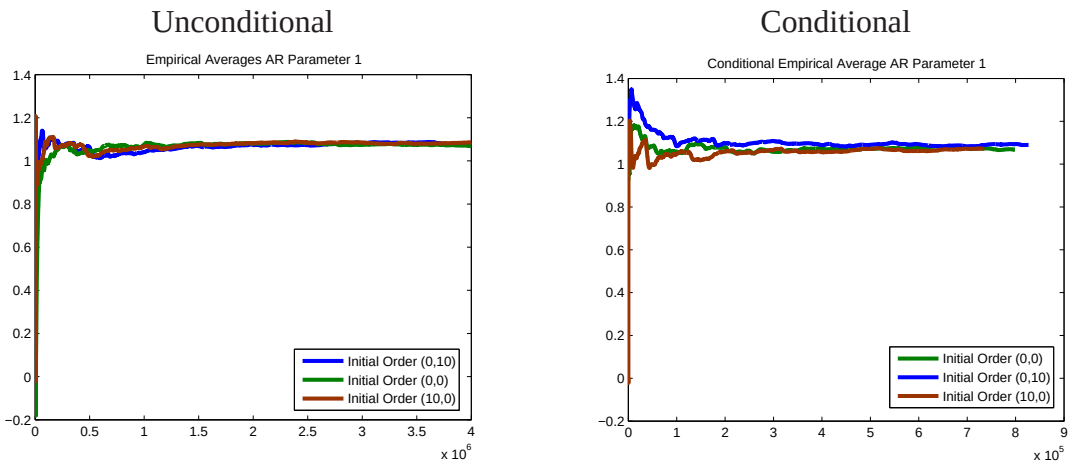

Figure 8: Convergence Diagnostics

Table 5 reports point estimates for the shock process parameters taken from the posterior distribution conditional on $(p, q) = (3, 0)$. Additionally, the first two autocorrelations of the exogenous process, z_t, implied by these point estimates are given. The first autocorrelation is higher than, though consistent with, the choice of Hansen (1985) following Kydland and Prescott (1982) to model the technology process with a near unit root.

Parameter	Mean	Median	Hansen
AR(1)	1.1689	1.1681	0.95
	(0.04)		
AR(2)	-0.0732	-0.0725	N/A
	(0.06)		
AR(3)	-0.1224	-0.1215	N/A
	(0.04)		
σ	0.5873	0.5733	0.712
	(0.08)		
$\rho(1)$	0.9804	0.9810	0.95
$\rho(2)$	0.9528	0.9542	0.9025

Table 5: Posterior Point Estimates and Autocorelations

5.4 US GDP Data: Correlation Structure

We now examine the variance and correlation structures implied by our posteriors and compare these with the data and the statistics implied by our baseline AR(1) model implied by Hansen

(1985).[18] The posterior matches the structure of the second moments of output quite well. As we estimated with real per capita GDP data, this is reassuring and indicates that the procedure does indeed provide a substantial improvement in fit.

Data	Hansen	Posterior Mode Model	Posterior Mode	90% Posterior Credible Set
2.8491	3.2574	2.8332	2.8182	2.1074 — 4.0965

Table 6: Standard Deviation of Output, in %

The standard deviations of output are in table 6. Both the standard deviation of model at the posterior mode of the ARMA order and parameter space and the posterior mode of the standard deviations line up very close to the statistic in the data, whereas the statistic of Hansen (1985) shows greater a difference from the value in the data. The 80% posterior credible set shows the extent of posterior uncertainty, which here is great enough to encompass all the point values reported.

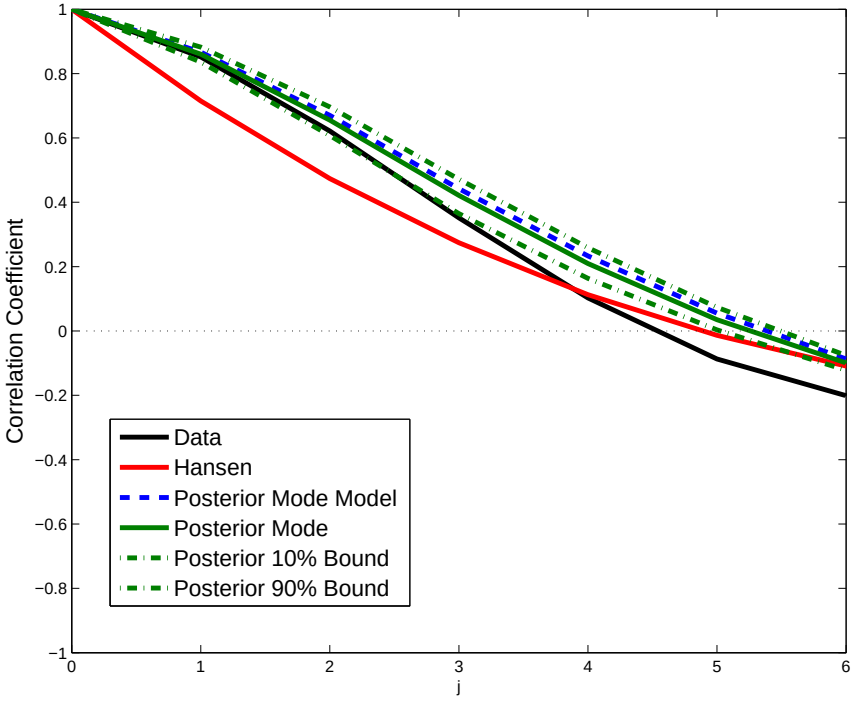

Figure 9: Comparison of Autocorrelations of Output

[18]Following Hansen (1985), we calculate the second moments for his model using an HP filtered (with the smoothing parameter, λ, set to 1600) version of model.

The first six autocorrelations tell a more certain story, however, and can be found in figure 9. Again, both the autocorrelations of the model at the posterior mode of the ARMA order and parameter space and the posterior mode of the autocorrelations match the statistic in the data very closely. The AR(1) structure imposed by Hansen (1985) forces a compromise, with the initial autocorrelation are somewhat lower and the later values somewhat higher than in the data.

The fit as implied by the point estimates of our posterior with respect to our observable series output is reassuring in that our application of the RJMCMC method is successfully doing what it should. With a mean zero normally distributed process, the second moments describe the stochastic properties of the process and our posterior brings the second moments of output from the RBC model closer to the data by selecting appropriate ARMA processes.

5.5 US GDP Data: Impulse Responses

With a posterior distribution over both models—i.e., orders p and q—and their parameters for the ARMA technology process, we plot impulse responses taking posterior uncertainty about the model into account. In the presence of MA components, this requires us to take a stand on which covariance equivalent representation we choose.[19] We will first examine the invertible or fundamental impulse responses associated with the posterior distribution. Then, we will allow the possibility of nonfundamental representations by sampling with a noninformative prior from the admissible (i.e., real valued) covariance equivalent representations and examine the resulting impulse responses.

