Predicting default – a dynamic Nelson-Siegel approach to forward intensities

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Credit risk

- Default prediction at core of credit risk
- Determinants of default risk relevant for
  - credit risk management
  - and financial regulation
- Probability of default is key input for
  - bond pricing
  - credit derivative pricing
  - credit ratings, etc.
Prediction approaches

- **Important early empirical work**
  Discriminant analysis, static logit models, single-period prediction
  

- **Dominant body relies on duration analysis**
  Proportional hazards models, dynamic logit models, single-period prediction
  
  Examples: Campbell et al. (2008), Chava et al. (2004), Hillegeist et al. (2004), Lane et al. (1986), Lee et al. (1996), and Shumway (2001)

Default prediction with Nelson-Siegel forward intensities
Prediction approaches

- **Most recent work**
  Duration analysis, default and other exit possibilities, *multi-period* prediction

Examples: Duan et al. (2013, 2012), Duffie et al. (2007), Orth (2013), and Prastyo et al. (2014)

Default prediction with Nelson-Siegel forward intensities
Motivation

Standard reduced form model

- Doubly stochastic Poisson process, $N_t$, with intensity $\lambda_t$
- Time of default $\tau$: first jump time of $N_t$
- Time-$t$-conditional survival and default probability:

\[
S_t(T) = P_t(\tau > T) = E_t \left\{ \exp \left( - \int_t^T \lambda_s \, ds \right) \right\}
\]  

(1)

\[
PD_t(T) = P_t(t < \tau \leq T) = E_t \left\{ \int_t^T \exp \left( - \int_t^s \lambda_u \, du \right) \lambda_s \, ds \right\}
\]  

(2)

- Two possibilities of solving (1) and (2)
  - $\lambda_t$ affine/quadratic: closed form
  - $\lambda_t$ not affine/quadratic: simulation

Default prediction with Nelson-Siegel forward intensities
Duffie et al. (2007)

- Specify $\lambda_t = \exp(c + \alpha^T u_t + \beta^T y_t)$
  - Macro factors: $u_t = (u_{1t}, \ldots, u_{kt})^T$
  - Firm-specific factors: $y_t = (y_{1t}, \ldots, y_{mt})^T$
  - Intercept $c, \alpha, \beta$ constant over time and across firms

- Simulation of $\lambda_t$ necessary for multi-period prediction
- Time series model for $u_t$ and $y_t$ required
- Model of $u_t$ and $y_t$ must be estimated
  - Large sample of companies ($n \approx 3000$)
  - High-dimensional problem

Default prediction with Nelson-Siegel forward intensities
Forward intensities

**Definition**

Let $F_t(s)$ be the time-$t$ conditional CDF of $\tau$ at $s > t$. Assume $F_t(s)$ as differentiable, then the forward intensity $\lambda_t(s)$ is given as

$$\lambda_t(s) = \frac{F'_t(s)}{1 - F_t(s)}.$$  \hfill (3)

- (1) and (2) contain an **instantaneous spot intensity**, $\lambda_t$
- Using $\lambda_t(s)$ we may compute $S_t(T)$ and $PD_t(T)$ right away

$$S_t(T) = \exp \left\{ - \int_t^T \lambda_t(s) \, ds \right\}$$  \hfill (4)

$$PD_t(T) = \int_t^T \exp \left\{ - \int_t^s \lambda_t(u) \, du \right\} \lambda_t(s) \, ds$$  \hfill (5)

Default prediction with Nelson-Siegel forward intensities
Duan et al. (2012)

- Model forward intensity directly,

\[ \lambda_t(s) = \exp\{c + \alpha(s)^\top u_t + \beta(s)^\top y_t\} \]

- Covariates \( u_t \), and \( y_t \) equivalent to Duffie et al. (2007)

- **No need for a time series model of** \( (u_t, y_t)^\top \)

- Loadings \( \alpha(s) \) and \( \beta(s) \) constant over time but dependent on prediction horizon

- Direct estimation of \( \alpha(s) \) and \( \beta(s) \) via ML (discrete)

