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Solving DSGE Portfolio Choice Models with Asymmetric Countries

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Abstract

This paper combines the bifurcation theory and the nonlinear moving average approximation to solve asymmetric DSGE models with portfolio choice. The proposed method can be viewed as a generalization of the workhorse routine developed by Devereux and Sutherland (2010, 2011) . Contrary to their approach, it can be used to obtain higher-order approximation of gross asset holdings capturing the direct effect of the presence of risk on agents' portfolios. The risk-adjusted net and gross asset positions are shown to be in line with the global solution. Hence, the proposed method is able to account for asymmetries, which may lead to an accuracy improvement in terms of Euler equation errors relative to the Devereux-Sutherland procedure.

Keywords: Country Portfolios, Solution Method, Asymmetric Countries

JEL Classification Numbers: E44, F41, G11

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1 Introduction

The explosion of cross-border gross asset positions over the last two decades, documented by Lane and Milesi-Ferretti (2001, 2007), has drawn researchers' attention to international portfolios. Obstfeld (2007) writes

> **3. Wanted: A general-equilibrium portfolio-balance model**
> In light of these important implications of international portfolios, it is imperative to understand how investors make asset allocation decisions for different asset classes across countries and currencies. [...] the need for such an approach [i.e. general equilibrium approach] has become acute as asset trade has expanded.

Investigating portfolio choice in a general equilibrium model under the assumption of incomplete markets is challenging, as such models are associated with indeterminacy in a certainty equivalent environment. As a consequence, standard local approximation methods cannot be applied. Furthermore, global methods suffer from the curse of dimensionality and cannot be employed in models with a richer state space. In response to these problems, new solution methods have been developed.[1]

The workhorse routine to solve a DGSE model with portfolio choice is a perturbation-based method developed by Devereux and Sutherland (2010, 2011), henceforth *DS*. It is fast, easy to implement and can be applied to a variety of models. Rabitsch et al. (2015) show that *DS* performs well in comparison to a global solution method, but they also find some scope for improvement in a setup with asymmetric countries. In particular, they document that 1) *DS* does not capture the direct effect of the presence of risk on portfolio holdings[2] and 2) approximates the policy function around net foreign positions equal to zero, even in presence of cross-country differences. Moreover, Rabitsch et al. (2015) show that iterative procedure proposed by Devereux and Sutherland (2009) to update net foreign position deteriorates the accuracy of the approximation. As a result, applying *DS* may yield unsatisfactory results, if, for instance, the focus lies on gross capital flows between developed and emerging market countries.

[1] See, among others, Schmedders et al. (2002), Devereux and Sutherland (2010, 2011), Tille and Van Wincoop (2010), Evans and Hnatkovska (2012), Stepanchuk and Tsyrennikov (2015) and Reiter (2015).

[2] See also Rabitsch and Stepanchuk (2014).

The aim of this paper is to improve upon the two shortcomings of DS. To this end, it combines the bifurcation theory and the nonlinear moving average approximation (Lan and Meyer-Gohde, 2013, 2014*a*). The use of the bifurcation approach overcomes the problem of indeterminacy of portfolio holdings, whereas policy functions approximated with the nonlinear moving average include risk correction evaluated at the stochastic steady state. The proposed technique can be viewed as a generalization of DS. That is, it yields the same results up to first order of accuracy but can also be used to compute higher-order approximations accounting for the presence of risk. The resulting risk correction of gross and net asset positions is in line with the solution provided by global methods. This indicates that asymmetries present in the model are captured already at the starting point of approximation. Moreover, the proposed technique is fast and can handle models with a richer state space. The time necessary to compute a solution of the model considered in this paper amounts to 1.846683 seconds.[3] Finally, the procedure can be easily incorporated in Dynare, a popular software platform for solving DSGE and OLG models.

Including second-order risk correction under the proposed method is shown to improve the quality of the approximation. First, the ergodic mean of gross asset holdings lies closer to its global solution counterpart with the largest discrepancy among the available assets amounting to 3.63 %. By contrast, this figure is nearly twice as large for DS. Second, accounting for the direct effect of risk has the potential to improve the accuracy of the approximation measured by Euler equation errors. The largest documented average accuracy gain is one order of magnitude, whereas the maximum improvement amounts to five orders.

This paper builds mostly on Judd and Guu (2001) who discuss theoretical foundations of bifurcation methods and employ them to solve a partial equilibrium model with portfolio choice. I aim at extending their methods to general equilibrium models. In this regard, my work is closely related to Winant (2014). He independently developed a bifurcation-based solution method for DSGE models with portfolio choice. The main difference between this paper and Winant (2014) is the use of nonlinear moving average. In particular, I show that using standard state space methods instead can lead to highly volatile portfolios.

Implementation of the proposed methodology is based on root-finding algorithms and fixed point iteration techniques. Therefore, this work is also related to the paper by

[3] All experiments are conducted on a desktop computer with Intel® Core™ i5-4690 CPU (3.5 GHz).

Tille and Van Wincoop (2010) who utilize iterative procedures to obtain an approximation to portfolio holdings. However, as their method is virtually the same as *DS* (the only difference being the way of implementing), it suffers from the two aforementioned drawbacks.

The rest of the paper is organized as follows. Section 2 presents the model which is used to explain and evaluate the proposed methodology. Section 3 discusses the key elements of the proposed method and the main steps of the solution algorithm. All results are discussed in section 4. Finally, section 5 concludes.

2 Motivating Example

This section presents the model used in the following to evaluate the proposed local approximation method. It is a version of a real two-country Lucas tree model with portfolio choice employed by Rabitsch et al. (2015). The choice of this particular model enables a direct comparison to the literature and thus speeds up the assessment of proposed method's potential to improve on existing techniques.

Economic environment. It is assumed that the world consists of two countries: Home (H) and Foreign (F). Each country is endowed with two types of income. They are labeled as "capital income" (Y^K) and "labor income" (Y^L) for convenience. Total GDP is thus simply the sum of both types of income, i.e. $Y_{it} = Y^K_{it} + Y^L_{it}$, with $i = \{H, F\}$ being the country index.
The logarithm of country $i's$ income streams follows an autoregressive process of order one:

$$\log(Y^K_{it}) = \rho_K \log(Y^K_{it-1}) + \epsilon^K_{it}, \tag{1}$$

$$\log(Y^L_{it}) = \rho_L \log(Y^L_{it-1}) + \epsilon^L_{it}. \tag{2}$$

Innovations are assumed to be normally distributed and independent across countries but correlation between shocks within a country is allowed to be non-zero, i.e., $\epsilon^j_{it} \sim N(0, \sigma^2_{i\epsilon})$ and $corr(\epsilon^j_{Ht}, \epsilon^j_{Ft}) = 0$, with $j \in \{K, L\}$. Moreover, I introduce asymmetries into the model, by assuming that foreign income stream is twice as volatile as the endowment in the home country. This assumption should capture the empirical observation that emerging market countries are characterized by higher macroeconomic risk (Aguiar and Gopinath, 2007). Thus, the foreign economy can be viewed as a developing country.

Following Lan and Meyer-Gohde (2013), the model is perturbed via future shocks. Hence, all future shocks are scaled by the perturbation parameter σ which governs the size of risk in the model. $\sigma = 0$ implies a deterministic setup, whereas $\sigma = 1$ refers to fully stochastic world.[4]

Household. Country i is populated by a representative household whose preferences are given by the following lifetime utility function:

$$E_0 \sum_{t=0}^{\infty} \phi_{it} \frac{C_{it}^{1-\gamma}}{1-\gamma}. \tag{3}$$

C_{it} stands for a single good consumption and ϕ_{it} is the endogenous discount factor, one of the mechanisms proposed by Schmitt-Grohé and Uribe (2003) to ensure stationarity of the approximate solution under the assumption of incomplete markets. Following Devereux and Sutherland (2011), the endogenous discount factor is given by:

$$\phi_{it} = \bar{\beta} {C_{Ait}}^{-\eta} \phi_{it-1}, \qquad \phi_{i0} = 1. \tag{4}$$

with $\bar{\beta}$ denoting the discount factor in the deterministic steady state. Note that endogenous discount factor does not depend on the individual consumption but on the economy average (C_{Ait}). This assumption prevents the agent from internalizing the effects of her savings choice on the discount factor and thus avoids further complications. In equilibrium, the individual consumption is equal to the aggregate level, as there exists one representative household in each country.

The representative household allocates its wealth between two internationally traded assets which represent claims on "capital income" of the respective country. Because of their definition, assets can be interpreted as equity shares. The resulting budget constraint of the agent in country i can be written as:

$$C_{it} + Q_{Ht}\theta_{it}^{H} + Q_{Ft}\theta_{it}^{F} = (Q_{Ht} + Y_{Ht}^{K})\theta_{it-1}^{H} + (Q_{Ft} + Y_{Ft}^{K})\theta_{it-1}^{F} + Y_{it}^{L}, \tag{5}$$

[4]One should distinguish between σ measuring the size of the risk, $\sigma_{i\epsilon}$ being the standard deviation of shocks in country i, and σ_{iY} denoting the resulting standard deviation of the income.

where Q_{it} denotes the price of claims on country $i's$ "capital income" whereas θ_{it}^H and θ_{it}^F stand for holdings of home and foreign assets respectively.

The household in country i maximizes its lifetime utility subject to the budget restriction. Solving this maximization problem yields the following Euler equations:

$$Q_{Ht} = E_t \left[\beta \frac{C_{it}^{\gamma-\eta}}{C_{it+1}^{\gamma}} (Q_{Ht+1} + Y_{Ht+1}^K) \right] \tag{6}$$

$$Q_{Ft} = E_t \left[\beta \frac{C_{it}^{\gamma-\eta}}{C_{it+1}^{\gamma}} (Q_{Ft+1} + Y_{Ft+1}^K) \right] \tag{7}$$

Market clearing. The goods market clears when

$$Y_{Ht} + Y_{Ft} = C_{Ht} + C_{Ft}. \tag{8}$$

The supply of each asset is normalized to unity, so that financial markets clear if

$$\theta_{Ht}^H + \theta_{Ft}^H = 1, \tag{9}$$

and

$$\theta_{Ht}^F + \theta_{Ft}^F = 1. \tag{10}$$

Note that, because of normalization of the asset supply to one, θ_{Ht}^H can be interpreted as the share of home equity held by home country.

3 Solution Methods

3.1 Preliminaries

Rewriting the model This section discusses solution methods that are employed to solve our example model. To apply local approximation techniques, it is helpful to rewrite the model such that gross asset positions are in zero net supply.[5] To this end, I follow Rabitsch et al. (2015), and define $\alpha_{Ht}^H \equiv \left(\theta_{Ht}^H - 1\right) Q_{Ht}$ and $\alpha_{Ht}^F \equiv \theta_{Ht}^F Q_{Ft}$ as funds invested in home and foreign assets

[5] See Devereux and Yetman (2010).

by the home country. With these definitions, the budget constraint of the home agent can be written as:

$$C_{Ht} + \alpha^{H}_{Ht} + \alpha^{F}_{Ht} = R_{Ht}\alpha^{H}_{Ht-1} + R_{Ft}\alpha^{F}_{Ht-1} + Y_{Ht}, \tag{11}$$

where

$$R_{it} = \frac{Q_{it} + Y^{K}_{it}}{Q_{it-1}} \tag{12}$$

is the rate of return on equity issued by country i. Similarly, the market clearing conditions for financial markets are given by:

$$\alpha^{H}_{Ht} = -\alpha^{H}_{Ft} \tag{13}$$

$$\alpha^{F}_{Ht} = -\alpha^{F}_{Ft} \tag{14}$$

According to (11), consumption in the deterministic steady state depends on the steady-state portfolio holdings. However, as explained below, the latter cannot be pinned down in a non-stochastic environment. This problem will be solved by applying a sequential procedure, where the Nth-order approximation of nonportfolio variables will be computed together with the (N-1)th-order approximation of asset holdings (Samuelson, 1970; Devereux and Sutherland, 2010, 2011). To this end, the budget constraint will be rewritten in terms of net foreign asset holdings (NFA_{Ht}):

$$C_{Ht} + NFA_{Ht} = R_{xt}\alpha^{H}_{Ht-1} + R_{Ft}NFA_{Ht-1} + Y_{Ht}, \tag{15}$$

with

$$NFA_{Ht} = \alpha^{H}_{Ht} + \alpha^{F}_{Ht} \tag{16}$$

and $R_{xt} \equiv R_{Ht} - R_{Ft}$ denoting the excess rate of return on home equity. Market clearing conditions (13) and (14) imply that $NFA_{Ht} = -NFA_{Ft}$.