In figure 10, we plot the impulse responses to a one standard deviation technology shock. We plot the invertible impulse associated with the model at the posterior mode of the ARMA order and parameter space against the pointwise posteriors (mode and 80% credible set) over all impulse responses weighted by posterior probabilities. To guarantee invertibility, we sample from the inverse partial autocorrelations analogously to our sampling from the partial autocorrelations for the AR components that guarantees stationarity. We also include the impulse response with Hansen's (1985) AR(1) technology assumption in the plot. The data driven selection of the specification of the shock process implies a different dynamic behavior of the model compared to Hansen's

[19] See Lippi and Reichlin (1994), Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007), and Alessi, Barigozzi, and Capasso (2011) for more on different MA representations in macroeconomic modeling.

calibration. Our RJMCMC procedure identifies hump-shaped impulse responses, a salient feature of the data identified in many empirical studies; e.g., Cogley and Nason (1995) identify a hump shaped response of output to transitory technology shocks using both an SVAR and a VEC model. In essence, the sluggishness of output in the data that is captured by frictions in more sophisticated models, see especially Sims (1998) for an early assessment, is relegated to the exogenous process by our procedure.

We now move beyond imposing fundamentalness in the sampled MA components. In admitting nonfundamental or noninvertible MA representations, we acknowledge that the covariance structure associated with our posterior distribution potentially implies several possible different structural representations. For an invertible or fundamental moving average representation, the roots, λ_{q_i}, of the MA polynomial

$$\gamma_i(1/\lambda) \doteq \lambda^{q_i} + \gamma_{i,1}\lambda^{q_i-1} \ldots + \gamma_{i,q} \tag{28}$$

must all be contained within the unit circle. That is, there exists no λ such that $\gamma_i(\lambda) = 0$ where $|\lambda| \geq 1$.[20] We follow Lippi and Reichlin (1994) and engage in a root-flipping procedure to construct admissible covariance equivalent representations. We proceed as follows.

Sampling From Admissible Covariance Equivalent Representations

1. For a given draw of order $q > 0$ for the MA component of the exogenous process, factor the MA polynomial as
   $$1 + \gamma_{i,1}L \ldots + \gamma_{i,q}L^{q_i} = (1 - \lambda_1 L)(1 - \lambda_2 L)\ldots\left(1 - \lambda_{q_i}L\right) \tag{29}$$
2. Enumerate all possible combinations of root flips, discarding any combination that would flip only one of complex conjugate pair of roots[21]
3. Draw an integer $n \in \{0, 1, \ldots, \tilde{n}\}$ from a uniform distribution, where $\tilde{n}$ is the number of admissible combinations of root flips
4. Flip the roots according to the combination enumerate with n, where a draw of 0 indicates that no root is flipped (i.e., the invertible or fundamental representation is drawn.

For example, if $n = 10$ is drawn and the number 10 was associated with flipping roots λ_2 and λ_3, the MA polynomial for calculating impulse responses becomes

$$\gamma_i(L) = (-\lambda_2)(-\lambda_3)\left(1 - \frac{1}{\lambda_2}L\right)\left(1 - \frac{1}{\lambda_3}L\right)(1 - \lambda_1 L)(1 - \lambda_4 L)\ldots\left(1 - \lambda_{q_i}L\right) \tag{30}$$

Drawing the covariance equivalent representation from a uniform distribution over all admissible covariance equivalent representations puts equal weight on each admissible representation, reflect-

[20] See, e.g., Hamilton (1994).

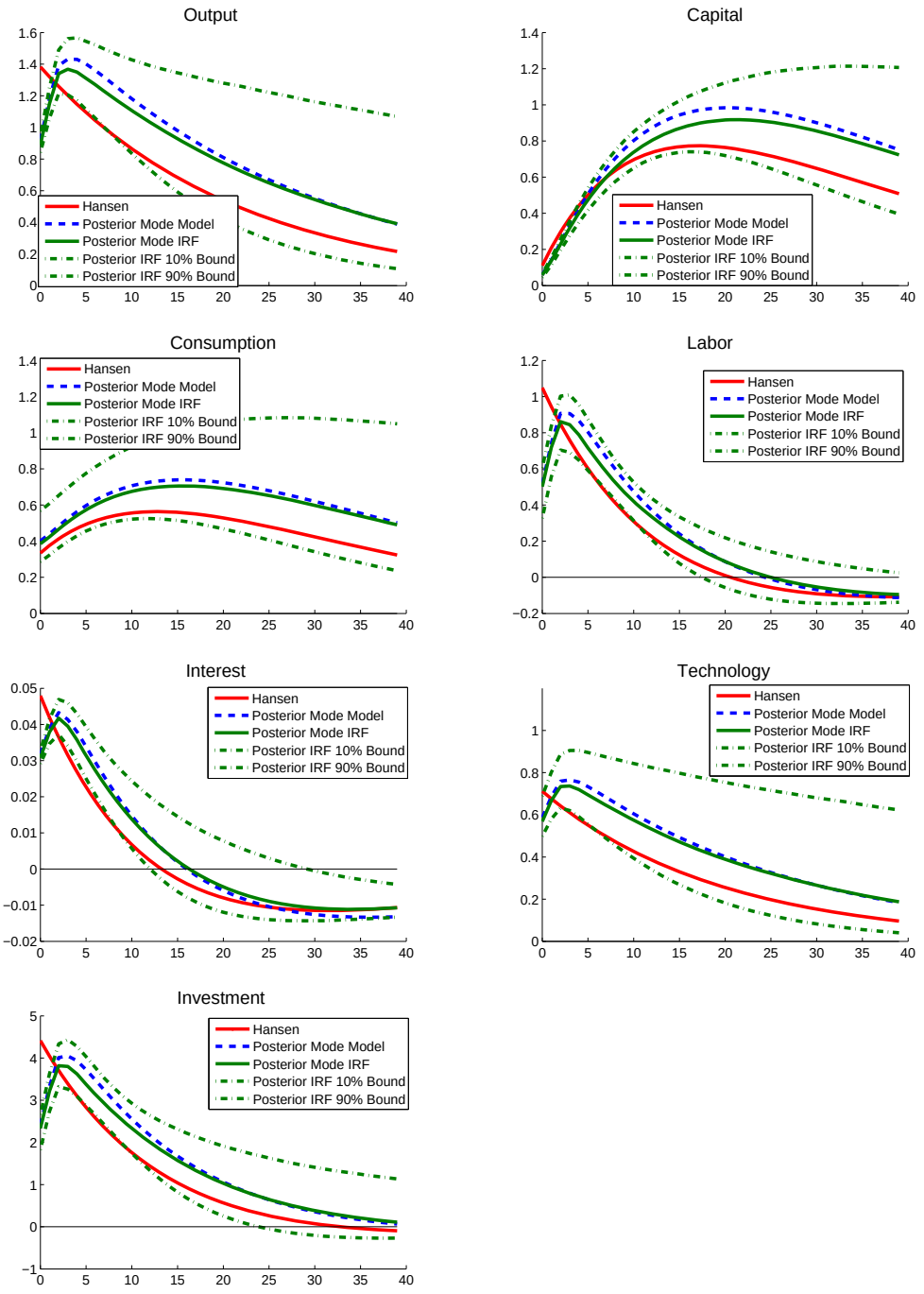

Figure 10: Impulse Responses to a One Standard Deviation Technology Shock
Invertibility of MA Components Imposed

ing our flat prior over the different representations over which DSGE theory is noninformative.