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Default prediction with Nelson-Siegel forward intensities
Example 1

Suppose $T - t = 36$ months, $\Delta t = 1$ month. Discretise (4):

$$S_t(T) = \exp \left\{ - \sum_{s=0}^{T-\Delta t} \lambda_t(s) \Delta t \right\} \quad (6)$$

- Number of $\lambda_t(s)$ to compute: $\frac{T-t}{\Delta t} = 36$
- Number of parameters per $\lambda_t(s)$: $(k + m + 1)$
- Total number of parameters: $36 \times (k + m + 1)$
Figure 1: Estimated default and other-exit forward intensities for $s \in [0, \ldots, 36]$.

Default prediction with Nelson-Siegel forward intensities
Motivation

Fitted forward intensity function

Figure 2: Estimated default forward intensity curve for a random company in February, 1991, in the Duan et al. (2012) data set.
Model shortfalls

- Only partial forward intensity curve is constructed
- No dynamic dependence structure of forward intensities, i.e. $\alpha(s)$ and $\beta(s)$ are constant over time
- High number of model parameters

Research goals

- Multi-period default prediction model
- Higher long-term accuracy than Duan et al. (2012)
- Less parameters
- A more dynamic intensity parameterisation

Default prediction with Nelson-Siegel forward intensities
Outline

1. Motivation ✓
2. Modelling Approach
3. Model Estimation
4. Outlook

Default prediction with Nelson-Siegel forward intensities
Market exit scenarios

- Duffie et al. (2007) and Duan et al. (2012) allow for two types of market exit
  - Company default
  - Other reasons (i.e. takeover)

- Necessary model adjustments
  - Time of default $\tau_D$: process $M_t$ with intensity $\lambda_t$
  - Time of other exit $\tau_O$: process $L_t$ with intensity $\phi_t$
  - Combined exit time $\tau_C$: $\min(\tau_D, \tau_O)$

\[
S_t(T) = \exp \left[ - \int_t^T \{ \lambda_t(s) + \phi_t(s) \} \, ds \right] 
\]

(st)

\[
PD_t(T) = \int_t^T \exp \left[ - \int_t^s \{ \lambda_t(s) + \phi_t(s) \} \, du \right] \lambda_t(s) \, ds
\]

Default prediction with Nelson-Siegel forward intensities
Intuition

Figure 3: Estimated default and other-exit forward intensities for a random company in December, 1991, using the Duan et al. (2012) model.
Figure 4: Conditional default intensity for Xerox on January 1, 2001, taken from Duffie et al. (2007).

Default prediction with Nelson-Siegel forward intensities
Term structure of intensity curve can be assumed smooth

- Forward curve of default intensities should be function of
  - Prediction horizon
  - Covariates $x_t = (u_t, y_t)^T$
  - Time $t$

Apply a dynamic curve model directly to forward intensities.
General dynamic curve model

- Intensities take some functional form:
  \[
  \lambda_t(s) = f(t, s, \alpha_t) \tag{9}
  \]
  \[
  \phi_t(s) = g(t, s, \beta_t), \tag{10}
  \]

  where \(\alpha_t, \beta_t\) are \((p \times 1)\) and \((q \times 1)\) vectors of factors

- \(\alpha_t\) and \(\beta_t\) determined through dimensionality reduction

- Various factor models are possible
  - Dynamic semiparametric factor model
  - Dynamic Nelson-Siegel model
  - ... 

Default prediction with Nelson-Siegel forward intensities
Dynamic Nelson-Siegel (DNS) model

- Dynamic version of Nelson et al. (1987) yield curve model by Diebold et al. (2006)
- Curve dynamics driven by three latent factors:
  - level: $L_t$
  - slope: $S_t$
  - curvature: $C_t$
- Latent factors identified as the first three principal components of yields

Default prediction with Nelson-Siegel forward intensities
Spot yield curve given by

\[ y_t(T) = L_t + S_t \left[ \frac{1 - \exp\{- (T - t) \delta\}}{(T - t) \delta} \right] + C_t \left[ \frac{1 - \exp\{- (T - t) \delta\}}{(T - t) \delta} - \exp\{- (T - t) \delta\} \right], \]

where \( \delta \) is called decay factor and \( T \) is the maturity.