The main focus of this paper lies on portfolio holdings reflected by α's. It is sufficient to obtain a solution for α^{H}_{Ht} to determine the entire asset holdings structure in the model.[6]. For this reason, I simplify the notation and denote α^{H}_{Ht} as α_t in what follows.

[6]All other α's can be computed via clearing conditions for financial markets and the definition of home net foreign assets.

Equilibrium The full equilibrium of the rewritten model is described by equations (6)-(7), (12), (15)-(16) for both home and foreign country, and market clearing conditions (8), (13)-(14). This gives 13 equations and 12 endogenous variables: $\alpha_H^H, \alpha_H^F, \alpha_F^H, \alpha_F^F, NFA_H, NFA_F, Q_H, Q_F, R_H, R_F, C_H, C_F$, with one equation being redundant by the Walras' law.

Model Solution The model solution can be represented either as a set of state space policy functions (see e.g., Jin and Judd, 2002; Schmitt-Grohé and Uribe, 2004) or as nonlinear moving average policy functions introduced by Lan and Meyer-Gohde (2013). The former approach uses a time-invariant mapping of state variables (y^{state}) and a vector of shocks (ϵ) to model variables (y):

$$y_t = g(\sigma, z_t) \tag{17}$$

where

$$z_t = [y_{t-1}^{state}, \epsilon_t]^{\mathrm{T}}$$

with "T" denoting a transpose.

By contrast, the nonlinear moving average represents a direct mapping of the history of shocks to model variables, i.e.,

$$y_t = y(\sigma, \epsilon_t, \epsilon_{t-1}, ..) \tag{18}$$

Note that size of risk enters as a separate argument in both cases, because it has a direct effect on the policy function.

Due to nonlinearities present in the model, an exact solution is not feasible and thus one must rely on approximation methods. Following Lan and Meyer-Gohde (2014*a*), the Mth-order Taylor approximation of the state space policy function around the deterministic steady state can be written as:[7].

$$y_t^{(M)} = \sum_{j=0}^{M} \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^i \sigma^i} \right] (z_t - \bar{z})^{\otimes[j]} \tag{19}$$

where $(z_t - \bar{z})^{\otimes[j]}$ denotes the jth fold Kronecker product of $(z_t - \bar{z})$ with itself. Due

[7]Consider a matrix valued function $A(x) : \mathbb{R}^{s\times 1} \rightarrow \mathbb{R}^{k\times l}$. Then, A_{x^i} denotes the ith derivative of A with respect to x evaluated at the deterministic steady state.

to the rewritten form of the underlying model the state space is reduced to $y_t^{state} = [Y_{Ht}^K, Y_{Ht}^L, Y_{Ft}^K, Y_{Ft}^L, NFA_{Ht}, Q_{Ht}, Q_{Ft}, \alpha_t]^{\mathrm{T}}$.

On the other hand, taking the Mth-order Taylor approximation of the nonlinear moving average policy function yields

$$y_t^{(M)} = \sum_{m=0}^{M} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} ... \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 i_2 ... i_m} \sigma^n \right] (\epsilon_{t-i_1} \otimes \epsilon_{t-i_2} ... \otimes \epsilon_{t-i_m}) \tag{20}$$

To facilitate the comparison with state space methods, I will exploit the recursive representation of the nonlinear moving average approximation. As argued by Lan and Meyer-Gohde (2014*a*), (20) can be rewritten as follows:

$$y^{(M)} = \sum_{m=0}^{M} \frac{1}{m!} y_{\sigma^m} + \sum_{m=1}^{M} dy_t^{(m)} \tag{21}$$

where $dy_t^{(m)} \equiv y_t^{(m)} - y_t^{(m-1)} - \frac{1}{m!} y_{\sigma^m}$, $m = 1, 2, ...$ denotes the *mth*-order increment in the nonlinear moving average approximation. Lan and Meyer-Gohde (2014*a*) derive a recursive representation for these increments and show that deterministic coefficients in (21) match the ones in the state space policy function. However, there exist differences in the risk correction. Firstly, constant risk adjustment terms in the nonlinear moving average approximation can be directly used to compute an approximation to the stochastic steady state, defined as a fixed point in the presence of risk ($\sigma = 1$), but in absence of shocks $\epsilon_t = 0$ (Meyer-Gohde, 2014). In particular, setting the history of shocks to zero yields the following expression for the stochastic steady state:

$$\bar{y}^{stoch} \approx \sum_{m=0}^{M} \frac{1}{m!} y_{\sigma^m} \tag{22}$$

with $\bar{y} \equiv y_{\sigma^0}$. By contrast, standard state space methods deliver one-step ahead risk adjustment. Therefore, solving for an approximation of the stochastic steady by using the state space representation is not as straightforward as in the case of the nonlinear moving average and involves iterative numerical procedures (see Juillard, 2011 and Coeurdacier et al., 2011).

Going beyond constant risk adjustment, Meyer-Gohde (2014) shows that time-varying risk correction of the nonlinear moving average can be related to first-order derivatives of the underlying policy functions evaluated at the stochastic steady state.

These properties allow the nonlinear moving average approximation to account for risk characteristics of the model as reflected by the starting point of the approximation. As a result, it is suitable to solve DSGE models with portfolio choice as risk considerations play a major role in this setup. The reason for this is that optimal asset holdings are determined by agents' hedging motives.

In the course of this paper, I discuss how one can pin down coefficients in the approximate solution for portfolio holdings. Moreover, I show that it matters for the solution whether the state space approach or the nonlinear moving average is being used.

3.2 Failure of Regular Perturbation Techniques

Solving the model example with perturbation methods involves two difficulties. First, risk is completely eliminated in the deterministic steady state. As the two assets differ only in their risk characteristics, they become then perfect substitutes and yield the same rate of return. This can be seen by investigating the Euler equations (6) and (7). They imply that $\bar{R}_H = \bar{R}_F$, with a bar over a variable standing for its steady state value. As a consequence, countries' gross asset positions cannot be uniquely pinned down in the non-stochastic steady state.

Second, even if indeterminacy of the approximation point is somehow resolved, the first-order approximation is not sufficient to determine the dynamics of portfolio holdings. The first-order approximation of the Euler equation implies:

$$E_t \left[\hat{R}^{(1)}_{Ht+1} \right] = E_t \left[\hat{R}^{(1)}_{Ft+1} \right]. \tag{23}$$

where a hat denotes log-deviations from the deterministic steady state. Thus, up to a first order of accuracy, all assets have the same expected rate of return and portfolio holdings are again indeterminate.[8] Consequently, higher-order perturbations are necessary to obtain approximate dynamics of portfolio holdings.

To explain general implications of the existence of portfolio choice for the perturbation approach, I will cast our example model in a more general form. In particular, as a member of a

[8] This is an implication of the certainty equivalence of first-order approximation (see Schmitt-Grohé and Uribe, 2004).

family of discrete time rational expectations models, it can be written as:

$$E_t\left[f\left(y_{t+1}, y_t, y_{t-1}, \epsilon_t\right)\right] = 0 \tag{24}$$

where $f : \mathbb{R}^{ny} \times \mathbb{R}^{ny} \times \mathbb{R}^{ny} \times \mathbb{R}^{ne} \to \mathbb{R}^{ny}$ is assumed to be analytic, $y_t \in \mathbb{R}^{ny}$ stands for the vector containing both endogenous and exogenous variables, and $\epsilon_t \in \mathbb{R}^{ne}$ is a vector of zero-mean iid shocks.

Standard local approximation methods are based on the Taylor Approximation and the Implicit Function Theorem (Judd, 1998). The idea is to insert policy functions (state space or non-linear moving average) into (24) and apply successive differentiation, where each derivative is evaluated at the non-stochastic steady state. Applying this procedure to find first-order coefficients in the state space policy function and postmultiplying the result with z_y yields:[9]

$$f_{y^+}(g_z z_y)^2 + f_y(g_z z_y) + f_z z_y = 0 \tag{25}$$

(25) is a matrix quadratic equation in $g_z z_y$ measuring the dependence of y on state variables. Note that $g_z z_y$ can be interpreted as a lead operator (henceforth, F) in the absence of shocks. Thus, multiplying (25) by $\hat{y}_{t-1}$ yields a second-order difference equation which in turn can be converted to a first-order system:

$$(DF - E)\hat{x}_t = 0 \quad with \quad D \equiv \begin{bmatrix} 0_{ny\times ny} & I_{ny} \\ f_{y^+} & 0_{ny\times ny} \end{bmatrix}, E \equiv \begin{bmatrix} I_{ny} & 0_{ny\times ny} \\ -f_y & -f_z z_y \end{bmatrix} \quad and \quad \hat{x}_t \equiv \begin{bmatrix} \hat{y}_t \\ \hat{y}_{t-1} \end{bmatrix} \tag{26}$$

A unique solution to (26) can be obtained by using the generalized Schur decomposition of D and E if the the pencil defined by those matrices, $P(z) = Dz - E$, is regular (Klein, 2000), i.e.,

$$\exists a \quad z: \quad |Dx - E| \neq 0$$

However, DSGE portoflio choice models are characterized by a collinear relationship among the Euler equations up to first order of accuracy. As a consequence, the above regularity

[9]Shifting the state space policy function (17) one period into the future yields $y_{t+1} = g^+(\sigma, z_{t+1})$ with $z_{t+1} = [y_t, \sigma\epsilon_{t+1}]^{\mathrm{T}}$ (Lan and Meyer-Gohde, 2014*b*).

condition is violated and there exists a matrix polynomial $\varphi(z)$, such that $\varphi(z)(Dz - E) = 0$ (King and Watson, 1998). Multiplying (26) by $\varphi(F)$ implies $0 = 0$. Thus, there exists infinitely many solutions to (25) with the result that standard perturbation methods cannot be applied to DSGE models embedding a portfolio choice problem.