Figure 11 contains the pointwise posteriors (mode and 80% credible set) over all impulse responses weighted by posterior probabilities and drawn, potentially, from nonfundamental covariance equivalent representations as outlined above. We plot these pointwise posteriors against the invertible representation of the model at the posterior mode over ARMA orders and their parameter values and against the impulse response with Hansen's (1985) AR(1) technology assumption. The admission of non-fundamental representations increases our uncertainty over the dynamic response of variables to a technology innovation, spreading the bounds of the 80% credible sets apart. Most of this spread is downward so that the number of periods for which the 80% credible set covers exclusively positive responses to a technology shock is greatly reduced.

Admitting non-fundamental moving average representations places a negative response of hours to a positive technology shock is contained in the credible set. Hence even this simplest real business cycle model with an estimated technology shock process can recreate this stylized observation of Galí (1999) and Francis and Ramey (2005). The conclusion, therefore, that the stochastic growth model is unable to generate this response to technology shocks would require a strong prior against the noninvertible moving average representations, e.g., against news shocks and policy announcement shocks. Though the majority of the posterior mass still lies in a region where the response of hours to technology is conventional, in line with the results in Chari, Kehoe, and McGrattan (2008) and Uhlig (2004).

In sum, the posterior mode model and the posterior distribution over impulse responses, both fundamental and admitting the possibility of non-fundamental moving average representations, as markedly different than those implied by the AR(1) assumption in Hansen's (1985) original study. The data clearly favors hump-shaped impulse responses and cannot rule out a drop in hours in response to a positive technology shock.

6 Conclusion

In this paper we present a novel approach to addressing misspecification in DSGE models. We relax the assumptions usually placed on the structure of exogenous processes, standard practice being AR(1) processes, and estimate generalized, ARMA(p, q) processes of unknown orders. Since

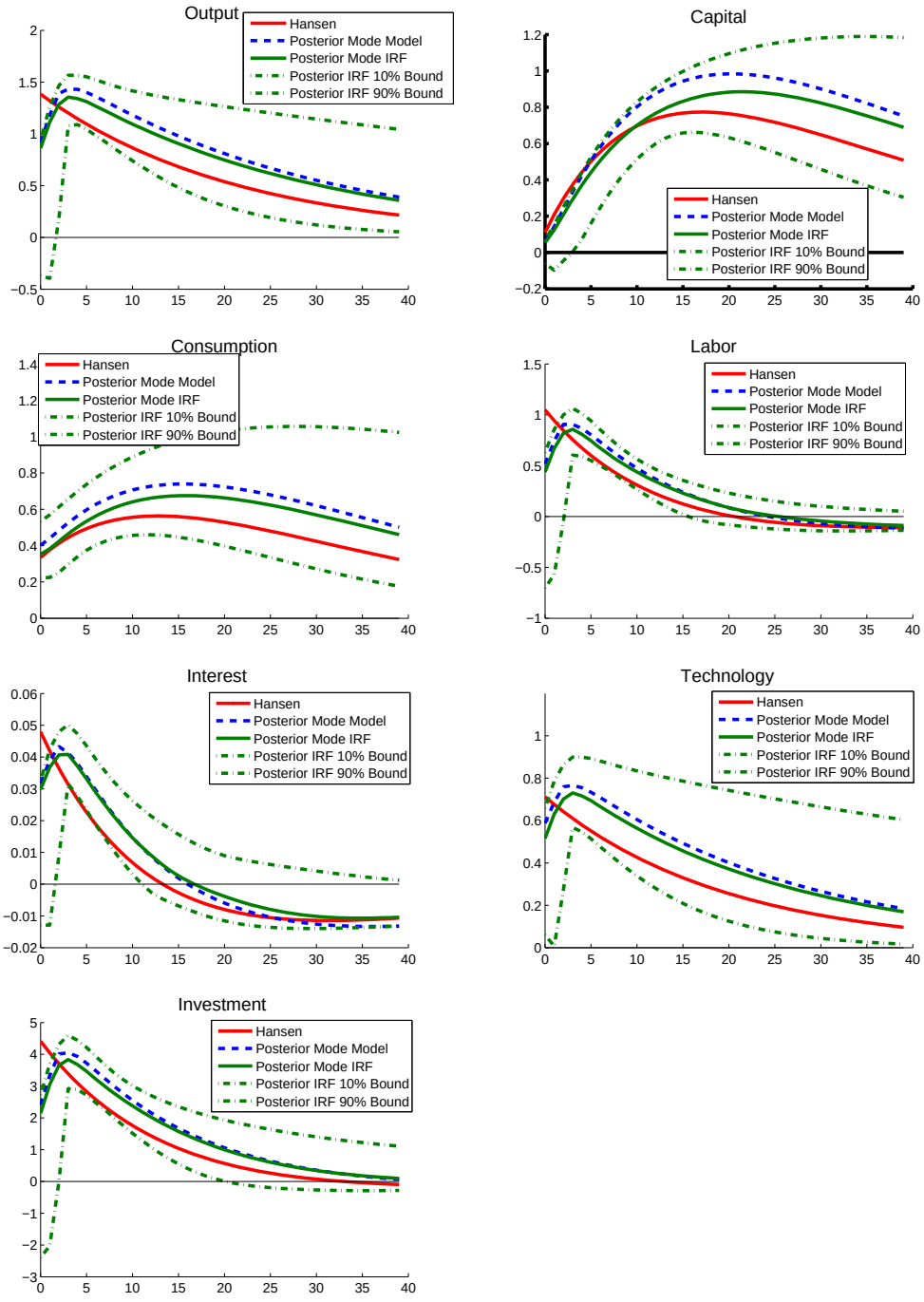

Figure 11: Impulse Responses to a One Standard Deviation Technology Shock
Invertibility of MA Components Not Imposed

theory provides no guidance on autocorrelation patterns of exogenous variables and the order of the these processes in DSGE models is seldom if ever estimated, the usual choice of the AR(1) structure on exogenous processes often lacks any empirical support. Our method treats the ARMA orders of shock processes as additional parameters to be estimated, enabling the researcher to identify those shock process structures which bring the model closer to the data.