The forward curve is given by

\[ F_t(T) = y_t(T) + y'_t(T)(T - t) \]

\[ = L_t + S_t \exp\{- (T - t) \delta\} + C_t \delta(T - t) \exp\{- (T - t) \delta\}. \]
Is DNS appropriate for forward intensities?

- Explained variance in Duan et al. (2012) intensity estimates
  - Mean: 99.96%
  - Median: 99.98%
  - Standard deviation: 0.05% across firms

Default prediction with Nelson-Siegel forward intensities
Is DNS appropriate for forward intensities?

Figure 5: Standardized PCs ($PC_1$, $PC_2$, $PC_3$) and $L_t$, $S_t$, $C_t$ of estimated default intensity curves for firms from Duan et al. (2012) sample.

Default prediction with Nelson-Siegel forward intensities
Forward intensity DNS specification

Model each forward intensity with a separate DNS forward curve

\[\lambda_t(s) = \alpha_{t1} + \alpha_{t2} \exp\{-(s - t)\delta_{\lambda}\} \]
\[+ \alpha_{t3} \delta_{\lambda} (s - t) \exp\{-(s - t)\delta_{\lambda}\}\]  \(\text{(14)}\)

\[\phi_t(s) = \beta_{t1} + \beta_{t2} \exp\{-(s - t)\delta_{\phi}\} \]
\[+ \beta_{t3} \delta_{\phi} (s - t) \exp\{-(s - t)\delta_{\phi}\}\]  \(\text{(15)}\)

Choice of forward version because of forward intensity curve shape

Default prediction with Nelson-Siegel forward intensities
Inclusion of observable covariates

- Common approach: $\alpha_t = (\alpha_1 t, \alpha_2 t, \alpha_3 t)^\top$ modelled as VAR, together with covariates $x_t$
- Our assumption: elements of $\alpha_t$ are affine functions of $x_t$

$$\alpha_t = A \begin{pmatrix} 1 \\ x_t \end{pmatrix}, \quad A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k+m+1} \\ \vdots & \ddots & \vdots \\ a_{3,1} & \cdots & a_{3,k+m+1} \end{pmatrix}$$

- $\phi_t$ modelled equivalently with factors $\beta_t$ and transformation matrix $B$

Default prediction with Nelson-Siegel forward intensities
Model advantages

- Entire term structure of intensities given at any time $t$
  - Distribution of default times computable
- Much fewer parameters than in Duan et al. (2012)
  - $2 \times 3 \times (k + m + 1) + 2$

Model disadvantages

- No dynamic dependence structure between intensities and covariates
- Guarantee for positive in-sample intensities, but possibility of negative out-of-sample ones

Default prediction with Nelson-Siegel forward intensities
Likelihood notation

- Sample of $n$ firms
- Overall sample period: $[t_0, t]$
- Firm $i$ observation period: $[t_{0i}, t_i]$, $t_i = \min(\tau_{Ci}, t)$
- Denote $\Upsilon(t) = \{t_1, t_2, \ldots, t_n\}$
- Prediction period: $(t, T]$
**Likelihood function**

- Firm $i$ survival duration likelihood

$$
\ell_i (A, B, \delta_\lambda, \delta_\phi) = \exp \left\{ - \int_{t_{0i}}^{t_i} \{ \lambda_{t_{0i}}(s) + \phi_{t_{0i}}(s) \} \, ds \right\} \times \{ 1(t_i = t) \\
+ 1(t_i = \tau_D) \lambda_{t_{0i}}(t_i) \\
+ 1(t_i = \tau_O) \phi_{t_{0i}}(t_i) \} \right. 
$$

where $\lambda_t(s)$ and $\phi_t(s)$ are given by (14) and (15)

- Sample likelihood

$$
\mathcal{L} \{ A, B, \delta_\lambda, \delta_\phi; \tau(t) \} = \prod_{i=1}^{n} \ell_i (A, B, \delta_\lambda, \delta_\phi) 
$$
Decomposed likelihood function