3.3 Devereux Sutherland Method

Devereux and Sutherland (2010, 2011) aim to obtain the following approximation of the portfolio solution:

$$\alpha_t^{(1)} = \bar{\alpha} + \tilde{g}^{\alpha}_{y^{state}} \hat{y}_t^{(1),state} \tag{27}$$

with '$\sim$' reflecting the fact that the coefficient measures the dependence on the current values of state variables and a hat denoting again the log-deviation from the deterministic steady state.[10] To this end, the authors decompose the model into a *portfolio equation* and a *macroeconomic part* (i.e., the remaining equations). The *portfolio equation* can be obtained by combining the Euler equations:

$$E_t\left[\left(C_{Ht+1}^{-\gamma} - C_{Ft+1}^{-\gamma}\right)(R_{Ht+1} - R_{Ft+1})\right] = 0. \tag{28}$$

In the *macroeconomic part* of the model, portfolio holdings appear only in the budget constraint (15) and are multiplied by the excess return. As mentioned above, this fact allows for a sequential solution strategy. To see this, consider the log-linearized version of the budget constraint:

$$N\hat{F}A_{Ht}^{(1)} = \frac{1}{\beta} N\hat{F}A_{Ht-1}^{(1)} + \frac{1}{\beta \bar{Y}_H} \bar{\alpha} \hat{R}_{xt}^{(1)} - \hat{C}_{Ht}^{(1)} + \hat{Y}_{Ht}^{(1)} \tag{29}$$

where $N\hat{F}A_{Ht} = \frac{NFA_{Ht}}{\bar{Y}_H}$ and $\hat{R}_{xt} = \hat{R}_{Ht} - \hat{R}_{Ft}$. Note that (29) does not include $\hat{\alpha}_t$ so that first-order nonportfolio variables depend only on the steady-state value of α. Moreover, since the expected excess return is zero up to a first-order accuracy, one can eliminate the expression $\frac{1}{\beta \bar{Y}_H} \bar{\alpha} \hat{R}_{xt}^{(1)}$ by introducing an auxiliary wealth shock ($\zeta_t \equiv \frac{1}{\beta \bar{Y}_H} \bar{\alpha} \hat{R}_{xt}^{(1)}$). The *macroeconomic part* can be then solved conditional on this shock. This approximate solution is in turn used to compute zero-order portfolio holdings. Devereux and Sutherland (2010) show that this procedure can be extended to determine first-order portfolio dynamics. In general, portfolio equation needs to be

[10] It does not matter whether the approximate solution links portfolio holdings to the current or past values of states, as both representations are equivalent. (27) follows the convention of *DS*.

approximated up to the order N+2, whereas the *macroeconomic part* to the (N+1)-th order, to be able to pin down the Nth-order component of portfolio holdings (Samuelson, 1970).

3.4 Bifurcation Methods

Standard perturbation techniques cannot be employed to solve DSGE models with portfolio choice as there are infinitely many optimal portfolio holdings when risk is eliminated (i.e. $\sigma = 0$). However, as long as some risk is present, there exists a unique solution, given that standard regulatory conditions are fulfilled (concavity of the objective function etc.). This change in the number of solutions, as the perturbation parameter varies, is an example of a *bifurcation*.

Definition (Bifurcation, Judd and Guu, 2001). *Suppose that $H(\alpha, \sigma)$ is analytic and $\alpha(\sigma)$ is implicitly defined by $H(\alpha(\sigma), \sigma) = 0$. One way to view equation $H(\alpha, \sigma) = 0$ is that for each σ it defines a collection of α that solves it. Bifurcation occurs if number of such α changes as we change σ.*

Bifurcation problems can be tackled by employing the bifurcation theory. In the following, I lay down its two building blocks.

Definition (Bifurcation Point, Zeidler, 1986). *(α_0, σ_0) is a bifurcation point of H iff the number of solutions α to $H(\alpha, \sigma) = 0$ changes as σ passes through σ_0, and there are at least two distinct parametric paths $(\alpha_{A,n}, \sigma_{A,n})$ and $(\alpha_{B,n}, \sigma_{B,n})$ which converge to (α_0, σ_0) as $n \to \infty$.*

Theorem (Bifurcation Theorem for $\mathbb{R}^n$). *Suppose H: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, H is analytic for (α, σ) in a neighborhood of (α_0, σ_0), and $H(\alpha, \sigma_0) = 0$ for all $\alpha \in \mathbb{R}^n$. Furthermore suppose that*

$$\begin{gathered} i)\ H_\alpha(\alpha_0, \sigma_0) = 0 \\ ii)\ H_{\sigma\sigma}(\alpha_0, \sigma_0) = 0 \\ iii)\ det\left(H_{\sigma\sigma\alpha}(\alpha_0, \sigma_0)\right) \neq 0 \end{gathered} .$$

Then (α_0, σ_0) is a bifurcation point and there is an open neighborhood N of (α_0, σ_0) and a function $h(\sigma)$: $\mathbb{R} \to \mathbb{R}^n$, such that h is analytic and $H(h(\sigma), \sigma) = 0$ for $(h(\sigma), \sigma) \in N$.

Proof. See Appendix A. □

The intuition behind the bifurcation theorem can be understood as follows. The original function H, characterized by a first-order singularity in a non-stochastic environment, is replaced by some other function, $\tilde{H}$. Given that $\sigma = 0$, this new function has a zero at the bifurcation point of H. Moreover, since the indeterminacy issue does not apply to $\tilde{H}$, the successive differentiation can be employed again. In the context of a DSGE model, it can be shown that

$$\tilde{H}(\alpha, \sigma) = \begin{cases} \frac{H(\alpha,\sigma)}{\sigma^2} & if \quad \sigma \neq 0 \\ \frac{\partial^2 H(\alpha,\sigma)}{(\partial\sigma)^2} & if \quad \sigma = 0 \end{cases} .$$

In the following, I will demonstrate the practical implementation of the bifurcation theory. To this end, I use firstly state-space methods and then discuss how the approach changes if we the use the nonlinear moving average instead.

3.4.1 State Space Approach

The bifurcation theorem cannot be directly applied to DSGE models with portfolio choice, because it requires that all endogenous variables are indeterminate at the approximation point.[11] This is true only for portfolio holdings, whereas all non-portfolio variables are pinned uniquely in the deteministic steady state. To overcome this problem, I follow Devereux and Sutherland (2010, 2011) and decompose the model into a *portfolio equation* and a *macroeconomic part*.[12] To this end, I will distinguish between several types of model variables and rewrite the model (24) as follows:

$$E_t \left[n \left(y_{t+1}^{fwd}, y_t^{fwd}, y_t^{state}, y_{t-1}^{state}, \alpha_t, \alpha_{t-1}, \epsilon_t \right) \right] = 0 \tag{30}$$

$$E_t \left[m(\mu_{t+1}) \otimes b(r_{t+1}) \right] = 0 \tag{31}$$

where n and b are vector-valued functions with dimensions $ny \times 1$ and $na \times 1$ respectively, whereas $\mathbb{R}$ is the range of m . All functions are assumed to be analytic in the neighborhood of the bifurcation point. Moreover, $y^{fwd} \in \mathbb{R}^{nyfwd}$ denotes forward looking variables, $y^{state} \in \mathbb{R}^{nystate}$ contains both endogenous and exogenous non-portfolio state variables and $\alpha_t \in \mathbb{R}^{na}$ represents gross asset holdings. Note the following link to (24): $ny = nyfwd + nystate + na$. Finally, (31) decomposes y^{fwd} into a vector of rates of returns, $r \in \mathbb{R}^{nr}$, and the remaining forward looking

[11] See condition i) of the bifurcation theorem.
[12] This is also the approach adopted by Winant (2014).

variables, $\mu \in \mathbb{R}^{nmu}$. Thus, $nyfwd = nr + nmu = na + 1 + nmu$. Given a guess for the policy function for α, a unique approximate solution to the real part (30) of the model can be obtained. The approximate solution can then be exploited to express the *portfolio equation* (31) in terms of portfolio holdings, perturbation parameter, future shocks and state variables of the model:

$$
\begin{aligned}
& H\left(\alpha_t, y_t^{state}, \sigma, \epsilon_{t+1} | g^{\alpha}_{guess}\right) \equiv \\
& E_t\left[m\left(g^{\mu}\left(\sigma, \alpha_t, y_t^{state}, \epsilon_{t+1} \middle| g^{\alpha}_{guess}\right)\right) \otimes b\left(g^{r}\left(\sigma, \alpha_t, y_t^{state}, \epsilon_{t+1} | g^{\alpha}_{guess}\right)\right)\right]
\end{aligned}
\tag{32}
$$

where "$|g^{\alpha}_{guess}$" indicates that policy functions for non-portfolio variables has been approximated given a guess for the policy function governing gross asset holdings. The function H defined in (32) fulfills all requirements stated by the bifurcation theorem.

To solve a decomposed version of a DSGE portfolio choice model, a fixed point needs to be found, i.e. $g^{\alpha} = g^{\alpha}_{guess}$. This can be done relatively easily by applying a recursive procedure given that Nth-order components of non-portfolio variables depend only on the (N-1)th-order component of gross asset holdings. As already explained, this is the case in the rewritten version of the example model. Likewise, I will assume throughout the general exposition that this condition is fulfilled.

Assumption (**Recursiveness**). *Nth order components of non-portfolio variables depend only on the (N-1)th order component of portfolio holdings.*

In the following, I will explain how to use the bifurcation theory to compute the first-order approximation of the optimal portfolio:

$$
\alpha_t^{(1)} = \bar{\alpha} + \tilde{g}^{\alpha}_{\sigma} + \tilde{g}^{\alpha}_{y^{state}} \hat{y}_t^{(1),state} \tag{33}
$$

Computing the bifurcation portfolio According to the bifurcation theorem, zero-order portfolio holdings ($\bar{\alpha}$) satisfy the following condition:

$$
\bar{H}_{\sigma\sigma} \equiv H_{\sigma\sigma}|_{\sigma=0, \hat{y}_t^{states}=0} = -2 b_r g^{r}_{\epsilon} \Sigma g^{\mu\top}_{\epsilon} m^{\top}_{\mu} = 0 \tag{34}
$$

where g_ϵ denotes vector of coefficients measuring the dependence on shock realizations and Σ stands for variance-covariance matrix of the underlying shock process. To evaluate (34), one requires the first-order approximation of non-portfolio variables, which in turn depends on the zero-order portfolio holdings. To solve the resulting root-finding problem standard nonlinear solvers can be applied. The iterative procedure can be summarized as follows:

Algorithm 1. *Computing the Bifurcation Portfolio*

1. *Select an error tolerance δ for the stopping criterion and an initial guess for $\bar{\alpha}$.*
2. *Solve the macroeconomic part of the model conditional on the guess.*
3. *Use results from step 2 to evaluate (34).*
4. *Check stopping criterion: if $|\bar{H}_{\sigma\sigma}| < \delta$, the guessed value of $\bar{\alpha}$ represents the bifurcation portfolio. Otherwise, update the guess (according to the numerical procedure used) and go back to step 2.*

Equation (34) coincides with the condition characterizing steady state portfolio holdings computed with *DS*.[13] Thus, I provide a formal proof that *DS* always yields the bifurcation point as steady-state portfolio holdings.

Computing first-order coefficients. Given $\bar{\alpha}$, the bifurcation theorem enables implicit differentiation to pin down first-order coefficients of the approximated policy function:

$$\tilde{g}^{\alpha}_{\sigma} = -\frac{1}{3}\bar{H}^{-1}_{\sigma\sigma\alpha}\bar{H}_{\sigma\sigma\sigma} \tag{35}$$

$$\tilde{g}^{\alpha}_{y^{state}} = -\bar{H}^{-1}_{\sigma\sigma\alpha}\bar{H}_{\sigma\sigma y^{state}} \tag{36}$$

The first-order dynamics of gross asset holdings is driven by time-varying risk components which are reflected by third derivatives of the portfolio equation.

To evaluate (35) and (36), the second-order approximation of non-portfolio variables is necessary.[14] It depends in turn on the first-order dynamics of portfolio holdings. Therefore, the problem at hand takes again the form of a fixed point search and can be solved by applying the following algorithm:

[13]See Devereux and Sutherland (2010), p. 1331, equation (21).
[14]See Appendix B for expressions of the respective derivatives of the portfolio equation.