The impulse responses implied by the estimated ARMA process for the technology shock using US GDP data with Hansen's (1985) specification of the canonical stochastic neoclassical growth model are markedly different than those generated under the original calibration. Our posterior clearly identifies hump-shaped impulse responses and cannot rule out a drop in hours in response to a positive technology shock.

Our method has the advantage that it will ultimately enable the analysis of a joint posterior over different specifications of the exogenous processes including their parameters as well as parameters of the model, as we are investigating in work in progress. This allows for the quantification of posterior uncertainty regarding the model parameters and all parameters of the exogenous processes including their orders, while maintaining the interpretability of these processes as structural. If one interprets the richer shock structure preferred by our method as a means of controlling for misspecification and, insofar as this misspecification is taken to be policy invariant, the generalized shocks should improve the accuracy of policy experiments while at the same time improving the fit of the model, as indicated in Del Negro and Schorfheide (2009).

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A Appendices

A.1 Detailed Derivation of Inflated Proposal Mapping

To choose an appropriate mapping $g_{pp'}$, it is useful to break the mapping into two parts according to the desired parameters P_p and the auxiliary parameters u. The mapping $g_{pp'}$ is given by

$$(P^{p'}, u') = g_{pp'}(P^p, u) = (g_{1pp'}(P^p, u), g_{2pp'}(P^p, u)) \tag{A-1}$$

and its inverse

$$(P^p, u) = g^{-1}_{pp'}(P^{p'}, u') = g_{p'p}(P^{p'}, u') = (g_{1p'p}(P^{p'}, u'), g_{2p'p}(P^{p'}, u') \tag{A-2}$$

Start with $g_{1pp'}$. Suppose now that the current state of the Markov chain is at $\varsigma = (p, P^p)$. Now with probability $\gamma_p(p'|p)$, a move to the model with order p' is proposed. Conditional on this proposal, we draw u from some proposal distribution $\gamma_{pp'}(u)$. Then, we introduce a deterministic mapping $g_{1pp'}$ that maps the current state and the auxiliary proposal u to the proposed new state such that $(p', P^{p'}) = (p', g_{1pp'}(P^p, u))$. Note that u is not part of the state of the chain.

Additionally, we have to find $g_{2pp'}$. In order to be able to easily verify adherence to detailed balance for a move from a state (p, P^p) to $(p', P^{p'}) = (p', g_{1pp'}(P^p, u))$ the vectors of Markov chain states and the random auxiliary proposal variables (P^p, u) and $(P^{p'}, u')$ must be of equal dimension and requiring $g_{pp'}$ to be a differentiable bijection lets us use a simple change-of-variables in the detailed balance equation. I.e., the kernel of the chain is now defined in terms of the auxiliary variable u together with the model indicator and the parameter vectors.

Armed with this structure it is now straightforward to derive the appropriate acceptance probability. The detailed balance condition holds if[22]

$$\int_{\mathcal{A}_p} \pi(p|y)\,\pi(P^p|p,y)\,Q\left(\varsigma, \mathcal{B}_{p'}\right) dP^p = \int_{\mathcal{B}_{p'}} \pi(p'|y)\,\pi\left(P^{p'}|p',y\right) Q\left(\varsigma', \mathcal{A}_p\right) dP^{p'} \tag{A-3}$$

for all subsets $\mathcal{A}_p$ and $\mathcal{B}_{p'}$ of the parameter spaces associated with autoregressive polynomials of order p and p' respectively. The posterior distribution $\pi(\varsigma|y)$ is factorized as $\pi(\varsigma|y) = \pi(p|y)\pi(P^p|p,y)$ and

$$\begin{aligned} Q\left(\varsigma, \mathcal{B}_{p'}\right) &= \int_{\mathcal{B}_{p'}} \gamma(\varsigma'|\varsigma)\alpha(\varsigma,\varsigma')d\varsigma' \\ &= \gamma_p(p'|p) \int \mathbb{1}(g_{1pp'}(P^p,u) \in \mathcal{B}_{p'})\alpha_{pp'}(P^p, g_{1pp'}(P^p,u)\gamma_{pp'}(P^p,u)du \end{aligned}$$

[22] See also Waagepetersen and Sorensen (2001).

The left hand side of (A-3) is then

$$\int_{\mathcal{A}_p} \pi(\varsigma|y)\, Q\left(\varsigma, \mathcal{B}_{p'}\right) dP^p = \int\int \mathbb{1}(P^p \in \mathcal{A}_p, g_{1pp'}(P^p,u) \in \mathcal{B}_{p'}) \pi(p|y)\, \pi(P^p|p,y) \times \tag{A-4}$$

$$\gamma_p(p'|p)\alpha_{pp'}(P^p, g_{1pp'}(P^p,u)\gamma_{pp'}(P^p,u) dP^p du \tag{A-5}$$

and the right hand side reads

$$\int_{\mathcal{B}_{p'}} \pi(\varsigma'|y)\, Q\left(\varsigma', \mathcal{A}_p\right) dP^{p'} = \int\int \mathbb{1}(P^{p'} \in \mathcal{B}_{p'}, g_{1p'p}(P^{p'},u') \in \mathcal{A}_p) \pi(p'|y)\, \pi\left(P^{p'}|p',y\right) \times \tag{A-6}$$

$$\gamma_p(p|p')\alpha_{p'p}(P^{p'}, g_{1p'p}(P^{p'},u'))\gamma_{p'p}(P^{p'},u') dP^{p'} du' \tag{A-7}$$

where $\gamma(\varsigma'|\varsigma)$ is again factorized as $\gamma_p(p|p)\gamma_{pp'}(P^p,u)$. The fact that $g_{pp'}$ is a differentiable bijection together with the dimension matching conditions enables a change of variable in (A-6) leading to

$$\int\int 1(g_{1pp'}(P^p,u) \in \mathcal{B}_{p'}, P^p \in \mathcal{A}_p) \pi(p'|y)\, \pi\left(g_{1pp'}(P^p,u)|p',y\right) \gamma_p(p|p')$$

$$\times \alpha_{p'p}(g_{1pp'}(P^p,u), P^p)\gamma_{p'p}(g_{1pp'}(P^p,u), g_{2pp'}(P^p,u))|g'_{pp'}(P^p,u)| dP^p du \tag{A-8}$$

where $dP^{p'} du' = |g'_{pp'}(P^p,u)| dP^p du$ and $|g'_{pp'}(P^p,u)|$ is the determinant of the Jacobian of $g_{pp'}$.