(17) decomposable into default and other-exit parts

\[
L \{A, B, \delta_\lambda, \delta_\phi; \Upsilon(t)\} = L \{A, \delta_\lambda; \Upsilon(t)\} \times L \{B, \delta_\phi; \Upsilon(t)\} \quad (18)
\]

\[
= \prod_{i=1}^{n} \ell^\lambda_i (A, \delta_\lambda) \times \prod_{i=1}^{n} \ell^\phi_i (B, \delta_\phi) \quad (19)
\]

Further decompositions likely possible

- Likelihoods for rows of A and B
- Conditional on \( \delta_\lambda \) and \( \delta_\phi \)

Default prediction with Nelson-Siegel forward intensities
Next steps

- Calibrate the model to time series of estimated forward intensities from Duan et al. (2012)
- Fit the model to raw firm data using the MLE approach
- Investigate a broader set of curve models
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Average intensity

- Time-\(t\) conditional CDF of \(\tau_C\) at \(s > t\): \(F_t(s)\)
- Define

\[
K_t(s) \equiv - \frac{\log \{1 - F_t(s)\}}{s - t} \quad (20)
\]

\[
= - \frac{\log E_t \{\exp \left(- \int_t^s \lambda_u + \phi_u \, du\right)\}}{s - t} . \quad (21)
\]

- Survival probability over \([t, s]\) is now given by

\[
P_t(\tau_C > s) = \exp \{-K_t(s)(s - t)\} . \quad (22)
\]
The concept of forward intensity

Assume $F_t$ and $K_t$ as differentiable.

The combined exit time $t$ forward intensity for time $s$ is defined as

$$\kappa_t(s) = \lambda_t(s) + \phi_t(s)$$

$$\equiv \frac{F'(s)}{1 - F_t(s)}$$

$$= K_t(s) + K'_t(s) (s - t).$$

Hence,

$$\exp\{-K_t(s)(s - t)\} = \exp\left\{-\int_t^s \kappa_t(u) \, du\right\}.$$  (24)
□ Forward default intensity censored by other forms of exit:

\[
\begin{align*}
\lambda_t(s) & \equiv e^{K_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{P_t(s < \tau_D = \tau_C \leq s + \Delta t)}{\Delta t} \\
&= e^{K_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{E_t \left\{ \int_s^{s+\Delta t} \exp \left( - \int_t^u \lambda_z + \phi_z \, dz \right) \lambda_u \, du \right\}}{\Delta t}
\end{align*}
\] (25)

□ Forward intensity of other forms of exit censored by default:

\[
\begin{align*}
\phi_t(s) & \equiv e^{K_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{P_t(s < \tau_{Oi} = \tau_C \leq s + \Delta t)}{\Delta t} \\
&= e^{K_t(s)(s-t)} \lim_{\Delta t \to 0} \frac{E_t \left\{ \int_s^{s+\Delta t} \exp \left( - \int_t^u \lambda_z + \phi_z \, dz \right) \phi_u \, du \right\}}{\Delta t}
\end{align*}
\] (27)

Default prediction with Nelson-Siegel forward intensities
Firm-specific decomposed likelihoods

\[ \ell^\lambda_i (A, \delta_\lambda) = \exp \left\{ - \int_{t_0}^{t_i} \lambda_{t_0i}(s) ds \right\} \times \]
\[ \{1(t_i = t) \]
\[ + 1(t_i = \tau_D) \lambda_{t_0i}(t_i) \]
\[ + 1(t_i = \tau_O) \} \]
\[ \text{(29)} \]

and

\[ \ell^\phi_i (B, \delta_\phi) = \exp \left\{ - \int_{t_0}^{t_i} \phi_{t_0i}(s) ds \right\} \times \]
\[ \{1(t_i = t) \]
\[ + 1(t_i = \tau_D) \]
\[ + 1(t_i = \tau_O) \phi_{t_0i}(t_i) \} \]
\[ \text{(30)} \]