Algorithm 2. *Computing First-Order Components of Portfolio Holdings*

1. *Select an error tolerance δ for the stopping criterion and an initial guess for $\tilde{g}^{\alpha}_{\sigma}(0)$ and $\tilde{g}^{\alpha}_{y^{state}}(0)$.*

2. *Solve the macroeconomic part of the model conditional on the guess $\tilde{g}^{\alpha}_{\sigma}(k)$ and $\tilde{g}^{\alpha}_{y^{state}}(k)$, where k is the iteration index.*

3. *Use results from step 2 to compute (35) and (36): $\tilde{g}^{\alpha}_{\sigma}(k+1)$ and $\tilde{g}^{\alpha}_{y^{state}}(k+1)$.*

4. *Check the stopping criterion: if $\|p(k+1) - p(k)\| < \delta\,(1 + \|p(k)\|)$ with $p \equiv [\tilde{g}^{\alpha}_{\sigma}, \tilde{g}^{\alpha}_{y^{state}}]^{\mathrm{T}}$, a fixed point has been reached. Otherwise, update the guess and go back to step 2.*

As shown in Appendix B, the first-order risk correction can be expressed as:

$$\tilde{g}^{\alpha}_{\sigma} = \bar{\tau}\Sigma_3 \tag{37}$$

with $\bar{\tau}$ denoting the skew tolerance at the bifurcation point, as in Judd and Guu (2001), and $\Sigma_3 \equiv E_t\left[\epsilon_t^{\otimes[3]}\right]$ referring to the matrix of third moments of the underlying shock structure. Note that (37) implies that $\tilde{g}^{\alpha}_{\sigma}$ is equal to zero under symmetrically distributed shocks. This result can be seen as an extension of the certainty equivalence of the first-order approximation documented by Schmitt-Grohé and Uribe (2004). Furthermore, it explains why Devereux and Sutherland (2010) consider only state variables in their first-order approximation, given that they assume a symmetric distribution.

Computing Second-Order Risk Correction Term The above procedure can be easily extended to pin down coefficients of higher-order approximations of portfolio holdings. The focus of this paper lies on the heterogeneity across countries implying different magnitudes of agents' precautionary motives which will be reflected by risk adjustment terms. Therefore, I will also discuss how the second-order risk correction can be obtained for gross asset holdings. In particular, the bifurcation theorem implies the following expression under normally distributed shocks:

$$\tilde{g}^{\alpha}_{\sigma\sigma} = -\frac{1}{6}\bar{H}^{-1}_{\sigma\sigma\alpha}\bar{H}_{\sigma\sigma\sigma\sigma} \tag{38}$$

The second-order risk adjustment of portfolio holdings is thus driven by forth-order accurate interaction between non-portfolio variables of the model.

The expression (38) can be evaluated with the help of the third-order approximation of non-portfolio variables. Their third-order components depend in turn on second-order asset holdings. Therefore, we face again a fixed point problem which can be solved by employing the following iterative routine:

Algorithm 3. *Computing Second-Order Risk Correction of Portfolio Holdings*

1. *Select an error tolerance δ for the stopping criterion and an initial guess for $\tilde{g}^{\alpha}_{\sigma\sigma}(0)$.*
2. *Solve the macroeconomic part of the model conditional on the guess $\tilde{g}^{\alpha}_{\sigma\sigma}(k)$, where k is the iteration index.*
3. *Use results from step 2 to evaluate (38): $\tilde{g}^{\alpha}_{\sigma\sigma}(k+1)$.*
4. *Check the stopping criterion: if $\|\tilde{g}^{\alpha}_{\sigma\sigma}(k+1) - \tilde{g}^{\alpha}_{\sigma\sigma}(k)\| < \delta\left(1 + \|\tilde{g}^{\alpha}_{\sigma\sigma}\|\right)$, a fixed point has been reached. Otherwise, update the guess and go back to step 2.*

3.4.2 Nonlinear Moving Average

If the nonlinear moving average is used instead of state space methods, the first-order approximation of portfolio holdings is given by:

$$\alpha^{(1)} = \bar{\alpha} + \alpha_{\sigma} + \tilde{g}^{\alpha,nlma}_{y^{state}} dy^{(1),state}_{t} \tag{39}$$

To combine the bifurcation theory with the nonlinear moving average approximation, one can exploit ideas presented in the previous section. In particular, the H function needs to be replaced by H^{nlma} defined as:

$$H^{nlma}\left(\sigma, \alpha_t, \epsilon_{t+1}, \epsilon_t, \epsilon_{t-1}, ...|\alpha_{guess}\right) = H\left[\sigma, \alpha_t, y^{state}_t\left(\sigma, \epsilon_t, \epsilon_{t-1}, ...\right)|\alpha_{guess}\right] \tag{40}$$

Implicit differentiation of (40) yields the following results. Firstly, it does not matter for the bifurcation portfolio, whether the nonlinear moving average or state space methods are being used. Since only the first-order approximation is necessary to compute zero-order asset holdings, it holds, due to certainty equivalence, that $\bar{H}^{nlma}_{\sigma\sigma} = \bar{H}_{\sigma\sigma}$. A similar result applies to the first-order risk adjustment term which is always zero under the normality assumption, no matter which representation of the policy function is being used. On the other hand, state space methods and the nonlinear moving average will imply different first-order dynamics, as reflected by $\tilde{g}^{\alpha}_{x}$. This can be seen by inspecting the following relationship: $\bar{H}^{nlma}_{\sigma\sigma y^{state}} = \bar{H}_{\sigma\sigma y^{state}} + \bar{H}_{y^{state}y^{state}} \left(I_{nystate} \otimes y^{state}_{\sigma\sigma} \right)$ with $I_{nystate}$ denoting the identity matrix with dimension $nystate \times nystate$. Finally, if we want to go beyond the first-order approximation, it can be shown that the second-order risk adjustment term implied by the nonlinear moving average is given by:

$$\alpha_{\sigma\sigma} = g^{\alpha,nlma}_{\sigma\sigma} + g^{\alpha,nlma}_{y^{state}} y^{state}_{\sigma\sigma} = g^{\alpha}_{\sigma\sigma} + \Delta + g^{\alpha,nlma}_{y^{state}} y^{state}_{\sigma\sigma} \tag{41}$$

Note that the one-step-ahead risk correction ($g^{\alpha,nlma}_{\sigma\sigma}$) differs from its state space counterpart as it includes the factor Δ. The reason for this is that excess returns (R_x) does not depend on state variables up to a first-order accuracy along the equilibrium path. However, this is no longer the case for higher-order approximations. Thus, Δ reflects the transition to the second order of accuracy.

4 Numerical Results

This section evaluates three perturbation methods: *DS*, bifurcation used together with the state space approach (henceforth: *BIF*) and a combination of bifurcation methods and the nonlinear moving average (henceforth: *BIFN*). As the nonlinear moving average approximation is automatically pruned, solutions obtained with *DS* and *BIF* are pruned as well for the sake of comparability. The second-order approximation will be pruned with the Kim et al. (2008) algorithm, whereas the procedure developed by Andreasen et al. (2013) will be used for the third-order approximation. An additional advantage of pruning the solution of BIF and DS is the possibility to represent all three methods in a unified state space. In particular, Lan and Meyer-Gohde (2014*a*) express the above pruning algorithms recursively in terms of approximation increments, exactly as in the case of the nonlinear moving average.

Parameter		Value
Discount factor in deterministic steady state	$\bar{\beta}$	0.95
Elasticity of the endogenous discount factor	η	0.001
Risk aversion	γ	2
Capital income share	$\frac{\bar{Y}_K}{\bar{Y}}$	0.3
Persistence	ρ	0.8
Volatility of output in Home	$\sigma_{Y_K}, \sigma_{Y_L}$	0.02
Volatility of output in Foreign	$\sigma_{Y_K^*}, \sigma_{Y_L^*}$	0.04
Correlation	$corr(Y_K, Y_L)$	0.2

Table 1: Calibration

4.1 Calibration

In all numerical exercises, I employ calibration used by Rabitsch et al. (2015). This allows me to use their global solution[15] as a benchmark for evaluating accuracy of the proposed technique. Table 1 reports the chosen parameter values. Almost all of them are commonly used in the macroeconomic literature. The only exception is the consumption elasticity of the endogenous discount factor which is set to 0.001, whereas the standard choice is 0.022 (Mendoza, 1991; Schmitt-Grohé and Uribe, 2003). A small value of η aims at minimizing the effect of this stationarity-inducing device on the predictions of the model.[16] Note that this assumption implies a high persistence of net foreign assets, as the corresponding eigenvalue is close to unity. This property of the model will make lengthy simulations necessary to compute underlying ergodic distributions.

4.2 First-Order Accurate Dynamics

Before considering the role of risk correction, I evaluate first-order differences between *BIF* and *BIFN* documented in the previous section. The latter technique yields the same first-order dynamics as *DS*. On the other hand, portfolio holdings implied by BIF are more volatile.[17]

[15] Rabitsch et al. (2015) use time iteration spline collocation algorithm to solve the model globally.

[16] See Rabitsch et al. (2015) for a more detailed discussion.

[17] In a symmetric setup, all three methods imply the same dynamics of portfolio holdings. The reason for this are identical precautionary motives of economic agents that offset each other.

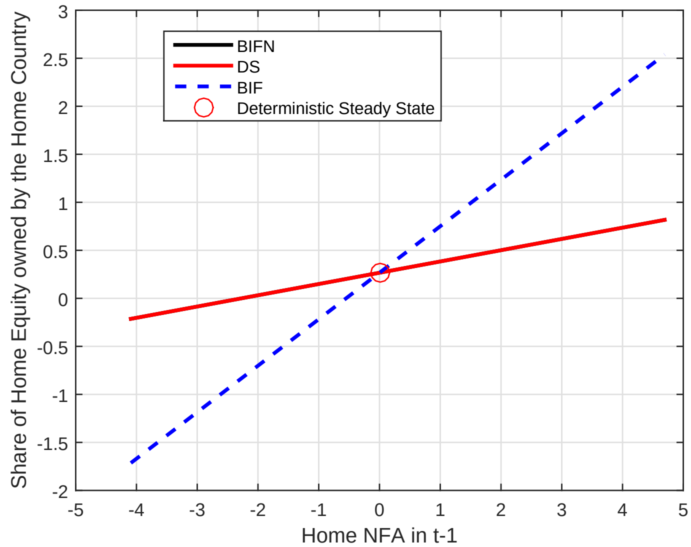


Figure 1: First-Order Accurate Share of Home Equity owned by the Home Country. *Policy functions are depicted in an interval based on the ergodic set of the home NFA implied by DS. The ergodic set is defined as an interval covering 95 % of the probability mass of the underlying distribution. It is determined by simulating 10 million periods and subsequently discretized by 1001 equidistant grid points.*

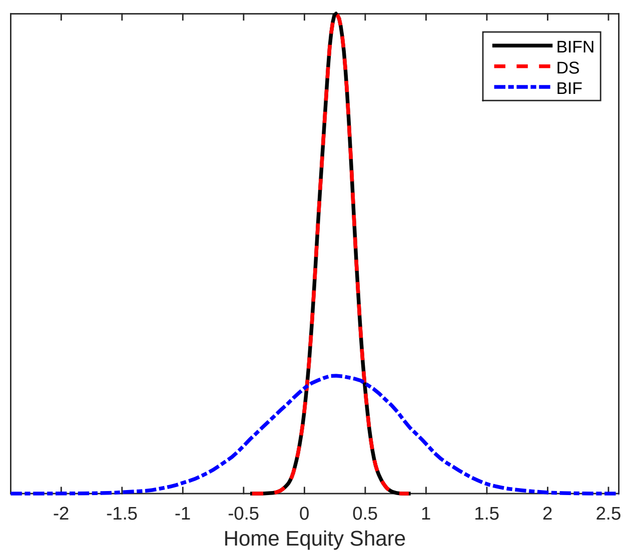


Figure 2: Ergodic Distribution of the Share of Home Equity owned by Home Country. *A proxy for the ergodic distribution is obtained by simulating 10 million of periods.*