By inspection of (A-4) and (A-8), the reversibility condition (A-3) is satisfied if

$$\pi(p|y)\, \pi(P^p|p,y)\, \gamma_p(p'|p)\alpha_{pp'}(P^p, g_{1pp'}(P^p,u))\gamma_{pp'}(P^p,u) =$$

$$\pi(p'|y)\, \pi\left(g_{1pp'}(P^p,u)|p',y\right) \gamma_p(p|p')\alpha_{p'p}(g_{1pp'}(P^p,u), P^p) \times$$

$$\gamma_{p'p}(g_{1pp'}(P^p,u), g_{2pp'}(P^p,u))|g'_{pp'}(P^p,u)| \tag{A-9}$$

Choosing the acceptance probability as large as possible, we have

$$\alpha_{pp'} = \min\left(1, \chi_{pp'}(\varsigma,\varsigma')\right) \tag{A-10}$$

with

$$\chi_{pp'}(\varsigma,\varsigma') = \underbrace{\frac{\mathcal{L}(\varsigma')}{\mathcal{L}(\varsigma)}}_{\text{Likelihood Ratio}} \underbrace{\frac{\rho(\varsigma')}{\rho(\varsigma)}}_{\text{Prior Ratio}} \underbrace{\frac{\gamma_p(p|p')\gamma_{p'p}(g_{pp'}(P^p,u))}{\gamma_p(p'|p)\gamma_{pp'}(P^p,u)} |g'_{pp'}(P^p,u)|}_{\text{Proposal Ratio}} \tag{A-11}$$

With our mapping $g_{pp'}$, in (9), $|g'_{pp'}(P^p,u)|$ is equal to one and (A-11) reduces to (14).[23]

A.2 Imposing Stationarity and Invertibility on ARMA(p,q) Sampling

To constrain sampling to these invertible and stationary regions of the parameters spaces of each model, we follow Barndorff-Nielsen and Schou (1973), Monahan (1984) and Jones (1987) and reparametrize the AR (and MA) polynomial in terms of its (inverse) partial autocorrelations (PACs).

[23]The posterior π is here written factorized as the product of likelihood and prior $\mathcal{L}(\varsigma)\rho(\varsigma)$ for correspondence with the general formulation of the detailed balance condition (3).

If the (inverse) partial autocorrelations are between -1 and 1 the process is (invertible) stationary.

First, we generalize the AR(p) model to an ARMA(p,q) as follows

$$\text{(A-12)} \quad y_t = P_1^{p,q} y_{t-1} + P_2^{p,q} y_{t-2} + \ldots + P_p^{p,q} y_{t-p} + \epsilon_t + Q_1^{p,q} \epsilon_{t-1} + \ldots + Q_q^{p,q} \epsilon_{t-q}, \quad \epsilon_t \sim \mathcal{N}\left(0, \sigma^2\right)$$

In order to recover the coefficients of the AR polynomials, the following algorithm is run

Recovering AR Coefficients from PACs

1. Introduce $p^k = \left(p_1^{(k)}, \ldots, p_k^{(k)}\right), k = 1, \ldots, p$
2. Draw $r = r_1, \ldots, r_p,$ for $r_i \in (0, 1)$ partial autocorrelations
3. Set $p_1^{(1)} = r_1$
4. Run the recursion
$$p_i^{(k)} = p_i^{(k-1)} - r_k p_{k-i}^{(k-1)}, i = 1, \text{ for } \ldots, k-1$$
with $p_k^{(k)} = r_k$ for $k = 2, \ldots, p$
5. Set $P^p = p^{(p)}$

The MA coefficients are recovered analogously, where the inverse partial autocorrelations substitute for the partial autocorrelations, r_i, in the foregoing. Ultimately, instead of proposing AR(MA) parameters directly, (inverse) partial autocorrelations are proposed in their place from which the parameters are then recovered. This will obviously necessitate the formulation of priors over (inverse) partial autocorrelations instead of parameters.

A.3 Class of DSGE Models with VARMA(p,q) Processes

We will consider linear(ized) DSGE models that can be expressed compactly as

$$\text{(A-13)} \quad 0 = E_t \left[\underset{n_x \times 1}{A X_{t+1}} + B X_t + C X_{t-1} + D \underset{n_z \times 1}{Z_t} \right]$$

where the vector X_t collects the endogenous variables and the vector Z_t the exogenous variables. Instead of the standard assumption of independent AR(1) processes for the elements of the vector Z_t,[24] we shall allow each element in Z_t to be driven by an independent ARMA(p,q) process, whose orders p and q along with whose parameters we shall estimate using the RJMCMC algorithm developed in section 2.

The method laid out in section 2 extends straigthforwardly to multiple autoregressive moving averages of finite order.[25] Specifically, we assume that each exogenous process can be represented

[24] Notable exceptions are Cúrdia and Reis (2010) and Chari, Kehoe, and McGrattan (2007), who let their vector of disturbances follow a vector AR(1) process, and Del Negro and Schorfheide (2009) and Smets and Wouters (2007), who let two of their seven disturbances follow ARMA(1,1).

[25] We will examine multiple ARMA processes instead of VARMA (vector autoregressive moving averages) both to

as a finite order ARMA[26]

$$z_{i,t} = \rho_{i,1} z_{t-1} + \rho_{i,2} z_{i,t-2} \ldots + \rho_{i,p_i} z_{i,t-p_i} + \gamma_{i,0}\epsilon_{i,t} + \gamma_{i,1}\epsilon_{i,t-1} \ldots + \gamma_{i,q_i}\epsilon_{i,t-q_i}, \quad \epsilon_{i,t} \sim \mathcal{N}\left(0, \sigma_i^2\right) \tag{A-14}$$

We assume that the processes in (A-14) are stationary and invertible, as we summarize in the following

Assumption A.1. *The roots of the polynomial*

$$\rho_i(\lambda) \doteq \lambda^{p_i} - \rho_{i,1}\lambda^{p_i-1} + \rho_{i,2}\lambda^{p_i-2} \ldots + \rho_{i,p_i} \tag{A-15}$$

are all inside the unit circle. That is, there exists no λ such that $\rho_i(\lambda) = 0$ where $|\lambda| \geq 1$.

Expressed in vector form, the exogenous processes can be collected as

$$Z_t = P_1 Z_{t-1} + P_2 Z_{t-2} \ldots + P_p Z_{t-p} + I \underset{n_z \times 1}{\epsilon_t} + Q_1 \epsilon_{t-1} \ldots + Q_q \epsilon_{t-q}, \quad \epsilon_t \sim \mathcal{N}(0, \Sigma) \tag{A-16}$$

where p is the highest autoregressive order ($p = \max(\{p_i\})$) and q the highest moving average order ($q = \max(\{q_i\})$) among the exogenous processes. The covariance matrix Σ is diagonal, collecting the variances of the individual processes along the diagonal—$\Sigma \doteq \text{diag}\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_{n_z}^2\right)$. The stationarity and invertibility of the individual processes in assumption A.1 transfers to the vector process (A-16), as we state formally as

Lemma A.2. *The latent roots of the λ matrix*

$$I_{n_z}\lambda^p - P_1\lambda^{p-1} + P_2\lambda^{p-2} \ldots + P_p \tag{A-17}$$

That is, there exists no λ such that $\det(P(\lambda)) = 0$ *where* $|\lambda| \geq 1$.

Proof. Follows directly from assumption A.1. □

A.4 Recursive Solution for DSGE Models with VARMA(p,q) Processes

We will solve for a recursive solution for the endogenous variables in the model (A-13) using a method of undetermined coefficients approach. Given (A-16) and (A-13), the state variables of the model are

$$\left\{X_{t-1}, Z_t, Z_{t-1}, \ldots, Z_{t-(\tilde{p}-1)}, \epsilon_t, \epsilon_{t-1}, \ldots, \epsilon_{t-(q-1)}\right\} \tag{A-18}$$

maintain the structural interpretation of the shock and to avoid the proliferation of parameters and reparameterizations, see Monahan (1984), needed to guarantee stationarity in vector processes.

[26] We adopt the convention that sums that terminate with an index smaller than that with which they began are empty sets. For example, if $p_i = 0$ in (A-14) for some i; $\sum_{j=1}^{-0} \rho_{i,j} z_{t-j} = \emptyset$ such that $z_{i,t}$ in this case would be

$$z_{i,t} = \gamma_{i,0}\epsilon_{i,t} + \gamma_{i,1}\epsilon_{i,t-1} \ldots + \gamma_{i,q_i}\epsilon_{i,t-q_i}$$

where $\tilde{p} = \max(p, 1)$,[27]

While we could redefine the model (A-13) to include the entire state vector (A-18) as endogenous variables to bring the model into the canonical form of, say, Sims (2001) or Klein (2000), doing so would significantly increase the computation costs involved in the QZ decomposition for the state transition and the Sylvester equation for the impact matrix of shocks. The solution for the endogenous variables is, accordingly, given by

$$X_t = \Lambda X_{t-1} + \Phi_0 Z_t + \Phi_1 Z_{t-1} \ldots + \Phi_{\tilde{p}-1} Z_{t-(\tilde{p}-1)} + \Theta_0 \epsilon_t + \Theta_1 \epsilon_{t-1} \ldots + \Theta_{q-1} \epsilon_{t-(q-1)} \tag{A-19}$$

where

$$\left\{\Lambda, \Phi_0, \Phi_1, \ldots, \Phi_{\tilde{p}-1}, \Theta_0, \Theta_1, \ldots, \Theta_{q-1}\right\} \tag{A-20}$$

are the unknown coefficients that we solve for.

We will make the following two assumptions that correspond to the Blanchard and Kahn's (1980) order and rank conditions to guarantee a unique stable solution. The order condition assumes a full set of latent roots with half inside and half outside the unit circle

Assumption A.3. *Order*

There exist $2n_x$ *latent roots of* $A\lambda^2 + B\lambda + C$*—that is,* $n_x + \text{rank}(A)$ *finite* $\lambda \in \mathbb{R} : \det A\lambda^2 + B\lambda + C = 0$ *as well as* $n_x - \text{rank}A$ *infinite* λ*—of which* n_x *lie inside the unit circle and* n_x *outside.*

We then assume that a solution, or solvent, can be constructed containing these stable roots

Assumption A.4. *Rank*

There exists an $\Lambda \in \mathbb{R}^{n_x \times n_x}$ *such that* $A\Lambda^2 + B\Lambda + C = 0$ *and* $|eig(\Lambda)| < 1$.

Thus, Λ is the unique solution to the matrix quadratic equation $A\Lambda^2 + B\Lambda + C = 0$ whose eigenvalues coincide with the stable latent roots of the quadratic λ matrix $A\lambda^2 + B\lambda + C$.[28]

Under the order and rank assumptions, as well as the stationarity assumption on the exogenous processes, the model (A-13) has a unique, stable solution, as we summarize in the following proposition

[27] This follows directly from (A-16) expressed in first order vector form

$$\left[Z_t' \quad Z_{t-1}' \quad \ldots \quad Z_{t-(\tilde{p}-1)}' \quad \epsilon_t' \quad \epsilon_{t-1}' \quad \ldots \quad \epsilon_{t-(q-1)}\right]' = PP\left[Z_{t-1}' \quad Z_{t-2}' \quad \ldots \quad Z_{t-p}' \quad \epsilon_{t-1}' \quad \epsilon_{t-1}' \quad \ldots \quad \epsilon_{t-q}\right]' + QQ\epsilon_t$$

for appropriate PP and QQ matrices. The left hand side of the foregoing is then the current exogenous state vector. The case $p = 0$ is permitted through $\tilde{p}$, which ensures Z_t remains on the left hand side of the foregoing despite the indexing convention laid out in footnote 26.

[28] See Lancaster (1966), Dennis, Jr., Traub, and Weber (1976), and Higham and Kim (2000) for detailed analysis of matrix polynomials and λ matrices, as well as Lan and Meyer-Gohde (2014) for an application to DSGE models.

Proposition A.5. *Let assumptions A.3, A.4, and A.2 hold. There exists a unique, stable solution (A-19) to (A-13). The coefficient Λ in (A-19) is the solvent of assumption A.4, the coefficients $\left\{\Theta_0, \Theta_1, \ldots, \Theta_{q-1}\right\}$ for $q > 0$ solve*

$$\begin{aligned}
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Theta_0 + \Phi_0 Q_1 + \Theta_1\right) + B\Theta_0 \\
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Theta_1 + \Phi_0 Q_2 + \Theta_2\right) + B\Theta_1 \\
&\vdots \\
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Theta_{q-2} + \Phi_0 Q_{q-1} + \Theta_{q-1}\right) + B\Theta_{q-2} \\
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Theta_{q-1} + \Phi_0 Q_q\right) + B\Theta_{q-1} \qquad \text{(A-21)}
\end{aligned}$$

and the coefficients $\left\{\Phi_0, \Phi_1, \ldots, \Phi_{p-1}\right\}$ solve

$$\begin{aligned}
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Phi_0 + \Phi_0 P_1 + \Phi_1\right) + B\Phi_0 + D \\
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Phi_1 + \Phi_0 P_2 + \Phi_2\right) + B\Phi_1 \\
&\vdots \\
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Phi_{p-2} + \Phi_0 P_{p-1} + \Phi_{p-1}\right) + B\Phi_{p-2} \\
\underset{n_x \times n_z}{0} &= A\left(\Lambda\Phi_{p-1} + \Phi_0 P_p\right) + B\Phi_{p-1} \qquad \text{(A-22)}
\end{aligned}$$

for $p > 0$ and Φ_0 solves

$$\underset{n_x \times n_z}{0} = A\Lambda\Phi_0 + B\Phi_0 + D$$

otherwise.