Figure 1 reports the first-order accurate share of home equity held by the domestic agent (θ_H^H) in an interval based on the ergodic set for home net foreign assets. All other state variables take their respective steady state values. Policy function obtained with *DS* and *BIFN* are indistinguishable, whereas *BIF* yields more variation of asset holdings. Figure 2 shows that higher volatility implied by *BIF* is not only a short-run outcome but is also reflected by the ergodic distribution of θ_H^H which is obtained by simulating 10 million periods. Compared to the other two methods, *BIF* implies a standard deviation that is roughly four times larger: 0.56 in contrast to 0.138. To understand this outcome, consider the respective derivative of H and H^{nlma} in the example model:[18]

$$
\begin{aligned}
\bar{H}_{\sigma\sigma y^{state}} = & \boxed{-2\gamma\left(g^{C_h}_{y^{state}_\epsilon} - g^{C_f}_{y^{state}_\epsilon}\right)\left(I_{nystate}\otimes\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}}\right)} \\
& \boxed{-2\gamma\left(g^{R_h}_{y^{state}_\epsilon} - g^{R_f}_{y^{state}_\epsilon}\right)\left(I_{nystate}\otimes\Sigma\left(g^{C_h}_\epsilon - g^{C_f}_\epsilon\right)^{\mathrm{T}}\right)} \\
& + 2\gamma^2 g^{C_h}_{y^{state}}\left(g^{C_h}_\epsilon\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}}\right) \\
& - 2\gamma^2 g^{C_f}_{y^{state}}\left(g^{C_f}_\epsilon\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}}\right) \\
& - \gamma\left(g^{C_h}_{y^{state}} - g^{C_f}_{y^{state}}\right)\left(g^{R_h}_\epsilon\Sigma g^{R_h\,\mathrm{T}}_\epsilon - g^{R_f}_\epsilon\Sigma g^{R_f\,\mathrm{T}}_\epsilon\right) \\
& - \gamma\left(g^{C_h}_{y^{state}} - g^{C_f}_{y^{state}}\right)\left(\boxed{g^{R_h}_{\sigma\sigma} - g^{R_f}_{\sigma\sigma}}\right) \\
& - \gamma\left(g^{C_h}_{y^{state}} - g^{C_f}_{y^{state}}\right)\left(g^{R_h}_{\epsilon\epsilon} - g^{R_f}_{\epsilon\epsilon}\right) vec(\Sigma)
\end{aligned}
$$

$$
\begin{aligned}
\bar{H}^{nlma}_{\sigma\sigma y^{state}} = & \boxed{-2\gamma\left(g^{C_h}_{y^{state}_\epsilon} - g^{C_f}_{x\epsilon}\right)\left(I_{nystate}\otimes\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}}\right)} \\
& \boxed{-2\gamma\left(g^{R_h}_{y^{state}_\epsilon} - g^{R_f}_{y^{state}_\epsilon}\right)\left(I_{nystate}\otimes\Sigma\left(g^{C_h}_\epsilon - g^{C_f}_\epsilon\right)^{\mathrm{T}}\right)} \\
& + 2\gamma^2 g^{C_h}_{y^{state}}\left(g^{C_h}_\epsilon\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}}\right) \\
& - 2\gamma^2 g^{C_f}_{y^{state}}\left(g^{C_f}_\epsilon\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}}\right) \\
& - \gamma\left(g^{C_h}_{y^{state}} - g^{C_f}_{y^{state}}\right)\left(g^{R_h}_\epsilon\Sigma g^{R_h\,\mathrm{T}}_\epsilon - g^{R_f}_\epsilon\Sigma g^{R_f\,\mathrm{T}}_\epsilon\right) \\
& - \gamma\left(g^{C_h}_{y^{state}} - g^{C_f}_{y^{state}}\right)\left(\boxed{R_{h,\sigma\sigma} - R_{f,\sigma\sigma}}\right) \\
& - \gamma\left(g^{C_h}_{y^{state}} - g^{C_f}_{y^{state}}\right)\left(g^{R_h}_{\epsilon\epsilon} - g^{R_f}_{\epsilon\epsilon}\right) vec(\Sigma)
\end{aligned}
$$

The green box points to the difference across the bifurcation approaches. *BIF* is represented on the left-hand side whereas the right panel shows the corresponding derivative under *BIFN*. The only difference between the two approaches is the second-order risk adjustment term of the excess rate of return on home assets. *BIF* includes one-step ahead risk adjustment given by state space methods. By contrast, *BIFN* considers the cumulative risk correction that can be linked to the stochastic steady state.

The red boxes highlight terms included also by *DS*. Why does the difference exist? Devereux and Sutherland (2010) eliminate the remaining terms by exploiting the second-order approximation of the expected future excess return:

$$
E_t\left[\hat{R}^{(2)}_{xt+1}\right] = -\frac{1}{2}\left(g^{R_h}_\epsilon\Sigma g^{R_h\,\mathrm{T}}_\epsilon - g^{R_f}_\epsilon\Sigma g^{R_f\,\mathrm{T}}_\epsilon\right) + \frac{1}{2}\gamma\left(g^{C_h}_\epsilon + g^{C_f}_\epsilon\right)\Sigma\left(g^{R_h}_\epsilon - g^{R_f}_\epsilon\right)^{\mathrm{T}} \tag{42}
$$

[18]Note that derivatives of a policy function, that has been shifted one period into the future, with respect to σ do not only include the risk correction term but also the coefficients on ϵ_{t+1} as the latter is scaled by σ. However, I will slightly abuse the notation by letting g_{σ^j} and y_{σ^j} denote only the risk correction in order to simplify the exposition.

Note that (42) implies that $E_t\left[\hat{R}_{xt+1}\right]$ is a constant up to a second-order accuracy. In general, this condition is not fulfilled by the state space approximate solution. To see this consider the pruned second-order approximation of $\hat{R}_x$:

$$\begin{aligned}\hat{R}^{(2)}_{xt+1} = &\left(g^{R_h}_{y^{state}} - g^{R_f}_{y^{state}}\right)\hat{y}^{(2),state}_t + \left(g^{R_h}_{\epsilon} - g^{R_f}_{\epsilon}\right)\sigma\epsilon_{t+1} + \frac{1}{2}\left(g^{R_h}_{\sigma\sigma} - g^{R_f}_{\sigma\sigma}\right)\sigma^2 \\ &+ \frac{1}{2}\left(g^{R_h}_{y^{state}y^{state}} - g^{R_f}_{y^{state}y^{state}}\right)\hat{y}^{(1),state}_t \otimes \hat{y}^{(1),state}_t + \frac{1}{2}\left(g^{R_h}_{\epsilon\epsilon} - g^{R_f}_{\epsilon\epsilon}\right)\sigma^2\epsilon_{t+1}\otimes\epsilon_{t+1} \\ &+ \left(g^{R_h}_{y^{state}\epsilon} - g^{R_f}_{y^{state}\epsilon}\right)\sigma\hat{y}^{(1),state}_t \otimes \epsilon_{t+1}\end{aligned} \tag{43}$$

(43) can be also rewritten in terms of approximation increments (Lan and Meyer-Gohde, 2014*a*):

$$\hat{R}^{(2)}_{xt+1} = d\hat{R}^{(1)}_{xt+1} + d\hat{R}^{(2)}_{xt+1} \tag{44}$$

$$d\hat{R}^{(1)}_{xt+1} = \left(g^{R_h}_{\epsilon} - g^{R_f}_{\epsilon}\right)\sigma\epsilon_{t+1} \tag{45}$$

$$\begin{aligned}d\hat{R}^{(2)}_{xt+1} = &\left(g^{R_h}_{y^{state}} - g^{R_f}_{y^{state}}\right)d\hat{y}^{(2),state}_t + \frac{1}{2}\left(g^{R_h}_{\sigma\sigma} - g^{R_f}_{\sigma\sigma}\right)\sigma^2 \\ &+ \frac{1}{2}\left(g^{R_h}_{y^{state}y^{state}} - g^{R_f}_{y^{state}y^{state}}\right)d\hat{y}^{(1),state}_t \otimes d\hat{y}^{(1),state}_t + \frac{1}{2}\left(g^{R_h}_{\epsilon\epsilon} - g^{R_f}_{\epsilon\epsilon}\right)\sigma^2\epsilon_{t+1}\otimes\epsilon_{t+1} \\ &+ \left(g^{R_h}_{y^{state}\epsilon} - g^{R_f}_{y^{state}\epsilon}\right)\sigma d\hat{y}^{(1),state}_t \otimes \epsilon_{t+1}\end{aligned} \tag{46}$$

Suppose now that we start in period $t = 0$ with $d\hat{y}^{(1),state}_0 = d\hat{y}^{(2),state}_0 = 0$. Then, because the risk correction term is included in the recursion (see 46), it holds that $E_0\left[\hat{R}^{(2)}_{x1}\right] \neq E_1\left[\hat{R}^{(2)}_{x2}\right]$. On the other hand, $E_t\left[\hat{R}^{(2)}_{xt+1}\right]$ implied by the nonlinear moving average is constant for all t.

As pointed by (Lan and Meyer-Gohde, 2014*a*), the state space approximation and the nonlinear moving approximation are asymptotically identical up to a second order of accuracy. Thus, the expected difference in log rates of return implied by the former is asymptotically constant. Moreover, if we initialize the state space approximation at this asymptotic point, then it will yield exactly the same predictions as the nonlinear moving average at every point in time. However, this approach requires ex-ante knowledge of the stochastic steady state.

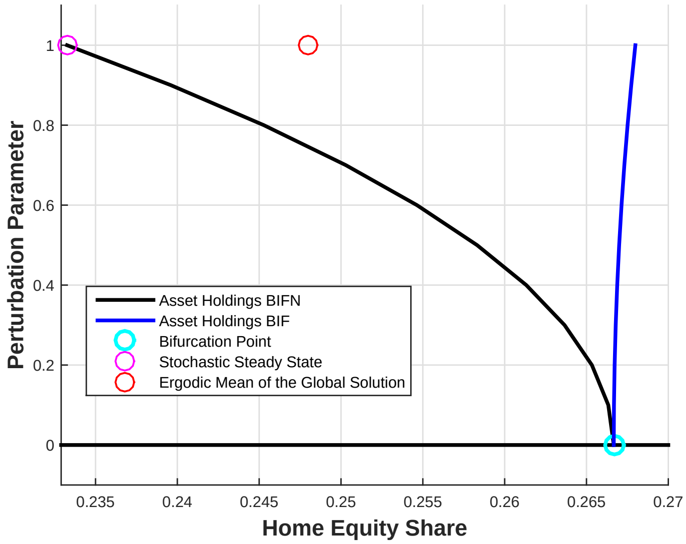


Figure 3: Risk-Adjusted Portfolio Holdings. $\sigma = 0$ *corresponds to the deterministic steady state whereas* $\sigma = 1$ *denotes fully stochastic environment. The ergodic mean of the global solution is taken from Rabitsch et al. (2015).*