Proof. Insert the solution (A-19) for X_t once and for X_{t+1} twice in (A-13), substitute (A-16) lagged forward once for the Z_{t+1} that arises when X_{t+1} is replaced with (A-19), and then collect coefficients on the state variables (A-18). As the solution (A-19) must hold for all values of the state variables, the coefficients just collected must all be zero. The resulting equations are those stated in the proposition. □

We can also calculate an infinite moving average representation for the solution, which will prove useful in the estimation exercise, allowing us to calculate the likelihood spectrally and to apply the closed form frequency domain representation of the HP filter (Hodrick and Prescott 1997) to treat the model with the filter while estimating. Taking the unique stable solution derived above

as given, we define the following λ matrices for the exogenous processes

$$P(\lambda) \doteq I_{n_z} - P_1\lambda - P_2\lambda^2 \ldots - P_p\lambda^p \tag{A-23}$$

$$Q(\lambda) \doteq I + Q_1\lambda \ldots + Q_q\lambda^q \tag{A-24}$$

and for the endogenous transfer function

$$\Phi(\lambda) \doteq \Phi_0 + \Phi_1\lambda \ldots + \Phi_{\tilde{p}-1}\lambda^{\tilde{p}-1} \tag{A-25}$$

$$\Theta(\lambda) \doteq \Theta_0 + \Theta_1\lambda \ldots + \Theta_{q-1}\lambda^{q-1} \tag{A-26}$$

Replacing λ with the lag or backshift operator L,[29] we can express X_t as an infinite moving average, as we summarize in the following proposition

Proposition A.6. *Let assumptions A.3, A.4, and A.2 hold. The unique, stable solution (A-19) to (A-13) for X_t in proposition A.5 has a unique infinite moving average representation given by*

$$X_t = \left(\underset{n_x \times n_x}{I} - \Lambda L\right)^{-1} \left[\Phi(L) P(L)^{-1} Q(L) + \Theta(L)\right] \epsilon_t \tag{A-27}$$

Proof. Invertibility of $\left(\underset{n_x \times n_x}{I} - \Lambda L\right)$ follows from proposition A.5 and that of $P(L)$ from lemma A.2. Uniqueness follows from the uniqueness of the homogenous representation from assumptions A.3 and A.4 and of the uniqueness of the inhomogenous representation from proposition A.5. □

A.5 Solving for the Coefficients in the Recursive Solution for DSGE Models with VARMA(p,q) Processes

For the sequence of coefficients $\{\Phi_i\}_{i=0}^{\tilde{p}-1}$ that measure the impact of the exogenous processes in Z_t on X_t we need to solve (A-22) or (A-23) if $p = 0$. This set of equations can be rewritten by recursive substitution as[30]

$$\Phi_{p-i} = \begin{cases} \sum_{j=1}^{i} \left(-(B + A\Lambda)^{-1} A\right)^j \Phi_0 P_{p-i+j} & \text{for } i = 1, 2, \ldots p-1 \\ \sum_{j=1}^{i} \left(-(B + A\Lambda)^{-1} A\right)^j \Phi_0 P_{p-i+j} - (B + A\Lambda)^{-1} D & \text{for } i = p \end{cases} \tag{A-28}$$

where the invertibility of $B + A\Lambda$ follows from assumptions A.3 and A.4.[31] Thus, given Φ_0 from the $i = p$ case we can recover the remaining matrices Φ_i.

For $i = p$, (A-28) is

$$\Phi_0 = \sum_{j=1}^{p} \left(-(B + A\Lambda)^{-1} A\right)^j \Phi_0 P_j - (B + A\Lambda)^{-1} D \tag{A-29}$$

[29] See, e.g., Sargent (1987).

[30] Starting with the last equation of (A-22). It is already in this form. Then proceed to the second-to-last equation and eliminate Φ_{p-1} in this equation using the last equation. Proceed thusly to the first equation.

[31] See Lan and Meyer-Gohde (2014).

or

$$\Phi_0 + \sum_{j=1}^{p} \left(-\left(B + A\Lambda\right)^{-1} A \right)^j \Phi_0 \left(-P_j \right) = -\left(B + A\Lambda\right)^{-1} D \tag{A-30}$$

which is linear in Φ_0, being a p'th generalized Sylvester equation of the form

$$x + \beta x \gamma_1 + \beta^2 x \gamma_2 \ldots + \beta^p x \gamma_J = \delta \tag{A-31}$$

where $x \doteq \Phi_0$ and $\beta \doteq -\left(B + A\Lambda\right)^{-1} A$.[32]

Proposition A.7. *A generalized Sylvester equation of the form*

$$x + \beta x \gamma_1 + \beta^2 x \gamma_2 \ldots + \beta^J x \gamma_J = \delta \tag{A-32}$$

can be solved recursively for $\underset{n_a \times n_b}{x}$ *as follows*

$$\tilde{x}_{i,\bullet} \left(\sum_{j=0}^{J} \gamma_j U_{i,i}^j \right) = \delta_{i,\bullet} - \left(\sum_{k=1}^{n_a - i} \sum_{j=0}^{J} \{U^j\}_{i,n_a+k} \tilde{x}_{n_a+k,\bullet} \gamma_j \right), \quad \text{for } i = n_a, n_a - 1, \ldots, 1 \tag{A-33}$$

where $\tilde{x} \doteq Q^\dagger x$, $QUQ^\dagger = \beta$ *with* U *upper diagonal and* Q *unitary is the complex Schur decomposition*[33] *of* β, † *indicates conjugate transposition, and* $_{c,d}$ *references the c'th row and d'th column of a matrix.*

Proof. With the Schur decomposition $QUQ^\dagger = \beta$, (A-31) can be rewritten as

$$x + QUQ^\dagger x \gamma_1 + \left(QUQ^\dagger\right)^2 x \gamma_2 \ldots + \left(QUQ^\dagger\right)^J x \gamma_J = \delta \tag{A-34}$$

The matrix Q is unitary, so $Q^\dagger = Q^{-1}$ reducing the foregoing to

$$x + QUQ^\dagger x \gamma_1 + QU^2Q^\dagger x \gamma_2 \ldots + QU^JQ^\dagger x \gamma_J = \delta \tag{A-35}$$

multiplying through with $Q^\dagger$ and using the definition $\tilde{x} \doteq Q^\dagger x$ gives

$$\tilde{x} + U\tilde{x}\gamma_1 + U^2\tilde{x}\gamma_2 \ldots + U^J\tilde{x}\gamma_J = Q^\dagger\delta \tag{A-36}$$