4.3 The Direct Effect of the Presence of Risk on Portfolio Holdings

One of the drawbacks of *DS* highlighted by Rabitsch et al. (2015) is the fact that it fails to capture the direct effect of the presence of risk on gross asset positions. Thus, the question arises whether higher-order risk correction may affect model implications in a significant way and thereby improve the quality of the local approximation. To tackle this question, I extend the first-order approximations of portfolio holdings, (33) and (39), by including the second-order risk correction:

$$\alpha_t = \bar{\alpha} + \tilde{g}^{\alpha}_{y^{state}} \hat{y}^{(1),state}_t + \frac{1}{2} \tilde{g}^{\alpha}_{\sigma\sigma} \tag{47}$$

and

$$\alpha_t = \bar{\alpha} + \tilde{g}^{\alpha,nlma}_{y^{state}} d\hat{y}^{(1),state}_t + \frac{1}{2} \alpha_{\sigma\sigma} \tag{48}$$

In contrast to the first-order risk adjustment, the second-order term is in general not equal to zero, even under the normality assumption, and its value depends on method being used. Figure 3 compares risk adjusted portfolio holdings for a particular size of risk, given that all state variables take their steady state values. The ergodic mean of the global solution, reported by

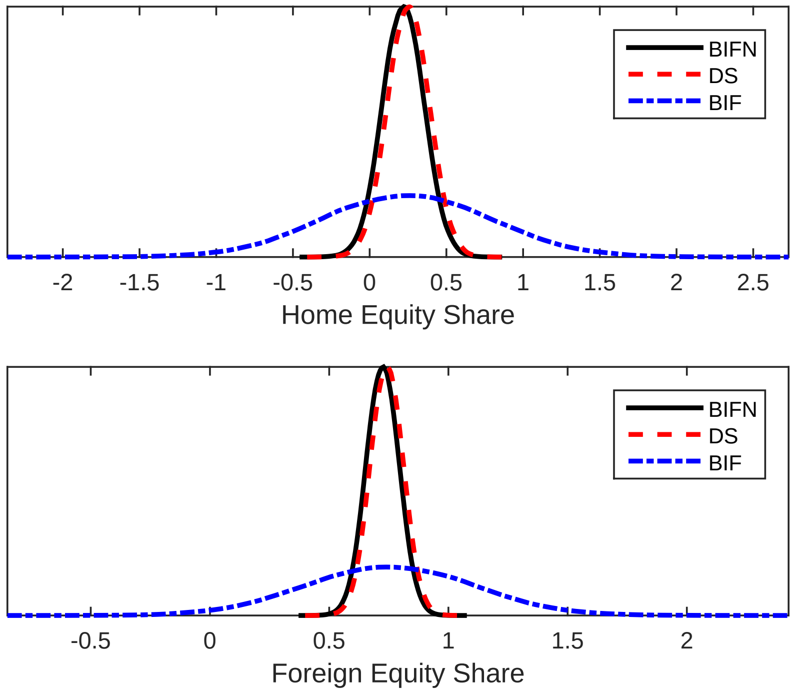


Figure 4: Ergodic Distributions of Portfolio Holdings (Home and Foreign assets) owned by Home Country. *A proxy for the ergodic distribution is obtained by simulating 10 million of periods.*

Rabitsch et al. (2015), is used as a benchmark.[19] As the size of risk goes asymptotically to zero and the bifurcation point is reached, home representative agent holds 26.7 % of home equity. This foreign equity bias is caused by the positive correlation between domestic "labor" and "capital income". According to Figure 3, *BIFN* correctly captures the sign of the direct effect of risk. Since the foreign country is subject to more volatile shocks, its precautionary motive is stronger and thus its long position in the home equity becomes larger as σ increases. By contrast, *BIF* fails to account for this effect and predicts that home country raises its holdings of the domestic equity.

The second-order accurate effect of risk on the ergodic distribution of asset holdings is visualized by Figure 4. Due to the stronger precautionary motive in the foreign country, the distribution under *BIFN* is slightly shifted to the left, compared to *DS*.

[19] Although ergodic mean and the risk-adjusted value are two distinct concepts, this comparison can determine whether heterogeneous precautionary motives, reflected by the ergodic mean, are also accounted for at the starting point of the approximation.

Ergodic Mean of GS	BIFN	BIF	DS plus Updating
-0.168	-0.2857	1.3021e-4	-6.19

Table 2: Risky NFA Positions. *The ergodic mean of the global solution (GS) is taken from Rabitsch et al. (2015). The value for NFA implied by the iterative DS procedure is taken from its working paper version. Entries for BIF and BIFN represent second-order risk correction terms.*

4.4 Non-zero Net Foreign Asset Positions

Another issue raised by Rabitsch et al. (2015) refers to the fact, that the approximation of an asymmetric two-country model is still computed at zero net foreign assets, although the presence of asymmetries implies most likely non-zero positions. Alternatively, Devereux and Sutherland (2009) propose an iterative procedure to update the value for net foreign assets at the approximation point. However, Rabitsch et al. (2015) show that this procedure reduces the accuracy of the local approximation. Constructing the approximation around a point with non-zero net foreign assets (e.g. stochastic steady state) is beyond the scope of this paper. Nevertheless, it is of interest to investigate whether *BIFN* can mitigate the problem by yielding correctly risk-adjusted net foreign assets. In particular, I propose to start with net position equal to zero and let model's risk characteristics endogenously determine the risk-adjusted net foreign assets that are used as a starting point for the approximation.

Table 2 gives risk-adjusted net asset positions implied by different methods. The ergodic mean of the global solution reported by Rabitsch et al. (2015) is used again as a benchmark. Mean net foreign liabilities of home country under the global solution represent 16.8 % of the steady state domestic output.[20] *BIFN* correctly captures the sign of the effect of risk and predicts a negative home net foreign asset position caused by a stronger precautionary motive in the foreign country. On the other hand, *BIF* yields slightly positive net assets. Though it is important to note that the net position reported for *BIF*, consistently with the state space approach, accounts only for one-step ahead constant risk correction and transits deterministically to the second-order accurate stochastic steady state (see Lan and Meyer-Gohde, 2014*a*). In general, differences in the implied gross assets between *BIF* and *BIFN* may lead to different values of net positions in the stochastic steady state. Yet numerical exercises show that this difference can be neglected in the case of

[20]Steady state output is normalized to 1.

	GS		BIFN		BIF		DS	
	mean	st. dev.	mean	st. dev.	mean	st. dev.	mean	st. dev.
NFA	-0.168	1.11	-0.1697	1.1569	-0.1686	1.2334	-0.1696	1.1569
θ_h^h	0.248	0.13	0.239	0.1379	0.2656	0.5622	0.2645	0.1379
θ_h^f	0.723	0.066	0.7278	0.0701	0.7346	0.3598	0.7357	0.0701

Table 3: Ergodic Moments. *Mean and standard deviation of the global solution are taken from Rabitsch et al. (2015). To obtain moments of local approximation methods, the model is simulated ten times. Each simulation consists of 10 million periods.*

our example model.[21] Nevertheless, in short-run simulations, constant risk correction term still differ.

According to *DS* with an updating procedure, home country's debt adds up to 619 % of the steady state output. Thus, the iterative algorithm overestimates the precautionary motive of the Foreign country and yields net foreign positions that differ greatly from the implications of the global solution.

4.5 Performance Evaluation

The analysis so far shows that *BIFN* can account for direct effects of the size of risk on both net and gross asset positions. In the following, I investigate whether capturing these effects improves the quality of the approximation. To this end, I compare ergodic moments implied by the different methods and conduct the Euler equation error test to measure their accuracy.

4.5.1 Simulated Moments

This section reports the ergodic moments of gross and net asset holdings implied by the three perturbation methods. As in the case of previous sections, the global solution, reported by Rabitsch et al. (2015), is used as a benchmark.

To obtain moments of local approximation techniques, the model is simulated 10 times. Each simulation contains 10 million observations. Table 4 reports the results of this exercise. First, as already discussed, (both home and foreign) equity holdings of home country implied by *BIF* are characterized by high volatility. The standard deviation of home equity share is more than four

[21]Still, *BIF* and *BIFN* yield different ergodic moments for net foreign position as presented in the next section.

	Average Errors			Maximum Errors		
	DS	BIF	BIFN	DS	BIF	BIFN
YKH	-4.5018	-4.5024	-5.6531	-3.2887	-3.2896	-5.1536
NFA	-5.5762	-4.5120	-5.5829	-5.0926	-4.0066	-5.0926
QH	-6.4219	-4.5417	-6.4699	-5.8697	-4.4380	-5.3270

Table 4: Euler Equation Errors

times greater than predicted by the global solution. This translates also into a higher volatility of net foreign asset positions. On the other hand, *DS* and *BIFN* yield second moments for both gross positions and the implied net foreign assets that are more in line with the global solution. Second, among local approximation methods considered in this study, the mean of portfolio holdings implied by *BIFN* is closest to its global solution counterpart. The largest discrepancy among the available assets amounts to 3.63 %. By contrast, this figure is nearly twice as large for *DS*.

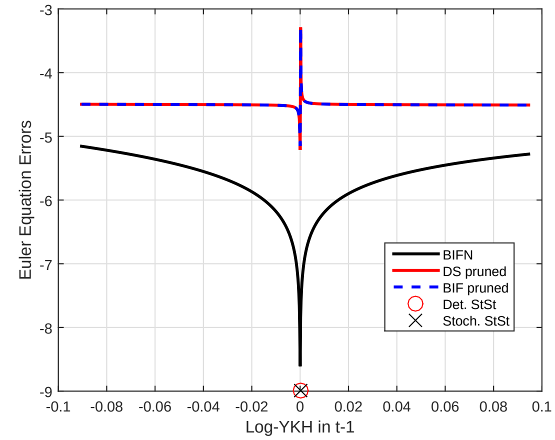
4.5.2 Euler Equation Error Test

The focus of this investigation lies on the importance of risk adjustment terms of portfolio holdings. In the underlying model, there is no Euler equation embedding asset positions explicitly. Therefore, I use pseudo Euler equation errors, proposed by Kazimov (2012), to measure the accuracy of local approximations. In particular, I directly introduce assets into Euler equations as follows:

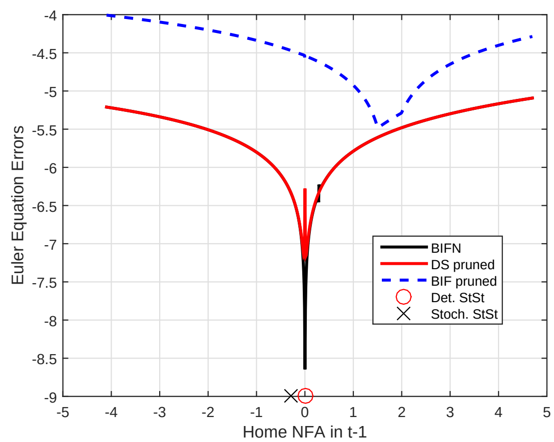
$$NFA_{Ht} = E_t\left[\left(\beta(C_{Ht})\left(\frac{C_{Ht}}{C_{Ht+1}}\right)^{\gamma}\right)(R_{Ht+1}\alpha_t + R_{Ft+1}(NFA_{Ht} - \alpha_t))\right] \tag{49}$$

$$NFA_{Ht} = E_t\left[\left(\beta(C_{Ft})\left(\frac{C_{Ft}}{C_{Ft+1}}\right)^{\gamma}\right)(R_{Ht+1}\alpha_t + R_{Ft+1}(NFA_{Ht} - \alpha_t))\right] \tag{50}$$

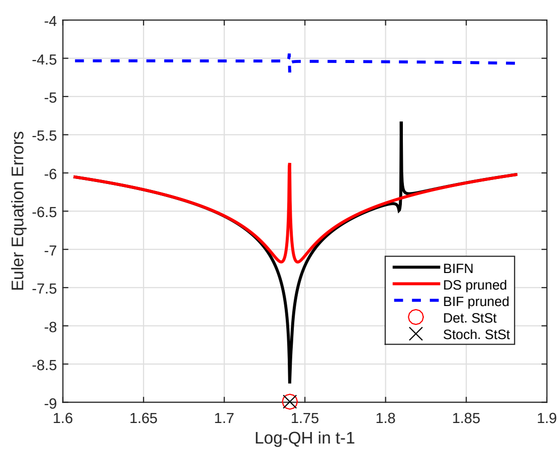
Equations (49) and (50) can be interpreted as home and foreign agent's portfolio Euler equation. The underlying idea is that the rate of return on an optimally constructed portfolio must obey similar restrictions as individual asset returns. In the following, I use the common logarithm of the absolute value of approximation errors as a measure of accuracy. According to this definition



(a) Home 'Capital Income'



(b) Home NFA



(c) Price of Home Equity

Figure 5: Euler Equation Errors *Euler Equation Errors are computed within intervals based on the respective ergodic sets under DS. All other state variables always take their steady state values. Each ergodic set is determined by simulating 10 million periods and subsequently discretized by 1001 equidistant grid points.*

an Euler equation error of -3 implies one dollar error for every thousand dollars spent. To obtain a scalar measure of accuracy, I average the errors associated with (49) and (50).