As U is upper diagonal, so is any power of U; thus given all rows of the matrix $\tilde{x}$ after some i, the i'th row of $\tilde{x}$, $\tilde{x}_{i,\bullet}$ solves

$$\sum_{j=0}^{J} U_{i,i}^j \tilde{x}_{i,\bullet} \gamma_j = \delta_{i,\bullet} - \left(\sum_{k=1}^{n_a - i} \sum_{j=0}^{J} \{U^j\}_{i,n_a+k} \tilde{x}_{n_a+k,\bullet} \gamma_j \right) \tag{A-37}$$

recognizing that $U_{i,i}$ is a scalar gives (A-33) which can be solved by multiplying on the right by the inverse of $\left(\sum_{j=0}^{J} \gamma_j U_{i,i}^j\right)$. □

Given Φ_0, the remaining sequence of coefficients $\{\Phi_i\}_{i=1}^{p-1}$ can be recovered recursively from (A-22) starting with Φ_{p-1} and working backwards to Φ_1. Likewise, given Φ_0, the sequence of

[32] For completeness, $\gamma_j \doteq -P_j$, for $j = 1, 2, \ldots, p$ and $\delta \doteq -\left(B + A\Lambda\right)^{-1} D$.

[33] See, e.g., Golub and Van Loan (1996).

coefficients $\{\Theta_i\}_{i=0}^{q-1}$ an be recovered recursively from (A-21) starting with Θ_{q-1} and working backwards to Θ_0.

A.6 DSGE Likelihood with VARMA(p,q) Processes

One difficulty in implementing likelihood methods lies in the evaluation of the likelihood function. As we will consider applying the HP filter to the model when it was applied to the data, the Kalman filter is less desirous here due to the availability of a closed form frequency domain representation for the HP filter, see King and Rebelo (1993). We follow an alternative approach based on the Toeplitz structure of the covariance of stationary time series that uses the iterative method of Meyer-Gohde (2010) for evaluating the likelihood function by treating the sample as a single draw from a multivariate normal distribution,[34] where the derivation of the sequence of autocovariances is done spectrally to enable us to apply the HP filter to the model while evaluating the likelihood function.

Consider now a linear combination of elements of X_t. I.e., the observables, given by

$$Y_t = \underset{n_y \times n_x}{\Upsilon^X} X_t \tag{A-38}$$

To evaluate the likelihood function, we will need to calculate the sequence of autocovariance matrices associated with the observables, Y_t,

$$\Gamma_0 \doteq E\left[Y_t Y_t'\right],\ \Gamma_1 \doteq E\left[Y_t Y_{t-1}'\right],\ \ldots \Gamma_n \doteq E\left[Y_t Y_{t-n}'\right] \tag{A-39}$$

Using the moving average representation of the observables

$$Y_t = \Upsilon^X \left(\underset{n_x \times n_x}{I} - \Lambda L\right)^{-1} \left[\Phi(L) P(L)^{-1} Q(L) + \Theta(L)\right] \epsilon_t \tag{A-40}$$

The autocovariances can be recovered, see, e.g., Sargent (1987), Hamilton (1994), and Uhlig (1999), through

$$\Gamma_n = \int_{-\pi}^{\pi} G(\omega) e^{i\omega n} d\omega \tag{A-41}$$

the inverse Fourier transformation of the spectral density of Y_t, $G(\omega)$ given by

$$\begin{aligned} G(\omega) = &\left[\Upsilon^X \left(\underset{n_x \times n_x}{I} - \Lambda e^{-i\omega}\right)^{-1} \left[\Phi\left(e^{-i\omega}\right) P\left(e^{-i\omega}\right)^{-1} Q\left(e^{-i\omega}\right) + \Theta\left(e^{-i\omega}\right)\right]\right] \\ &\times \Sigma \left[\Upsilon^X \left(\underset{n_x \times n_x}{I} - \Lambda e^{i\omega}\right)^{-1} \left[\Phi\left(e^{i\omega}\right) P\left(e^{i\omega}\right)^{-1} Q\left(e^{i\omega}\right) + \Theta\left(e^{i\omega}\right)\right]\right]' \end{aligned} \tag{A-42}$$

As we will also consider applying the HP filter to the model as well as to the data, we can use

[34] Similarly to Leeper and Sims (1994) and Schmitt-Grohé and Uribe (2010).

closed form representation of the HP filter in the frequency domain, see King and Rebelo (1993), given as

$$HP(\lambda,\omega) = \frac{4\lambda\left(1-\cos(\omega)\right)^2}{1+4\lambda\left(1-\cos(\omega)\right)^2} \tag{A-43}$$

where λ is the HP smoothing parameter and ω a frequency. In this case, the autocovariances of the HP filtered observables can be recovered through

$$\Gamma_n = \int_{-\pi}^{\pi} HP(\lambda,\omega)^2 G(\omega) e^{i\omega n} d\omega \tag{A-44}$$

Given the assumptions of linearity and stationarity behind proposition A.5 and that of the normality of the innovations ϵ_t, T observations on Y_t are normally distributed with mean zero and non-singular block Toeplitz covariance matrix

$$\Psi = \begin{bmatrix} \Gamma_0 & \Gamma_1' & \dots & \Gamma_{T-2}' & \Gamma_{T-1}' \\ \Gamma_1 & \Gamma_0 & \dots & \Gamma_{T-3}' & \Gamma_{T-2}' \\ \vdots & & \ddots & & \vdots \\ \Gamma_{T-2} & \Gamma_{T-3} & \dots & \Gamma_0 & \Gamma_1' \\ \Gamma_{T-1} & \Gamma_{T-2} & \dots & \Gamma_1 & \Gamma_0 \end{bmatrix} \tag{A-45}$$

with the autocovariance matrices, Γ_n, given by (A-41) or (A-44) depending on whether the HP filter was used and the log-likelihood of a vector of parameters ς given the data is thus

$$\mathcal{L}(\varsigma|Y) = -0.5pTln\left(2\pi\right) - 0.5ln\left(det\left(\Psi(\vartheta)\right)\right) - 0.5Y'\Psi(\vartheta)^{-1}Y \tag{A-46}$$

where $X = [Y_1'Y_2'\dots Y_T']'$.

Given (A-45), only two potentially challenging quantities need to be calculated: $ln\left(det\left(\Psi(\vartheta)\right)\right)$ and $X'\Psi(\vartheta)^{-1}X$, which we calculate using the recursive block-Levinson type algorithm of Meyer-Gohde (2010).

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