Figure 5 evaluates the performance of local approximations within an interval based on the ergodic set (under *DS*) of home "capital income", home net foreign asset and price of home equity, respectively. All other state variables always take their steady state values. According to the figure, *BIFN* performs uniformly better over the entire ergodic set of home "capital income", with maximal improvement being roughly five orders of magnitude. On the other hand, the advantage of BIFN is less pronounced for net foreign assets and price of home equity. Compared to *DS*, *BIFN* performs significantly better in the immediate neighborhood of the deterministic steady state. However, there exists also a small subset where *DS* is associated with lower approximation errors. Table 4 shows that in case of net foreign assets and price of home equity, *BIFN* and *DS* perform similarly on average. In the case of the home "capital income", *BIFN* leads to a significant improvement.

On the other hand, it is apparent from Figure 5 and Table 4 that *BIF* is outperformed by its competitors. This result reflects the excessive volatility of portfolio holdings implied by this method. Therefore, the bifurcation theory should be applied together with an approximation including risk correction at the stochastic steady state, e.g the nonlinear moving average approximation.

5 Conclusion

I propose a combination of bifurcation methods and nonlinear moving average approximation (*BIFN*) as a technique to solve asymmetric DSGE models with portfolio choice. The use of the bifurcation theory overcomes the problem of indeterminacy of portfolio holdings, whereas the nonlinear moving average accounts for risk correction at the stochastic steady state.

The main advantage of the proposed method is the fact that it can be used to compute higher-order approximation of portfolio holdings. Thereby, it can account for the direct effect of the presence of risk on both, gross and net asset holdings. This is reflected by the starting point of the approximation as well as by the moments of the implied ergodic distribution. Moreover, *BIFN* improves accuracy of the approximation measured by Euler equation errors relative to the workhorse routine developed by Devereux and Sutherland (2010, 2011). The biggest documented average accuracy gain is of one order of magnitude, whereas the maximum improvement amounts to five orders.

As a local approximation method, *BIFN* can be applied to investigate a variety of issues in macro-finance within the DSGE framework with a large state space. In particular, it can be to tackle economic questions that require at least third-order approximation of the non-portfolio variables. One example is the analysis of channels through which time-varying risk affects country portfolios. There exists a large body of empirical literature documenting the importance of changes in the global risk perception for international capital flows (Milesi-Ferretti and Tille, 2011; Forbes and Warnock, 2012; Ahmed and Zlate, 2014). A portfolio choice DSGE model can then be employed to uncover the underlying economic mechanism and to conduct an welfare-based evaluation of a range of possible policy responses.

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Appendix

A. Proof of the Bifurcation Theorem

Proof. The bifurcation theorem can be proven by dividing the H-function by its singularity (See Zeidler, 1986 and Judd and Guu, 2001). Define the following function $\tilde{H}$:

$$\tilde{H}(y,\sigma) = \begin{cases} \frac{H(\alpha,\sigma)}{\sigma^2} & if \quad \sigma \neq 0 \\ \frac{\partial^2 H(\alpha,\sigma)}{(\partial\sigma)^2} & if \quad \sigma = 0 \end{cases} .$$

Since H is analytic, and $H(\alpha,\sigma) = 0$ for all α, it follows that $H(\alpha,\sigma) = \tilde{H}(\alpha,\sigma)\sigma^2$ and $\tilde{H}$ is analytic in (α, σ). Implicit differentiation yields:

$$H_{\sigma\sigma}|_{\sigma=0} = \tilde{H}|_{\sigma=0} \tag{51}$$

$$H_{\sigma\sigma\alpha}|_{\sigma=0} = \tilde{H}_{\alpha}|_{\sigma=0} \tag{52}$$

Therefore, to obtain a root of $\tilde{H}|_{\sigma=0}$, $H_{\sigma\sigma}|_{\sigma=0}$ must be set equal to the zero vector. Moreover, IFT can be applied to $\tilde{H}$ if and only if $det\left(H_{\sigma\sigma\alpha}(\alpha_0,\sigma_0)\right) \neq 0$. □

B. Computing Derivatives of the Portfolio Equation - State Space Approach

Bi. General Relationship

Recall the portfolio equation:

$$H\left(\alpha_t, y_t^{state}, \sigma, \epsilon_{t+1}|g_{guess}^{\alpha}\right) \equiv$$
$$E_t\left[m\left(g^{\mu}\left(\sigma,\alpha_t,y_t^{state},\epsilon_{t+1}|\, g_{guess}^{\alpha}\right)\right)\otimes b\left(g^{r}\left(\sigma,\alpha_t,y_t^{state},\epsilon_{t+1}|g_{guess}^{\alpha}\right)\right)\right]$$

Furthermore, let $\tilde{H}\left(\alpha_t, y_t^{state}, \sigma, \epsilon_{t+1}|g_{guess}^{\alpha}\right)$ be the function determined by the bifurcation theorem. Then, the following holds:

$$H\left(\alpha_t, y_t^{state}, \sigma, \epsilon_{t+1}|g_{guess}^{\alpha}\right) = \tilde{H}\left(\alpha_t, y_t^{state}, \sigma, \epsilon_{t+1}|g_{guess}^{\alpha}\right)\sigma^2 \tag{53}$$

The policy function for α is defined as:

$$\tilde{H}\left(g^{\alpha}\left(y_t^{state},\sigma\right),y_t^{state},\sigma,\epsilon_{t+1}|g^{\alpha}\right)=0 \tag{54}$$

Bii. First-Order Coefficients

Implicit differentiation yields:

$$\tilde{g}^{\alpha}_{y^{state}}=-\bar{\tilde{H}}_{\alpha}^{-1}\bar{\tilde{H}}_{y^{state}} \tag{55}$$

$$\tilde{g}^{\alpha}_{\sigma}=-\bar{\tilde{H}}_{\alpha}^{-1}\bar{\tilde{H}}_{\sigma} \tag{56}$$

To find the corresponding derivatives of $\tilde{H}$, I implicitly differentiate (53). As a result, the following relationships are obtained: $\bar{\tilde{H}}_{\sigma}=\frac{1}{6}\bar{H}_{\sigma\sigma\sigma}$, $\bar{\tilde{H}}_{y^{state}}=\frac{1}{2}\bar{H}_{\sigma\sigma y^{state}}$, and $\bar{\tilde{H}}_{\alpha}=\frac{1}{2}\bar{H}_{\sigma\sigma\alpha}$. Inserting these expressions into (55) and (56) yields (35) and (36).

In the last step, exact expressions of the respective derivatives of the H-function need to be found. They will be pruned to avoid unnecessary higher-order terms that may lead to an explosive behavior and thus, deteriorate accuracy of the approximation. In particular, to obtain the first-order approximation of gross asset holdings, the derivatives will include only third-order components. This approach follows the ideas of Samuelson (1970) and also underlies the procedure of Devereux and Sutherland (2010, 2011).

Implicit differentiation and omitting components of order higher than three yields:

$$\begin{aligned}\bar{H}_{\sigma\sigma\sigma}=&\,3\left[\left(m_{\mu\mu}\left(g^{\mu}_{\epsilon}\otimes g^{\mu}_{\epsilon}\right)\right)\otimes\left(b_r g^{r}_{\epsilon}\right)\right]\Sigma_3\\&+3\left[\left(m_{\mu}g^{\mu}_{\epsilon}\right)\otimes\left(b_{rr}\left(g^{r}_{\epsilon}\otimes g^{r}_{\epsilon}\right)\right)\right]\Sigma_3\\&+3\left[\left(m_{\mu}g^{\mu}_{\epsilon\epsilon}\right)\otimes\left(b_r g^{r}_{\epsilon}\right)\right]\Sigma_3\\&+3\left[\left(m_{\mu}g^{\mu}_{\epsilon}\right)\otimes\left(b_r g^{r}_{\epsilon\epsilon}\right)\right]\Sigma_3\end{aligned} \tag{57}$$

where $\Sigma_3\equiv E_t\left[\epsilon_t^{\otimes[3]}\right]$ denotes a matrix of third moments of the underlying shock structure.

Let $p = [y^{state}, \alpha]^\top$ be a $np \times 1$ vector, then

$$\begin{aligned}
\bar{H}_{\sigma\sigma p} = & 2\left[\left(m_{\mu\mu}\left(g^{\mu}_{p} \otimes g^{\mu}_{\epsilon}\right)\right) \otimes \left(b_r g^{r}_{\epsilon}\right)\right]\left[I_{np} \otimes vec(\Sigma)\right] \\
& + 2\left[\left(m_\mu g^{\mu}_{\epsilon p}\right) \otimes \left(b_r g^{r}_{\epsilon}\right)\right]\left[I_{np} \otimes vec(\Sigma)\right] \\
& + 2\left[\left(b_r g^{r}_{\epsilon p}\right) \otimes \left(m_\mu g^{\mu}_{\epsilon}\right)\right]\left[I_{np} \otimes vec(\Sigma)\right] \\
& + \left[\left(m_\mu g^{\mu}_{p}\right) \otimes \left(b_{rr}\left(g^{r}_{\epsilon} \otimes g^{r}_{\epsilon}\right)\right)\right]\left[I_{np} \otimes vec(\Sigma)\right] \\
& + \left[\left(m_\mu g^{\mu}_{p}\right) \otimes \left(b_r g^{r}_{\epsilon\epsilon}\right)\right]\left[I_{np} \otimes vec(\Sigma)\right] \\
& + \left[\left(m_\mu g^{\mu}_{p}\right) \otimes \left(b_r g^{r}_{\sigma\sigma}\right)\right]
\end{aligned} \tag{58}$$

where I_{np} stands for the identity matrix of dimension $np \times np$ and Σ denotes the variance-covariance matrix of the underlying shock structure. Note that combining (56) with the above derivatives yields the expression for the first-order risk correction term presented in (37) with the skew tolerance given by:

$$\begin{aligned}
\tau = & 3\bar{H}^{-1}_{\sigma\sigma\alpha}\left[\left(m_{\mu\mu}\left(g^{\mu}_{\epsilon} \otimes g^{\mu}_{\epsilon}\right)\right) \otimes \left(b_r g^{r}_{\epsilon}\right)\right. \\
& + 3\left(m_\mu g^{\mu}_{\epsilon}\right) \otimes \left(b_{rr}\left(g^{r}_{\epsilon} \otimes g^{r}_{\epsilon}\right)\right) \\
& + 3\left(m_\mu g^{\mu}_{\epsilon\epsilon}\right) \otimes \left(b_r g^{r}_{\epsilon}\right) \\
& \left. + 3\left(m_\mu g^{\mu}_{\epsilon}\right) \otimes \left(b_r g^{r}_{\epsilon\epsilon}\right)\right]
\end{aligned}$$

Biii. Second-Order Risk Correction

Given the normality assumption, the second-order risk correction term can be computed as:

$$\tilde{g}^{\alpha}_{\sigma\sigma} = -\bar{\tilde{H}}^{-1}_{\alpha}\bar{\tilde{H}}_{\sigma\sigma} \tag{59}$$

To find the second derivative of $\tilde{H}$ with respect to σ, I implicitly differentiate (53). As a result, the following relationship is obtained: $\bar{\tilde{H}}_{\sigma\sigma} = \frac{1}{12}\bar{H}_{\sigma\sigma\sigma\sigma}$. Furthermore, applying the procedure described in the previous section yields:

$$
\begin{aligned}
\bar{H} = & \, 4\left[\left(m_{\mu\mu}\left(K_{nmu,nmu}+2I_{nmu^2}\right)\left(g^{\mu}_{\epsilon}\otimes g^{\mu}_{\sigma\sigma}\right)\right)\otimes\left(b_r g^{r}_{\epsilon}\right)\right] vec\left(\Sigma\right) \\
& +12\left[\left(m_{\mu} g^{\mu}_{\sigma\sigma\epsilon}\right)\otimes\left(b_r g^{r}_{\epsilon}\right)\right] vec\left(\Sigma\right) \\
& +6\left[\left(m_{\mu\mu}\left(g^{\mu}_{\epsilon}\otimes g^{\mu}_{\epsilon}\right)\right)\otimes\left(b_r g^{r}_{\sigma\sigma}\right)\right] vec\left(\Sigma\right) \\
& +6\left[\left(m_{\mu} g^{\mu}_{\sigma\sigma}\right)\otimes\left(b_r\left(g^{r}_{\epsilon}\otimes g^{r}_{\epsilon}\right)\right)\right] vec\left(\Sigma\right) \\
& +6\left[\left(m_{\mu} g^{\mu}_{\epsilon\epsilon}\right)\otimes\left(b_r g^{\mu}_{\sigma\sigma}\right)\right] vec\left(\Sigma\right) \\
& +6\left[\left(m_{\mu} g^{\mu}_{\sigma\sigma}\right)\otimes\left(b_r g^{\mu}_{\epsilon\epsilon}\right)\right] vec\left(\Sigma\right) \\
& +6\left(m_{\mu}\otimes b_r\right)\left(g^{\mu}_{\sigma\sigma}\otimes g^{\mu}_{\sigma\sigma}\right) \\
& +4\left[\left(m_{\mu} g^{\mu}_{\sigma\sigma}\right)\otimes\left(b_{rr}\left(K_{nr,nr}+2I_{nr^2}\right)\left(g^{r}_{\epsilon}\otimes g^{r}_{\sigma\sigma}\right)\right)\right] vec\left(\Sigma\right) \\
& +12\left[\left(m_{\mu} g^{\mu}_{\epsilon}\right)\otimes\left(b_r g^{r}_{\sigma\sigma\epsilon}\right)\right] vec\left(\Sigma\right) \qquad (60)
\end{aligned}
$$

with I_n denoting an $n \times n$ identity matrix and $K_{n,n}$ being a commutation matrix with dimension $n^2 \times n^2$ (Magnus and Neudecker, 1979).

C. Computing Derivatives of the Portfolio Equation - Nonlinear Moving Average

Ci. General Relationship

Recall the following relationship:

$$
H^{nlma}\left(\sigma,\alpha_t,\epsilon_{t+1},\epsilon_t,\epsilon_{t-1},...\right) = H\left[\sigma,\alpha_t,y^{state}_t\left(\sigma,\epsilon_t,\epsilon_{t-1},...\right)\right] \qquad (61)
$$

Moreover, note that the function determined by the bifurcation theorem is defined by:

$$
H^{nlma}\left(\sigma,\alpha_t,\epsilon_{t+1},\epsilon_t,\epsilon_{t-1},...|\alpha_{guess}\right) = \tilde{H}^{nlma}\left(\sigma,\alpha_t,\epsilon_{t+1},\epsilon_t,\epsilon_{t-1},...|\alpha_{guess}\right)\sigma^2 \qquad (62)
$$

Cii. First-Order Coefficients

The starting point for the computation of the first-order coefficients on state variables is the derivation of the coefficients on current shock realizations. According to the bifurcation theorem, the first-order coefficients on ϵ_{t-j} are given by:[22]

$$\alpha_j = -\Phi \bar{\bar{H}}_j^{nlma} \tag{63}$$

where Φ denotes the inverse of $\bar{\bar{H}}_\alpha^{nlma}$. Implicit differentiation of (62) yields $\bar{\bar{H}}_j^{nlma} = \frac{1}{2}\bar{H}_{\sigma\sigma j}^{nlma}$ and $\bar{\bar{H}}_\alpha^{nlma} = \frac{1}{2}\bar{H}_{\sigma\sigma\alpha}^{nlma}$. Moreover, differentiating (61) leads to:

$$\bar{H}_{\sigma\sigma j}^{nlma} = \bar{H}_{\sigma\sigma y^{state}} y_j^{state} + \bar{H}_{y^{state}y^{state}} \left(I_{nystate} \otimes y_{\sigma\sigma}^{state}\right) y_j^{state} \tag{64}$$

Combining (63) and (64) as well as exploting the fact that $\bar{H}_{\sigma\sigma\alpha} = \bar{H}_{\sigma\sigma\alpha}^{nlma}$ yields:

$$\alpha_j = -\bar{\bar{H}}_{\sigma\sigma\alpha}^{-1} \left[\bar{H}_{\sigma\sigma y^{state}} + \bar{H}_{y^{state}y^{state}} \left(I_{nystate} \otimes y_{\sigma\sigma}^{state}\right)\right] y_j^{state} \tag{65}$$

Since we are interested in first-order coefficients, (65) has to be equal to $g_{y^{state}}^{\alpha,nlma} y_j^{state}$. Thus,

$$g_{y^{state}}^{\alpha,nlma} = -\bar{\bar{H}}_{\sigma\sigma\alpha}^{-1} \left[\bar{H}_{\sigma\sigma y^{state}} + \bar{H}_{y^{state}y^{state}} \left(I_{nystate} \otimes y_{\sigma\sigma}^{state}\right)\right] = -\bar{\bar{H}}_{\sigma\sigma\alpha}^{-1} \bar{\bar{H}}_{\sigma\sigma y^{state}}^{nlma} \tag{66}$$

with

$$\begin{aligned}
\bar{\bar{H}}_{\sigma\sigma y^{state}}^{nlma} = 2 &\left[\left(m_{\mu\mu}\left(g_{y^{state}}^{\mu} \otimes g_\epsilon^\mu\right)\right) \otimes \left(b_r g_\epsilon^r\right)\right] \left[I_{nystate} \otimes vec(\Sigma)\right] \\
&+ 2\left[\left(m_\mu g_{\epsilon y^{state}}^{\mu}\right) \otimes \left(b_r g_\epsilon^r\right)\right] \left[I_{nystate} \otimes vec(\Sigma)\right] \\
&+ 2\left[\left(b_r g_{\epsilon y^{state}}^{r}\right) \otimes \left(m_\mu g_\epsilon^\mu\right)\right] \left[I_{nystate} \otimes vec(\Sigma)\right] \\
&+ \left[\left(m_\mu g_{y^{state}}^{\mu}\right) \otimes \left(b_{rr}\left(g_\epsilon^r \otimes g_\epsilon^r\right)\right)\right] \left[I_{nystate} \otimes vec(\Sigma)\right] \\
&+ \left[\left(m_\mu g_{y^{state}}^{\mu}\right) \otimes \left(b_r g_{\epsilon\epsilon}^r\right)\right] \left[I_{nystate} \otimes vec(\Sigma)\right] \\
&+ \left[\left(m_\mu g_{y^{state}}^{\mu}\right) \otimes \left(b_r r_{\sigma\sigma}\right)\right]
\end{aligned} \tag{67}$$

The first-order risk correction can be obtained by:

$$\alpha_\sigma = -\Phi \bar{\bar{H}}_\sigma^{nlma} \tag{68}$$

[22]Note that α_t does not depend on ϵ_{t+1}.

Implicit differentiation of (62) yields $\bar{\tilde{H}}^{nlma}_{\sigma} = \frac{1}{6}\bar{H}^{nlma}_{\sigma\sigma\sigma}$ and $\bar{\tilde{H}}^{nlma}_{\alpha} = \frac{1}{2}\bar{H}^{nlma}_{\sigma\sigma\alpha}$. Differentiating (61) and exploiting the certainty equivalency of first-order approximation ($y_{\sigma} = 0$) yields:

$$\bar{H}^{nlma}_{\sigma\sigma\sigma} = \bar{H}_{\sigma\sigma\sigma} \tag{69}$$

Therefore, both *BIF* and *BIFN* yield the same first-order risk adjustment term, i.e. $\alpha_{\sigma} = g^{\alpha}_{\sigma}$.

Ciii. Second-Order Risk Correction

Applying the procedure from the previous sections leads to the following expressions, given normally distributed shocks:

$$\alpha_{\sigma\sigma} = -\Phi\bar{\tilde{H}}^{nlma}_{\sigma\sigma} \tag{70}$$

$$\bar{\tilde{H}}^{nlma}_{\sigma\sigma\sigma\sigma} = \frac{1}{12}\bar{H}^{nlma}_{\sigma\sigma\sigma\sigma} \tag{71}$$

$$\bar{H}^{nlma}_{\sigma\sigma\sigma\sigma} = \bar{H}_{\sigma\sigma\sigma\sigma} + 6\bar{H}_{\sigma\sigma y^{state}} y^{state}_{\sigma\sigma} + 3\bar{H}_{y^{state}y^{state}}\left(y^{state}_{\sigma\sigma} \otimes y^{state}_{\sigma\sigma}\right) \tag{72}$$

Combining the three equations yields:

$$\alpha_{\sigma\sigma} = g^{\alpha,nlma}_{\sigma\sigma} + g^{\alpha,nlma}_{y^{state}} y^{state}_{\sigma\sigma} = g^{\alpha}_{\sigma\sigma} + \Delta + g^{\alpha,nlma}_{y^{state}} y^{state}_{\sigma\sigma} \tag{73}$$

with

$$\begin{aligned}\Delta \equiv -\Phi \Big[& \Big(\left(h_{\mu\mu}\left(g^{\mu}_{\epsilon} \otimes g^{\mu}_{\epsilon}\right)\right) \otimes \left(b_r g^{r}_{y^{state}}\right)\Big) vec(\Sigma) + \left(h_{\mu} g^{\mu}_{\sigma\sigma}\right) \otimes \left(b_r g^{r}_{y^{state}} y^{state}_{\sigma\sigma}\right) \\ & + \Big(\left(h_{\mu} g^{\mu}_{\epsilon\epsilon}\right) \otimes \left(b_r g^{r}_{y^{state}} y^{state}_{\sigma\sigma}\right)\Big) vec(\Sigma) + 2\Big(\left(h_{\mu} g^{\mu}_{\epsilon}\right) \otimes \left(b_{rr}\left(g^{r}_{y^{state}} y^{state}_{\sigma\sigma} \otimes g^{r}_{\epsilon}\right)\right)\Big) vec(\Sigma)\Big]\end{aligned}$$

Δ accounts for the transition from the first to the second order of accuracy as excess returns do not depend on state variables up to first-order approximation.

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